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FURTHER RESULT OF A TYPE OF SYSTEM OF COMPLEX DIFFERENCE EQUATIONS

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Abstract

Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the form of a type of system of difference equations and extend some result of solutions of difference equations to systems of complex difference equations.

1. Introduction

We use the standard notation of the Nevanlinna theory of meromorphic functions (see, e.g., [1]). In addition, we denote the order of $f(z)$ by $\rho(f)$.

Recently, Laine et al. [7]; Heittokangas et al. [5]; Halburd and Korhonen [6]; Chen and Ho [2]; Chen [3]; Chiang and Feng [4] etc.

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investigated the existence or growth of solutions of complex difference equations, Gao [9-13] investigated the existence or growth of solutions of systems of complex difference equations, they obtain some results.

In 2005, Laine et al. [7] considered the following difference equations:

$$\sum_{j=1}^{n} \alpha_j(z) f(z + c_j) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{1}$$

where the coefficients $\alpha_j(z)$ are non-vanishing small functions relative to f and where P, Q are relatively prime polynomials in f over the field of small functions relative to f. He obtained the following theorem:

Theorem A ([7]). *Suppose that* $c_1, \cdots, c_n$ *are distinct, non-zero complex numbers and that f is a transcendental meromorphic solution of Equation* (1), $q = \deg_f^Q > 0$, $n = \max\{p, q\} = \max\{\deg_f^P, \deg_f^Q\}$ *and that, without restricting generality, Q is a monic polynomial. If there exists* $\alpha \in [0, n)$ *such that for all r sufficiently large,*

$$\overline{N}\Big(r, \sum_{j=1}^{n} \alpha_j(z) f(z + c_j)\Big) \leq \alpha \overline{N}(r + C, f(z)) + S(r, f(z)),$$

where $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$, *then either the order* $\rho(f) = \infty$, *or*

$$Q(z, f(z)) \equiv (f(z) + h(z))^q,$$

where $h(z)$ *is a small meromorphic function.*

Our question arises whether or not the assertion of Theorem A remains valid, if we replace the difference equations with system of complex difference equations. In this paper, we first consider the problem of the growth of solution of system of complex difference equations of the form:

$$\begin{cases} \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j) = \dfrac{P_1(z, w_2)}{Q_1(z, w_2)}, \\ \sum_{j=1}^{n} \beta_j(z) w_2(z + c_j) = \dfrac{P_2(z, w_1)}{Q_2(z, w_1)}, \end{cases} \tag{2}$$

where the coefficients $\alpha_j(z), \beta_j(z)$ are non-vanishing small functions relative to w_1, w_2 and P_1, Q_1 are relatively prime polynomials in w_2, P_2, Q_2 are relatively prime polynomials in w_1. We assume that $n_1 = \max\{p_1, q_1\}$, $n_2 = \max\{p_2, q_2\}$, Q_1, Q_2 are monic polynomial.

Definition 1. Let $(w_1(z), w_2(z))$ be a set of meromorphic solutions of (2). If the component w_k of (w_1, w_2) satisfies the following condition:

$$\limsup_{r\to\infty,\, r\notin I} \frac{S(r)}{T(r, w_k)} = 0, \quad k = 1, 2,$$

we say that w_k is an admissible component of a set of solutions of (2), where I is a set of finite linear measure.

Let $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. We obtain the following result:

Theorem 1. *Let* $(w_1(z), w_2(z))$ *be admissible meromorphic solution of* (1), $\rho(w_i) < +\infty$, $i = 1, 2$. *If there exist* $\alpha, \beta \in [0, n)$ *such that for all* r *sufficiently large*,

$$\overline{N}\left(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j)\right) \le \alpha \overline{N}(r + C, w_1(z)) + S(r, w_1),$$

$$\overline{N}\left(r, \sum_{j=1}^{n} \beta_j(z) w_2(z + c_j)\right) \le \beta \overline{N}(r + C, w_2(z)) + S(r, w_2),$$

then

$$Q_1(z, w_2) \equiv (w_2(z) + h_1(z))^{q_1}, \quad Q_2(z, w_1) \equiv (w_1(z) + h_2(z))^{q_2},$$

at least one of them will be true, where $h_1(z), h_2(z)$ *are small meromorphic functions.*

Example 1. $(w_1(z), w_2(z)) = (\frac{1}{z+1}, \frac{1}{z-1})$ is an admissible solution of the following system of complex difference equations:

$$\begin{cases} w_1(z-1) = \dfrac{w_2(z)}{w_2(z)+1}, \\ w_2(z+1) = \dfrac{-w_1(z)}{w_1(z)-1}, \end{cases}$$

w_1, w_2 satisfy the conditions in Theorem 1, $q_1 = q_2 = 1$. It shows that Theorem 1 holds.

2. Some Lemmas

Lemma 1 ([8]). *Let $w(z)$ be a meromorphic function and let ϕ be given by*

$$\phi = w^n + a_{n-1}w^{n-1} + \cdots + a_0,$$

$$T(r, a_j) = S(r, w), \quad j = 0, \cdots, n-1.$$

Then either

$$\phi \equiv (w + \frac{a_{n-1}}{a_n})^n,$$

or

$$T(r, w) \leq \overline{N}(r, \frac{1}{\phi}) + \overline{N}(r, w) + S(r, w).$$

Lemma 2 ([7]). *Let w be a non-constant meromorphic function and let $P(z, w)$, $Q(z, w)$ be two polynomials in w with meromorphic coefficients small relative to w. If P and Q have no common factors of positive degree in w over the field of small functions relative to w, then*

$$\overline{N}(r, \frac{1}{Q(z, w)}) \leq \overline{N}(r, \frac{P(z, w)}{Q(z, w)}) + S(r, w).$$

3. Proof of Theorem 1

Proof. Suppose

$$Q_1(z, w_2) \not\equiv (w_2(z) + h_1(z))^{q_1}, \quad Q_2(z, w_1) \not\equiv (w_1(z) + h_2(z))^{q_2}.$$

Then by Lemma 1, we obtain

$$T(r, w_2) \le \overline{N}(r, \frac{1}{Q_1}) + \overline{N}(r, w_2) + S(r, w_2) \le \overline{N}(r, \frac{P_1}{Q_1}) + \overline{N}(r, w_2) + S(r, w_2),$$

$$T(r, w_2) - \overline{N}(r, w_2) \le \overline{N}(r, \frac{P_1}{Q_1}) + S(r, w_2)$$

$$= \overline{N}(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j)) + S(r, w_2)$$

$$\le \alpha \overline{N}(r + C, w_1(z)) + S(r, w_1) + S(r, w_2).$$

Similarly, we obtain

$$T(r, w_1) - \overline{N}(r, w_1) \le \beta \overline{N}(r + C, w_2(z)) + S(r, w_1) + S(r, w_2),$$

$\rho(w_i) < \infty$, $i = 1, 2$ implies that

$$S(r, w_1(z + c_j)) = S(r, w_1), \quad S(r, w_2(z + c_j)) = S(r, w_2).$$

Hence

$$T(r, w_2(z + c_j)) - \overline{N}(r, w_2(z + c_j)) \le \alpha \overline{N}(r + C, w_1(z + c_j)) + S(r, w_1) + S(r, w_2),$$

$$T(r, w_1(z + c_j)) - \overline{N}(r, w_1(z + c_j)) \le \beta \overline{N}(r + C, w_2(z + c_j)) + S(r, w_1) + S(r, w_2).$$

By Lemma 2 and the system (2), we have

$$n_1 T(r, w_2) = T(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j)) + S(r, w_1)$$

$$= T(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j)) + S(r, w_1)$$

$$- \overline{N}(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j)) + \overline{N}(r, \sum_{j=1}^{n} \alpha_j(z) w_1(z + c_j))$$

$$\leq \sum_{j=1}^{n} [T(r, w_1(z + c_j)) - \overline{N}(r, w_1(z + c_j))]$$

$$+ \alpha \overline{N}(r + C, w_1(z)) + S(r, w_1)$$

$$\leq \sum_{j=1}^{n} \beta \overline{N}(r + C, w_2(z + c_j)) + \alpha \overline{N}(r + C, w_1(z))$$

$$+ S(r, w_1) + S(r, w_2)$$

$$\leq \sum_{j=1}^{n} \beta \overline{N}(r + 2C, w_2(z)) + \alpha \overline{N}(r + C, w_1(z))$$

$$+ S(r, w_1) + S(r, w_2),$$

i.e.,

$$n_1 T(r, w_2) \leq n\beta \overline{N}(r + 2C, w_2(z)) + \alpha \overline{N}(r + 2C, w_1(z)) + S(r, w_1) + S(r, w_2).$$

Similarly,

$$n_2 T(r, w_1) \leq n\alpha \overline{N}(r + 2C, w_1(z)) + \beta \overline{N}(r + 2C, w_2(z)) + S(r, w_1) + S(r, w_2).$$

Hence

$$\begin{cases} T(r, w_2) \leq \dfrac{n\beta}{n_1} \overline{N}(r + 2C, w_2(z)) + \dfrac{\alpha}{n_1} \overline{N}(r + 2C, w_1(z)) + S(r, w_1) + S(r, w_2), \\ T(r, w_1) \leq \dfrac{n\alpha}{n_2} \overline{N}(r + 2C, w_1(z)) + \dfrac{\beta}{n_2} \overline{N}(r + 2C, w_2(z)) + S(r, w_1) + S(r, w_2). \end{cases}$$

$$\begin{cases} T(r, w_2) - \overline{N}(r, w_2(z)) \le \dfrac{n\beta}{n_1}\overline{N}(r+2C, w_2(z)) + \dfrac{\alpha}{n_1}\overline{N}(r+2C, w_1(z)) \\ \qquad\qquad - \overline{N}(r, w_2(z)) + S(r, w_1) + S(r, w_2), \\ T(r, w_1) - \overline{N}(r, w_1(z)) \le \dfrac{n\alpha}{n_2}\overline{N}(r+2C, w_1(z)) + \dfrac{\beta}{n_2}\overline{N}(r+2C, w_2(z)) \\ \qquad\qquad - \overline{N}(r, w_1(z)) + S(r, w_1) + S(r, w_2). \end{cases}$$

$$\begin{cases} T(r, w_2) - \overline{N}(r, w_2(z)) \le \dfrac{\beta(n+m)}{n_1}\overline{N}(r+2mC, w_2(z)) + \dfrac{m\alpha}{n_1}\overline{N}(r+2mC, w_1(z)) \\ \qquad\qquad - m\overline{N}(r, w_2(z)) + S(r, w_1) + S(r, w_2), \\ T(r, w_1) - \overline{N}(r, w_1(z)) \le \dfrac{\alpha(n+m)}{n_2}\overline{N}(r+2mC, w_1(z)) + \dfrac{m\beta}{n_2}\overline{N}(r+2mC, w_2(z)) \\ \qquad\qquad - m\overline{N}(r, w_1(z)) + S(r, w_1) + S(r, w_2). \end{cases}$$

We can also obtain

$$\begin{cases} \overline{N}(r, w_2(z)) \le \dfrac{\beta(n_1+m)}{mn_1}\overline{N}(r+2mC, w_2(z)) + \dfrac{\alpha}{n_1}\overline{N}(r+2mC, w_1(z)) \\ \qquad\qquad + S(r, w_1) + S(r, w_2), \\ \overline{N}(r, w_1(z)) \le \dfrac{\alpha(n_2+m)}{mn_2}\overline{N}(r+2mC, w_1(z)) + \dfrac{\beta}{n_2}\overline{N}(r+2mC, w_2(z)) \\ \qquad\qquad + S(r, w_1) + S(r, w_2). \end{cases}$$

For sufficiently large m, we see that

$$\frac{1}{\eta_1} = \beta\frac{(n_1+m)}{mn_1} = \beta\left(\frac{1}{m} + \frac{1}{n_1}\right) < 1,$$

$$\frac{1}{\eta_2} = \alpha\frac{(n_2+m)}{mn_2} = \alpha\left(\frac{1}{m} + \frac{1}{n_2}\right) < 1.$$

$$\overline{N}(r, w_2(z)) \le \frac{1}{\eta_1}\overline{N}(r+2mC, w_2(z)) + \frac{\alpha}{n_1}\overline{N}(r+2mC, w_1(z)) + S(r, w_1) + S(r, w_2),$$

$$\overline{N}(r+2mC,\ w_2(z)) \le \frac{1}{\eta_1}\overline{N}(r+4mC,\ w_2(z)) + \frac{\alpha}{n_1}\overline{N}(r+4mC,\ w_1(z))$$

$$+\ S(r+2mC,\ w_1) + S(r+2mC,\ w_2).$$

Hence

$$\overline{N}(r,\ w_2(z)) \le \frac{1}{\eta_1^l}\overline{N}(r+2lmC,\ w_2(z)) + \frac{1}{\eta_1^{l-1}}\frac{\alpha}{n_1}\overline{N}(r+2lmC,\ w_1(z))$$

$$+\ S(r+2(l-1)mC,\ w_1) + S(r+2(l-1)mC,\ w_2).$$

Similarly, we obtain

$$\overline{N}(r,\ w_1(z)) \le \frac{1}{\eta_2^l}\overline{N}(r+2lmC,\ w_1(z)) + \frac{1}{\eta_2^{l-1}}\frac{\beta}{n_2}\overline{N}(r+2lmC,\ w_2(z))$$

$$+\ S(r+2(l-1)mC,\ w_1) + S(r+2(l-1)mC,\ w_2).$$

But $\overline{N}(r+2lmC,\ w_i(z)) = \overline{N}(r,\ w_i(z)) + S(r,\ w_i),\ i = 1, 2.$ Thus

$$\overline{N}(r,\ w_1(z)) \le \frac{1}{\eta_2^l}\overline{N}(r,\ w_1(z)) + \frac{1}{\eta_2^{l-1}}\overline{N}(r,\ w_2(z)) + S(r,\ w_1) + S(r,\ w_2), \tag{3}$$

$$\overline{N}(r,\ w_2(z)) \le \frac{1}{\eta_1^l}\overline{N}(r,\ w_2(z)) + \frac{1}{\eta_1^{l-1}}\overline{N}(r,\ w_1(z)) + S(r,\ w_1) + S(r,\ w_2). \tag{4}$$

Dividing through (3), (4) by $T(r,\ w_1(z))$ and $T(r,\ w_2(z))$, respectively,

$$\frac{\overline{N}(r,\ w_1(z))}{T(r,\ w_1(z))} \le \frac{1}{\eta_2^l}\frac{\overline{N}(r,\ w_1(z))}{T(r,\ w_1(z))} + \frac{1}{\eta_2^{l-1}}\frac{\overline{N}(r,\ w_2(z))}{T(r,\ w_1(z))} + \frac{S(r,\ w_2)}{T(r,\ w_1)} + o(1), \tag{5}$$

$$\frac{\overline{N}(r,\ w_2(z))}{T(r,\ w_2(z))} \le \frac{1}{\eta_1^l}\frac{\overline{N}(r,\ w_2(z))}{T(r,\ w_2(z))} + \frac{1}{\eta_1^{l-1}}\frac{\overline{N}(r,\ w_1(z))}{T(r,\ w_2(z))} + \frac{S(r,\ w_1)}{T(r,\ w_2)} + o(1). \tag{6}$$

Combining (5) and (6), we have

$$(\eta_1^l - 1 + o(1))(\eta_2^l - 1 + o(1))\frac{\overline{N}(r, w_2(z))}{T(r, w_2(z))}\frac{\overline{N}(r, w_1(z))}{T(r, w_1(z))}$$

$$\leq \eta_1\eta_2\frac{\overline{N}(r, w_2(z))}{T(r, w_2(z))}\frac{\overline{N}(r, w_1(z))}{T(r, w_1(z))} + \frac{\overline{N}(r, w_2(z))}{T(r, w_1(z))}\frac{S(r, w_1(z))}{T(r, w_2(z))} + \frac{\overline{N}(r, w_1(z))}{T(r, w_2(z))}\frac{S(r, w_2(z))}{T(r, w_1(z))}.$$

According to w_1, w_2 are admissible, taking the upper limit as $r \to \infty$, we obtain

$$(\eta_1^l - 1)(\eta_2^l - 1)\limsup_{r\to\infty}\frac{\overline{N}(r, w_2(z))}{T(r, w_2(z))}\frac{\overline{N}(r, w_1(z))}{T(r, w_1(z))}$$

$$\leq (\eta_1\eta_2)\limsup_{r\to\infty}\frac{\overline{N}(r, w_2(z))}{T(r, w_2(z))}\frac{\overline{N}(r, w_1(z))}{T(r, w_1(z))},$$

that is,

$$(\eta_1^l - 1)(\eta_2^l - 1) \leq \eta_1\eta_2.$$

This is a contradiction to the $\eta_i > 1$, $i = 1, 2$, $l > 1$.

This implies that

$$Q_1(z, w_2) \equiv (w_2(z) + h_1(z))^{q_1}, \quad Q_2(z, w_1) \equiv (w_1(z) + h_2(z))^{q_2},$$

at least one of them will be true.

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