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FURTHER RESULTS ON APPROXIMATION BY DOUBLE SINGULAR INTEGRAL OPERATORS WITH RADIAL KERNELS

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Abstract

This paper contains some results on the point-wise convergence of a family of singular integral operators with radial kernels given in the following form:

$$L_\lambda(f;\, x,\, y) = \iint\limits_{\mathbb{R}^2} f(t,\, s)H_\lambda(t-x,\, s-y)\,dtds, \quad (x,\, y) \in \mathbb{R}^2, \quad \lambda \in \Lambda,$$

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where Λ is a set of non-negative numbers with accumulation point λ_0. Here, the functions $f : \mathbb{R}^2 \to \mathbb{R}$ are measurable in the sense of Lebesgue.

1. Introduction

Approximation with linear integral operators or suitable functions finds frequent application in many areas of mathematics including Fourier analysis. The following integral operator is a familiar example:

$$L_n(f; x) = \int_{-\pi}^{\pi} f(t)K_n(t, x)dt, \quad x \in (-\pi, \pi), n \in \mathbb{N}, \tag{1}$$

where $K_n : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ denotes a family of periodic kernels satisfying usual conditions for approximate identities.

In [16], Taberski studied the point-wise approximation of periodic and integrable functions on $\langle -\pi, \pi \rangle$, where $\langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed or open interval. The work used a two parameter family of singular integral operators of the form:

$$L_\lambda(f; x) = \int_{-\pi}^{\pi} f(t)K_\lambda(t - x)dt, \quad x \in \langle -\pi, \pi \rangle, \lambda \in \Lambda, \tag{2}$$

where $K_\lambda : \mathbb{R} \to \mathbb{R}_0^+$ denotes a family of periodic kernels satisfying suitable conditions and Λ is a given set of non-negative numbers with accumulation point λ_0. Following Taberki's line, [4], [9], and [6] obtained more general results. As concerns the study of integral operators in several settings, the reader may see also, e.g., [1, 2] and [13]-[22].

Taberski [14] advanced his analysis to double singular integral operators depending on three parameters of the form:

$$L_\lambda(f; x, y) = \iint_Q f(t, s)K_\lambda(t - x, s - y)dtds, \quad (x, y) \in Q, \tag{3}$$

where $Q = \langle -\pi, \pi\rangle \times \langle -\pi, \pi\rangle$ is an arbitrary closed, semi-closed or open region and $K_\lambda : \mathbb{R}^2 \to \mathbb{R}_0^+$ stands for a family of kernels satisfying some conditions. Those results were later used by Siudut [11, 12] presenting considerable theorems. Then, Rydzewska [8] improved the results of [15]. In recent papers [18]-[20], the kernel function was chosen as a radial function and the domain of integration was replaced by an arbitrary region $\langle a, b\rangle \times \langle c, d\rangle$. In the light of these modifications, some point-wise approximation theorems are presented. Also, we shall mention the papers [7] and [15] which are important works on weighted approximation by singular integral operators. Those contain detailed information regarding the characterization of the weight functions with examples.

The current manuscript deals with point-wise convergence of a family of singular integral operators with radial kernels given in the following form:

$$L_\lambda(f; x, y) = \iint\limits_{\mathbb{R}^2} f(t, s) H_\lambda(t - x, s - y)\, dt ds, \quad (x, y) \in \mathbb{R}^2, \lambda \in \Lambda, \tag{4}$$

where Λ is a set of non-negative numbers with accumulation point λ_0 at a p-generalized Lebesgue point of $f \in L_{p,\varphi}(\mathbb{R}^2)$. Here, $L_{p,\varphi}(\mathbb{R}^2)$ is the collection of all measurable functions for which $\left|\frac{f}{\varphi}\right|^p$ is integrable on $\mathbb{R}^2 (1 \le p < \infty)$ provided $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ is a weight function, which is measurable and bounded on any bounded subset of $\mathbb{R}^2$.

2. Preliminaries

In this section, we introduce the main definitions which have vital roles in this work.

Definition 1. A function $H \in L_1(\mathbb{R}^2)$, is said to be *radial*, if there exists a function $K : \mathbb{R}_0^+ \to \mathbb{R}$ such that $H(t, s) = K(\sqrt{t^2 + s^2})$ almost everywhere [3].

Now, we give another characterization of p-generalized Lebesgue point for the functions of two variables.

Definition 2. A p-generalized Lebesgue point of a locally (or globally) integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ is a point $(x_0, y_0) \in \mathbb{R}^2$ satisfying

$$\lim_{(h,k)\to(0,0)} \left(\frac{1}{h^{\alpha+1}k^{\beta+1}} \int_0^h \int_0^k |g(t + x_0, s + y_0) - g(x_0, y_0)|^p \, dtds \right)^{\frac{1}{p}} = 0,$$

$$1 \le p < \infty, \; 0 \le \alpha, \beta < 1.$$

If one takes $\alpha = \beta$, then the above definition reduces to definition of p-generalized Lebesgue point given in [20]. For some examples concerning this point, see [18].

Definition 3 (Class A_φ)**.** Let $H_\lambda : \mathbb{R}^2 \to \mathbb{R}_0^+$ be a radial function, i.e., there exists a function $K_\lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $H_\lambda(t, s) := K_\lambda \left(\sqrt{t^2 + s^2}\right)$ holds almost everywhere on $\mathbb{R}^2$ for each fixed $\lambda \in \Lambda$. Further, let $H_\lambda : \mathbb{R}^2 \to \mathbb{R}_0^+$ be a family of radial kernels which are integrable on $\mathbb{R}^2$, for each fixed $\lambda \in \Lambda$ and the weight function $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ is bounded and measurable on any bounded subset of $\mathbb{R}^2$ and satisfies the following inequality:

$$\varphi(u + t, v + s) \le \varphi(u, v)\varphi(t, s), \quad (t, s) \in \mathbb{R}^2, \quad (u, v) \in \mathbb{R}^2. \tag{5}$$

We say that $H_\lambda(t, s)$ belongs to class A_φ, if the following conditions are satisfied:

(a) At a p-generalized Lebesgue point $(x_0, y_0) \in \mathbb{R}^2$ of the weight function φ, the following relation:

$$\lim_{(x,y,\lambda)\to(x_0,y_0,\lambda_0)} \frac{1}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda\left(\sqrt{(t - x)^2 + (s - y)^2}\right) dtds = 1$$

holds.

(b) $\forall \xi > 0$,

$$\lim_{(x, y, \lambda) \to (x_0, y_0, \lambda_0)} \sup_{\xi \leq \sqrt{(t^2 + s^2)}} \left[\varphi(t, s) K_\lambda \left(\sqrt{(t - x)^2 + (s - y)^2} \right) \right] = 0.$$

(c) $\forall \xi > 0$,

$$\lim_{(x, y, \lambda) \to (x_0, y_0, \lambda_0)} \left[\iint_{\xi \leq \sqrt{(t^2 + s^2)}} \varphi(t, s) K_\lambda \left(\sqrt{(t - x)^2 + (s - y)^2} \right) dt ds \right] = 0.$$

(d) $K_\lambda \left(\sqrt{t^2 + s^2} \right)$ is monotonically increasing with respect to t on $(-\infty, 0]$ and similarly, $K_\lambda \left(\sqrt{t^2 + s^2} \right)$ is monotonically increasing with respect to s on $(-\infty, 0]$ for any $\lambda \in \Lambda$. Analogously, $K_\lambda \left(\sqrt{t^2 + s^2} \right)$ is bimonotonically increasing with respect to (t, s) on $[0, \infty) \times [0, \infty)$ and $(-\infty, 0] \times (-\infty, 0]$ and bimonotonically decreasing with respect to (t, s) on $[0, \infty) \times (-\infty, 0]$ and $(-\infty, 0] \times [0, \infty)$ for any $\lambda \in \Lambda$.

(e) $\|\varphi K_\lambda\|_{L_1(\mathbb{R}^2)} \leq M < \infty$, for all $\lambda \in \Lambda$.

(f) For fixed $(t_0, s_0) \in \mathbb{R}^2$, $K_\lambda \left(\sqrt{t_0^2 + s_0^2} \right)$ tends to infinity as λ tends to λ_0.

Note that, throughout this paper, we suppose that the kernel $H_\lambda(t, s)$ belongs to class A_φ.

Remark 1. If the function $g : \mathbb{R}^2 \to \mathbb{R}$ is bimonotonic on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$, then the equality given by

$$\begin{aligned} V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) &= \bigvee_{\alpha_1}^{\alpha_2} \bigvee_{\beta_1}^{\beta_2} (g(t, s)) \\ &= |g(\alpha_1, \beta_1) - g(\alpha_1, \beta_2) - g(\alpha_2, \beta_1) + g(\alpha_2, \beta_2)| \end{aligned} \tag{6}$$

holds [14, 5].

3. Existence of the Operator

The following lemma gives the existence of the operators defined by (4). For this kind of existence theorems, see [21, 20].

Lemma 1. *If* $f \in L_{p,\varphi}(\mathbb{R}^2)$, *then the operators* $L_\lambda(f;\, x,\, y)$ *define a continuous transformation acting on* $L_{p,\varphi}(\mathbb{R}^2)$.

Proof. Since $L_\lambda(f;\, x,\, y)$ is linear, it is sufficient to show that the norm given in the form

$$\|L_\lambda\|_\varphi = \sup_{f \neq 0} \frac{\|L_\lambda(f;\, x,\, y)\|_{L_{p,\varphi}(\mathbb{R}^2)}}{\|f\|_{L_{p,\varphi}(\mathbb{R}^2)}},$$

is bounded. Here the norm $\|f\|_{L_{p,\varphi}(\mathbb{R}^2)}$ given by

$$\|f\|_{L_{p,\varphi}(\mathbb{R}^2)} = \left(\iint_{\mathbb{R}^2} \left| \frac{f(t,\, s)}{\varphi(t,\, s)} \right|^p dtds \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

See, for example, [7, 15].

Now, let $1 < p < \infty$. In view of (5) and generalized Minkowski inequality [13], we have

$$\|L_\lambda(f;\, x,\, y)\|_{L_{1,\varphi}(\mathbb{R}^2)}$$

$$= \left(\iint_{\mathbb{R}^2} \left(\frac{1}{\varphi(x,\, y)} \left| \iint_{\mathbb{R}^2} f(t,\, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2} \right) dtds \right| \right)^p dxdy \right)^{\frac{1}{p}}$$

$$\leq \iint_{\mathbb{R}^2} K_\lambda\left(\sqrt{t^2 + s^2} \right) \varphi(t,\, s) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x,\, s+y)}{\varphi(t+x,\, s+y)} \right|^p dxdy \right)^{\frac{1}{p}} dtds$$

$$= \|\varphi K_\lambda\|_{L_1(\mathbb{R}^2)} \leq M \|f\|_{L_{p,\varphi}(\mathbb{R}^2)}.$$

Thus, the proof is completed for $1 < p < \infty$. One may easily prove the assertion for the case $p = 1$ by using similar method. Thus, this case is omitted. Hence the proof is finished. □

4. Point-wise Convergence

The following theorem gives a point-wise convergence of the integral operators of type (4) at a common p-generalized Lebesgue point of $f \in L_{p,\varphi}(\mathbb{R}^2)$ and the weight function $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$. It should be noticed that local integrability of $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ is guaranteed by the properties of it.

Theorem 1. *If* $(x_0, y_0) \in \mathbb{R}^2$ *is a common p-generalized Lebesgue point of* $f \in L_{p,\varphi}(\mathbb{R}^2)$ *and* φ, *then*

$$\lim_{(x, y, \lambda)\to(x_0, y_0, \lambda_0)} L_\lambda(f; x, y) = f(x_0, y_0),$$

on any set Z on which the function

$$\sup_{(t,s)\in B_\delta} \varphi(t, s)\left\{\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta} K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right)\left|\left\{|x_0 - t|^{\alpha+1}\right\}_t'\right|\right.$$

$$\times\left|\left\{|y_0 - s|^{\beta+1}\right\}_s'\right| dtds + 2|y_0 - y|^{\beta+1}\int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|)\left|\left\{|x_0 - t|^{\alpha+1}\right\}_t'\right| dt$$

$$+ 2|x_0 - x|^{\alpha+1}\int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|)\left|\left\{|y_0 - s|^{\beta+1}\right\}_s'\right| ds$$

$$\left. + 4K_\lambda(0)|x_0 - x|^{\alpha+1}|y_0 - y|^{\beta+1}\right\}, \tag{7}$$

where $B_\delta := \left\{(t, s) : (t - x_0)^2 + (s - y_0)^2 < \delta^2, (x_0, y_0) \in \mathbb{R}^2\right\}$, *is bounded as* (x, y, λ) *tends to* (x_0, y_0, λ_0).

Proof. Suppose that $(x_0, y_0) \in \mathbb{R}^2$ is a p-generalized Lebesgue point of function $f \in L_{p,\varphi}(\mathbb{R}^2)$. Besides, we may assume that $0 < x_0 - x < \delta/2$ and $0 < y_0 - y < \delta/2$ for $0 < \delta < \infty$. Therefore, for all given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all h and k satisfying $0 < h, k \leq \delta$, we have the inequality:

$$\int_{x_0}^{x_0+h} \int_{y_0-k}^{y_0} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p dtds < \varepsilon^p h^{\alpha+1} k^{\beta+1}. \tag{8}$$

In view of condition (a), we have

$$|L_\lambda(f; x, y) - f(x_0, y_0)|$$

$$\leq \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds$$

$$+ \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \left| \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds - \varphi(x_0, y_0) \right|$$

$$= I_1 + I_2.$$

Let $1 < p < \infty$. Using Hölder's inequality, see [10], for the term I_1, we have

$$|L_\lambda(f; x, y) - f(x_0, y_0)|$$

$$\leq \left(\iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds \right)^{\frac{1}{p}}$$

$$\times \left(\iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds \right)^{\frac{1}{q}} + I_2.$$

Since whenever for m, n being positive numbers the inequality $(m+n)^p \le 2^p(m^p + n^p)$ holds, see [10], by taking p-th power of both sides, we have

$$|L_\lambda(f;\, x,\, y) - f(x_0,\, y_0)|^p$$

$$\le 2^p \iint_{\mathbb{R}^2} \left|\frac{f(t,\, s)}{\varphi(t,\, s)} - \frac{f(x_0,\, y_0)}{\varphi(x_0,\, y_0)}\right|^p \varphi(t,\, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds$$

$$\times \left(\iint_{\mathbb{R}^2} \varphi(t,\, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds\right)^{p/q}$$

$$+ 2^p \left|\frac{f(x_0,\, y_0)}{\varphi(x_0,\, y_0)}\right|^p \left|\iint_{\mathbb{R}^2} \varphi(t,\, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds - \varphi(x_0,\, y_0)\right|^p$$

$$= 2^p I_{11} I^* + 2^p I_{21}.$$

In view of condition (a) of class A_φ, $I_{21} \to 0$ and $I^* \to \varphi(x_0,\, y_0)^{p/q} < \infty$ as $(x,\, y,\, \lambda)$ tends to $(x_0,\, y_0,\, \lambda_0)$, respectively. It is easy to see that I_{11} can be written in the form:

$$I_{11} = \left\{\iint_{\mathbb{R}^2 \setminus B_\delta} + \iint_{B_\delta}\right\} \left|\frac{f(t,\, s)}{\varphi(t,\, s)} - \frac{f(x_0,\, y_0)}{\varphi(x_0,\, y_0)}\right|^p \varphi(t,\, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2}\right) dtds$$

$$= I_{111} + I_{112},$$

where $B_\delta := \left\{(t,\, s) : (t-x_0)^2 + (s-y_0)^2 < \delta^2,\, (x_0,\, y_0) \in \mathbb{R}^2\right\}$. Since

$$I_{111}2^{-p} \leq \sup_{(t,s)\in\mathbb{R}^2\setminus B_\delta}\left[\varphi(t, s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)\right]\|f\|^p_{L_{p,\varphi}(\mathbb{R}^2)}$$

$$+\left|\frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\right|^p \iint_{\mathbb{R}^2\setminus B_\delta}\varphi(t, s)K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dtds$$

holds, by condition (b) and (c) of class A_φ, $I_{111} \to 0$ as $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$.

Let us prove that I_{112} tends to zero as (x, y, λ) tends to (x_0, y_0, λ_0). Since $\varphi(t, s)$ is bounded on arbitrary bounded subsets of $\mathbb{R}^2$, it is bounded on B_δ. Therefore, it is easy to see that the following inequality holds for I_{112}:

$$I_{112} < \sup_{(t,s)\in B_\delta}\varphi(t, s)\int_{x_0-\delta}^{x_0+\delta}\int_{y_0-\delta}^{y_0+\delta}\left|\frac{f(t, s)}{\varphi(t, s)}-\frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\right|^p K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dtds$$

$$= \sup_{(t,s)\in B_\delta}\varphi(t, s)I^*_{112}.$$

Splitting the integral I^*_{112} into four parts, we have

$$I^*_{112} \leq \left\{\int_{x_0}^{x_0+\delta}\int_{y_0-\delta}^{y_0} + \int_{x_0-\delta}^{x_0}\int_{y_0-\delta}^{y_0}\right\}\left|\frac{f(t, s)}{\varphi(t, s)}-\frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\right|^p K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dtds$$

$$+\left\{\int_{x_0-\delta}^{x_0}\int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta}\int_{y_0}^{y_0+\delta}\right\}\left|\frac{f(t, s)}{\varphi(t, s)}-\frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\right|^p$$

$$\times K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right)dtds$$

$$= I^*_{1121} + I^*_{1122} + I^*_{1123} + I^*_{1124}.$$

Let us consider the integral I^*_{1121}.

We define the variations:

$$E_1(u, v) := \begin{cases} \bigvee_{u}^{x_0+\delta-x} \bigvee_{y_0-\delta-y}^{v} (K_\lambda(\sqrt{t^2+s^2})), & x_0 - x \le u < x_0 + \delta - x, \\ & y_0 - \delta - y < v \le y_0 - y, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_2(u) := \begin{cases} \bigvee_{u}^{x_0+\delta-x} \left(K_\lambda\left(\sqrt{t^2 + (y_0 - \delta - y)^2}\right)\right), & x_0 - x \le u < x_0 + \delta - x, \\ 0, & \text{otherwise.} \end{cases}$$

$$E_3(v) := \begin{cases} \bigvee_{y_0-\delta-y}^{v} \left(K_\lambda\left(\sqrt{(x_0 - x + \delta)^2 + s^2}\right)\right), & y_0 - \delta - y < v \le y_0 - y, \\ 0, & \text{otherwise.} \end{cases}$$

Taking above variations and (8) into account and applying bivariate integration by parts method to last inequality, we have

$$I_{1121}^* \le -\varepsilon \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{y_0-y} \left[E_1(t,s) + E_2(t) + E_3(s) + K_\lambda\left(\sqrt{(x_0 - x + \delta)^2 + (y_0 - \delta - y)^2}\right)\right]$$

$$\times \left\{(t - x_0 + x)^{\alpha+1}\right\}_t' \left\{(y_0 - s - y)^{\beta+1}\right\}_s' dtds$$

$$= \varepsilon^p (i_1 + i_2 + i_3 + i_4).$$

Using Remark 1 and condition (d) of class A_φ, we get

$$i_1 + i_2 + i_3 + i_4 = -\int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{0} K_\lambda\left(\sqrt{t^2+s^2}\right) \left\{(t - x_0 + x)^{\alpha+1}\right\}_t'$$

$$\times \left\{(y_0 - s - y)^{\beta+1}\right\}_s' dtds$$

$$+ \int_{x_0 - x}^{x_0 - x + \delta} \int_{0}^{y_0 - y} \left(K_\lambda \left(\sqrt{t^2 + s^2} \right) - 2K_\lambda (|t|) \right) \left\{ (t - x_0 + x)^{\alpha+1} \right\}_t'$$

$$\times \left\{ (y_0 - s - y)^{\beta+1} \right\}_s' dtds.$$

Hence the following inequality holds for I_{1121}^*:

$$I_{1121}^* \le \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \left| \left\{ (t - x_0)^{\alpha+1} \right\}_t' \right| \left| \left\{ (y_0 - s)^{\beta+1} \right\}_s' \right| dtds$$

$$+ 2\varepsilon^p |y_0 - y|^{\beta+1} \int_{x_0}^{x_0+\delta} K_\lambda (|t - x|) \left| \left\{ |x_0 - t|^{\alpha-1} \right\}_t' \right| dt.$$

Analogous computations for I_{1122}^*, I_{1123}^*, and I_{1124}^* yields

$$I_{1122}^* \le \varepsilon^p \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \left| \left\{ (x_0 - t)^{\alpha+1} \right\}_t' \right| \left| \left\{ (y_0 - s)^{\beta+1} \right\}_s' \right| dtds$$

$$+ 2\varepsilon^p |y_0 - y|^{\beta+1} \int_{x_0-\delta}^{x_0} K_\lambda (|t - x|) \left| \left\{ |x_0 - t|^{\alpha-1} \right\}_t' \right| dt$$

$$+ 2\varepsilon^p |x_0 - x|^{\alpha+1} \int_{y_0-\delta}^{y_0} K_\lambda (|s - y|) \left| \left\{ |y_0 - s|^{\beta+1} \right\}_s' \right| ds$$

$$+ 4\varepsilon^p K_\lambda(0) |x_0 - x|^{\alpha+1} |y_0 - y|^{\beta+1};$$

$$I_{1123}^* \le \varepsilon^p \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} K_\lambda \left(\sqrt{(t-x)^2 + (s-y)^2} \right) \left| \left\{ (x_0 - t)^{\alpha+1} \right\}_t' \right| \left| \left\{ (s - y_0)^{\beta+1} \right\}_s' \right| dtds$$

$$+ 2\varepsilon^p |x_0 - x|^{\alpha+1} \int_{y_0}^{y_0+\delta} K_\lambda (|s - y|) \left| \left\{ |y_0 - s|^{\beta+1} \right\}_s' \right| ds;$$

$$I_{1124}^{*} \leq \varepsilon^{p} \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) \left|\left\{(t-x_0)^{\alpha+1}\right\}_t'\right| \left|\left\{(s-y_0)^{\beta+1}\right\}_s'\right| dtds.$$

Hence the following inequality is obtained for I_{112}:

$$I_{112} \leq \varepsilon^{p} \sup_{(t,s)\in B_\delta} \varphi(t, s) \left\{ \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) \right.$$

$$\times \left|\left\{|x_0 - t|^{\alpha+1}\right\}_t'\right| \left|\left\{|y_0 - s|^{\beta+1}\right\}_s'\right| dtds$$

$$+ 2|y_0 - y|^{\beta+1} \int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|) \left|\left\{|x_0 - t|^{\alpha+1}\right\}_t'\right| dt$$

$$+ 2|x_0 - x|^{\alpha+1} \int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|) \left|\left\{|y_0 - s|^{\beta+1}\right\}_s'\right| ds$$

$$\left. + 4K_\lambda(0)|x_0 - x|^{\alpha+1}|y_0 - y|^{\beta+1} \right\}.$$

The remaining part of the proof is obvious by the hyphothesis (7). The case $0 < x - x_0 < \delta/2$ and $0 < y - y_0 < \delta/2$ for $0 < \delta < \infty$ can be proved in similar way. Besides, the proof of the assertion for $p = 1$ is similar to the above one and thus is omitted. Therefore, the proof is completed. □

5. Rate of Convergence

In this section, we give a theorem about the rate of point-wise convergence.

Theorem 2. *Assume that the hypothesis of Theorem* 1 *is satisfied. Let*

$$\Delta(\lambda, \delta, x, y) = \sup_{(t,s)\in B_\delta} \varphi(t, s) \left\{ \int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} K_\lambda\left(\sqrt{(t-x)^2+(s-y)^2}\right) \right.$$

$$\times \left|\left\{|x_0 - t|^{\alpha+1}\right\}_t'\right| \left|\left\{|y_0 - s|^{\beta+1}\right\}_s'\right| dtds$$

$$+ 2|y_0 - y|^{\beta+1} \int_{x_0-\delta}^{x_0+\delta} K_\lambda(|t-x|) \left| \left\{ |x_0 - t|^{\alpha+1} \right\}_t' \right| dt$$

$$+ 2|x_0 - x|^{\alpha+1} \int_{y_0-\delta}^{y_0+\delta} K_\lambda(|s-y|) \left| \left\{ |y_0 - s|^{\beta+1} \right\}_s' \right| ds$$

$$\left. + 4K_\lambda(0)|x_0 - x|^{\alpha+1}|y_0 - y|^{\beta+1} \right\},$$

for $\delta > 0$ *and the following assumptions are satisfied*:

(i) $\Delta(\lambda, \delta, x, y) \to 0$ *as* $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$ *for some* $\delta > 0$.

(ii) *Letting* $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$, *we have*

$$\left| \frac{1}{\varphi(x_0, y_0)} \iint_{\mathbb{R}^2} \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2} \right) dtds - 1 \right| = o(\Delta(\lambda, \delta, x, y)).$$

(iii) *For every* $\xi > 0$,

$$\sup_{\xi \le \sqrt{(t^2+s^2)}} \left[\varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2} \right) \right] = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$.

(iv) $\forall \xi > 0$,

$$\iint_{\xi \le \sqrt{(t^2+s^2)}} \varphi(t, s) K_\lambda\left(\sqrt{(t-x)^2 + (s-y)^2} \right) dtds = o(\Delta(\lambda, \delta, x, y))$$

as $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$.

Then at each common p-generalized Lebesgue point of $f \in L_{p,\varphi}(\mathbb{R}^2)$ *and* φ, *we have as* $(x, y, \lambda) \to (x_0, y_0, \lambda_0)$

$$|L_\lambda(f; x, y) - f(x_0, y_0)|^p = o(\Delta(\lambda, \delta, x, y)).$$

Proof. The assertion is obvious by the hypothesis of Theorem 1. □

Example 1. Let $\Lambda = (0, \infty)$, $\lambda_0 = 0$, $\varphi(t, s) = (1 + |t|)(1 + |s|)$ and

$$H_\lambda(t, s) = \frac{1}{4\pi\lambda} e^{\frac{-(t^2+s^2)}{4\lambda}}.$$

To verify that $H_\lambda(t, s)$ satisfies the hypotheses of Theorems 1 and 2, see [12]. Also, the given weight function satisfies the hypotheses of Theorems 1 and 2.

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