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CONSTITUTIVE THEORIES FOR INTERNAL POLAR THERMOELASTIC SOLID CONTINUA

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Abstract

In recent papers [1, 2] the authors presented *an internal polar continuum theory* for solid continua (under the name *a polar continuum theory*) in which varying internal rotations and conjugate moments between neighboring material points that exist in all deforming homogeneous and isotropic solid continua were incorporated in the derivations of the conservation and balance laws. It was shown that this theory leads to a more complete thermodynamic framework as it incorporates the additional physics due to varying internal rotations that is completely neglected in the currently used thermodynamic framework. The currently used thermodynamic framework for solid continua is a subset of the internal polar continuum theory presented in [1, 2].

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Just as in non-internal polar thermoelastic solid continua, in internal polar thermoelastic solid continua also (both compressible and incompressible), the mechanical deformation is reversible, hence in such continua there is no mechanism of conversion of mechanical energy into any other forms, hence the rate of mechanical work does not contribute to the rate of entropy production. Thus, in internal polar thermoelastic solid continua the rate of mechanical work equilibrates with the rate of change of kinetic energy and the rate of change of strain energy. In this paper, this aspect of the physics is utilized to derive alternate forms of the first and second laws of thermodynamics applicable to internal polar thermoelastic solid continua. In these derivations of the alternate forms, the strain energy density is removed from the entropy inequality so that the resulting entropy inequality is purely a statement of the rates of entropies. This form is meritorious in demonstrating that the constitutive theories for the stress tensor and moment tensor are independent of the entropy inequality, hence the constitutive theories for the stress tensor and moment tensor have no thermodynamic restrictions. In this paper we consider both forms of the entropy inequality, the entropy inequality containing strain energy density and the one in its absence and other approaches and their applicability in the derivations of the constitutive theories for internal polar thermoelastic solid continua. The solid continua is assumed homogeneous and isotropic. The deformation and strain are assumed to be small.

1. Introduction

The constitutive theories presented here are for an internal polar thermoelastic solid continua for which the conservation and balance laws were derived and presented by Surana et. al. [1,2]. For the sake of completeness of the work presented in this paper, we summarize the continuum theory for internal polar thermoelastic solid continua presented in references [1, 2]. In complex deformation of a solid continua the deformation gradients vary between a material point and its neighbors. Polar decomposition of the Jacobian of deformation or its decomposition into symmetric and skew symmetric tensors shows that varying Jacobian of deformation results in varying rotations between the material points. These varying rotations arise due to varying deformation of the continua, hence are internal to the deforming matter and are explicitly defined by the deformation, hence do not require additional external rotational degrees of freedom.

If the internal varying rotations are resisted by the continua, then there must exist internal moments corresponding to these. The internal rotations and the corresponding moments can result in additional energy storage. This physics is all internal to the deforming continua (hence does not require consideration of external rotational degrees of freedom and associated external moments) and is neglected in the presently used continuum theories for solid continua. The internal polar continuum theory presented in references [1, 2] considers internal varying rotations and associated conjugate moments in the derivation of conservation and balance laws, thus the theory presented in references [1, 2] is an internal polar continuum theory but is clearly different from the micropolar continuum theories published currently in which the material points have six external degrees of freedom, i.e. rotations are three additional external degrees of freedom. The internal polar continuum theory only accounts for internal rotations and associated moments that exist as a consequence of deformation but are neglected in the present theories. This theory is called "internal polar continuum theory" as it considers internal rotations and moments in a deforming solid continua. A comprehensive literature review of micropolar theories, stress couple theories, rotation gradient theories, strain gradient theories with applications to bending, vibrations, beams, plates and shells were presented in references [1, 2], hence are not repeated here. Readers can refer to references [1, 2] for details.

In this paper we consider derivations of constitutive theories for internal polar thermoelastic solid continua. The theories in this paper only consider small deformation and small strains, hence the theories are basis (co- and contravariant and Jaumann rates) independent. If the deforming solid continua is in thermodynamic equilibrium, then the constitutive theories must satisfy conservation and balance laws. It is well known [3] that conservation of mass, balance of momenta (linear and angular), moments of moments and the first law of thermodynamics provide no mechanisms for deriving the constitutive theories. Thus, we generally explore the use of the second law of thermodynamics (entropy inequality) for conditions that may help in the derivations of the constitutive theories. In the case of internal polar thermoelastic solid continua, the rate of mechanical work does not result in rate of entropy production. Since the rate of specific internal energy (e) contains rate of strain energy, the entropy inequality expressed in terms of Helmholtz free energy density (Φ) naturally contains rate of strain energy density. This form of the entropy inequality indeed can be used to derive constitutive theories for stress tensor, moment tensor, and heat vector. These theories may have some thermodynamic restrictions imposed on them if these are warranted due to the entropy inequality. In an alternate approach we could redefine modified specific internal energy ($\underset{\sim}{e}$) and modified Helmholtz free energy density ($\underset{\sim}{\Phi}$) that do not contain strain energy density by just simply removing strain energy density from e, thus from Φ. The results are the alternate forms of the energy equation and the entropy inequality that are free of strain energy density. This entropy inequality is indeed a statement of the rate of entropies and is completely free of rates of strain energy densities. Both forms of the entropy inequality are explored in this paper for deriving the constitutive theories. The entropy inequality in $\underset{\sim}{\Phi}$ provides no mechanism for deriving constitutive theories for the quantities that are responsible for rate of work in thermoelastic solids. However, it may provide conditions for deriving constitutive theory for some other dependent variables in the constitutive theories, such as the heat vector. In the derivations of the constitutive theories presented here we use both forms of the entropy inequality and for each we consider: (i) Conservation and balance laws to determine the dependent variables in the constitutive theories. (ii) The argument tensors of these are decided based on the most general choice of tensors that are admissible based on the physics of internal polar thermoelastic solid continua. (iii) We explore the possibility of using the entropy inequality, theory of generators and invariants, strain energy density function, complementary strain energy density function, and Taylor series expansion in the derivations of the constitutive theories and discuss their merits and shortcomings. (iv) Derivations of material coefficients for each valid approach. (v) Comparisons of the constitutive theories resulting from the various approaches.

2. Conservation and balance laws for internal polar solid continua

The conservation and balance laws for isotropic, homogeneous, compressible as well as incompressible internal polar solid continua in which the rate of mechanical work results in rate of dissipation, hence rate of entropy production as well as rate of strain energy were derived in references [1,2] and are given below (for small deformation, small strain case). In Lagrangian or referential description we have the following.

Conservation of mass

$$\rho_0 = |J|\rho \tag{2.1}$$

Balance of linear momenta

$$\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla} \cdot ({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) = 0 \tag{2.2}$$

Balance of angular momenta

$$m_{mk,m} - \epsilon_{ijk}({}_a\sigma_{ij}) = 0 \tag{2.3}$$

Balance of moments of moments (or couples)

$$\epsilon_{ijk} m_{ij} = 0 \tag{2.4}$$

First law of thermodynamics

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} - d_s - d_m = 0 \tag{2.5}$$

Second law of thermodynamics

$$\rho_0 \left(\frac{D\Phi}{Dt} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} - d_s - d_m \leq 0 \tag{2.6}$$

in which

$$d_s = \operatorname{tr}([{}_s\sigma][D]) = \operatorname{tr}\left([{}_s\sigma][{}^d_s\dot{J}]\right) \tag{2.7}$$

$$d_m = \operatorname{tr}\left([m][{}^\Theta D]\right) = \operatorname{tr}\left([m][{}^\Theta_s\dot{J}]\right) \tag{2.8}$$

d_s is the rate of mechanical work due to the symmetric part of the Cauchy stress tensor and d_m is the rate of mechanical work due to the Cauchy moment tensor. Equations (2.5) and (2.6) are derived using the first and second laws of thermodynamics for internal polar solid continua in which rate of mechanical work may result in rate of strain energy and rate of dissipation i.e. rate of entropy production. From the energy equation and the entropy inequality we note that

(a) $[{}_s\sigma]$ and $[D]$ constitute a conjugate pair resulting in rate of work.

(b) $[m]$ and $[{}^\Theta D]$ also constitute a conjugate pair resulting in rate of work.

(c) From (2.5) and (2.6) we note that as far as rate of work is concerned $[{}^\Theta D]$ and $[D]$ have nothing to do with $[{}_s\sigma]$ and $[m]$ respectively i.e. $[{}^\Theta D]$ and $[D]$ are not conjugate to $[{}_s\sigma]$ and $[m]$.

(d) The inference drawn in (a)–(b) is critical in deriving the constitutive theories for internal polar thermoelastic solid continua.

(e) In the case of internal polar thermoelastic solid continua, the deformation process is reversible, hence the rate of mechanical work can only result in rate of strain energy i.e. in such solid continua the rate of entropy production due to rate of mechanical work is zero.

3. Alternate forms of the first and second laws of thermodynamics

Since in internal polar thermoelastic solids the rate of external work only results in rate of strain energy, hence does not influence the rate of entropy production, alternate forms of the first and second laws of thermodynamics can be derived in which the rate of strain energy is eliminated. This form of the entropy inequality is truly a statement that contains the rates of entropy as its original intent. Specifically $[{}_s\sigma]$ and $[m]$ and their conjugate rates only result in rates of strain energy densities i.e. d_s and d_m in (2.8) are rates of strain energy associated with $[{}_s\sigma]$ and $[m]$.

Let s be the total strain energy, then

$$\begin{aligned} d_s + d_m &= \dot{s} \\ \text{Let} \quad \rho_0 \underset{\sim}{e} &= \rho_0 e - s \\ \rho_0 \underset{\sim}{\Phi} &= \rho_0 \Phi - s \end{aligned} \tag{3.1}$$

In which $\dot{s}$ is the rate of total strain energy. $\underset{\sim}{e}$ and $\underset{\sim}{\Phi}$ are the modified specific internal energy and the modified Helmholtz free energy densities that are free of strain energy density. From (3.1), taking material derivative we obtain

$$\begin{aligned} \rho_0 \dot{\underset{\sim}{e}} &= \rho_0 \dot{e} - \dot{s} = \rho_0 \dot{e} - d_s - d_m \\ \rho_0 \dot{\underset{\sim}{\Phi}} &= \rho_0 \dot{\Phi} - \dot{s} = \rho_0 \dot{\Phi} - d_s - d_m \end{aligned} \tag{3.2}$$

Substituting $\rho_0 \dot{e} = \rho_0 \dot{\underset{\sim}{e}} + d_s + d_m$ and $\rho_0 \dot{\Phi} = \rho_0 \dot{\underset{\sim}{\Phi}} + d_s + d_m$ from (3.2) into (2.5) and (2.6), we obtain

$$\rho_0 \frac{De}{Dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \tag{3.3}$$

$$\rho_0\left(\frac{D\Phi}{Dt}+\eta\frac{D\theta}{Dt}\right)+\frac{q_i g_i}{\theta}\leq 0 \tag{3.4}$$

4. Remarks

(1) Using (3.3) and (3.4), we now have (2.1)–(2.4) and (3.3), (3.4) as conservation and balance equations for internal polar thermoelastic solid continua.

(2) Equations (3.3) and (3.4) are completely free of rate of strain energy.

(3) The form of the entropy equality in (3.4) is completely unaffected by the rate of mechanical work as it should be if the rate of mechanical work does not result in rate of entropy production, which is indeed the case in internal polar thermoelastic solids.

(4) We keep in mind that at the onset of the derivation of the entropy inequality and subsequently it is indeed a statement of rates of entropies (see equation (3.47) in reference [1]). It is only after using the energy equation and after introducing the Helmholtz free energy density in the original form of the entropy inequality (equation (3.47) in reference [1]) that we introduce the rate of strain energy in it. The presence of strain energy indeed is out of place in the entropy inequality as substantiated by the onset of the derivation and subsequently [1, 3] and before introduction of e and Φ in it.

(5) The form of the entropy inequality (3.4) establishes that if the entropy inequality is truly a statement that contains only the rates of entropies (as (3.4) does), then it has no mechanism for deriving constitutive theories for $[{}_s\sigma]$ and $[m]$ (established later as dependent variables in the constitutive theories) as for internal polar thermoelastic solid continua these do not influence entropy production.

(6) In (2.6), Φ contains strain energy densities and (2.6) also contains d_s and d_m which are strain energy rates due to $[{}_s\sigma]$ and $[m]$, thus it is not surprising to eventually find that constitutive theories for $[{}_s\sigma]$ and $[m]$ can be derived using Φ as Φ is a function of strain energy due to $[{}_s\sigma]$ and $[m]$. Introducing Φ containing strain energy densities in the entropy inequality is intentional so that constitutive theories for $[{}_s\sigma]$ and $[m]$ are possible using Φ. The fact is

that the constitutive theories for $[{}_s\sigma]$ and $[m]$ are related to the corresponding strain energies. The entropy inequality (3.4) clearly shows that if the entropy inequality is purely a statement of the rates of entropy, it contains no mechanism for deriving constitutive theories for $[{}_s\sigma]$ and $[m]$ for internal polar thermoelastic solids.

(7) In this paper we consider both forms of the entropy inequality ((2.6) and (3.4)) in the derivations of the constitutive theories for internal polar thermoelastic solids.

5. Derivations of the constitutive theories

In this section we present derivations of the constitutive theories for internal polar thermoelastic solids using conservation and balance equations (2.1)–(2.6) *(approach I)* as well as (2.1)–(2.4) and (3.3) and (3.4) *(approach II)*. The only difference in the two being the choice of energy equation and the entropy inequality, either (2.5) and (2.6) or (3.3) and (3.4).

5.1. Approach I

We consider conservation and balance equations (2.1)–(2.6) in the derivation of the constitutive theories presented in this section.

5.1.1. Dependent variables in the constitutive theories and their argument tensors

By examining the conservation and balance laws (2.1)–(2.4) it is rather straight forward to conclude the choice of the following as dependent variables in the constitutive theories for internal polar thermoelastic solid continua: Φ, η, $[{}_s\sigma]$, $[m]$ and $\{q\}$. The choices of $\{g\}$ (due to heat vector $\{q\}$) and θ as argument tensors is rather obvious. Since ($[{}_s\sigma]$, $[D]$ or $[{}^d_s\dot{J}]$) and ($[m]$, $[{}^\Theta D]$ or $[{}^\Theta_s\dot{J}]$) are conjugate due to rate of work, choices of $[{}^d_s\dot{J}]$ and $[{}^\Theta_s\dot{J}]$ are necessary as argument tensors. Thus, based on the principle of equipresence [3–5] we must choose $[{}^d_sJ]$, $[{}^\Theta_sJ]$, $\{g\}$, θ as argument tensors of all dependent variables in the constitutive theories for the internal polar thermoelastic solid continua. Hence, we have

$$\begin{aligned}
\Phi &= \Phi\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\eta &= \eta\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
{}_{s}\boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma}\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\boldsymbol{m} &= \boldsymbol{m}\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\boldsymbol{q} &= \boldsymbol{q}\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)
\end{aligned} \tag{5.1}$$

However, we know that $([{}_{s}\sigma], [{}^{d}_{s}J])$ and $([m], [{}^{\Theta}_{s}J])$ are work conjugate and that $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ are not work conjugate with $[m]$ and $[{}_{s}\sigma]$, hence in (5.1), $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ must be eliminated from the argument lists of $[m]$ and $[{}_{s}\sigma]$. Appearance of both $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ as arguments of Φ and η is essential as Φ contains strain energy density and both $[{}^{d}_{s}J]$ and $[{}^{\Theta}_{s}J]$ are work conjugate to $[{}_{s}\sigma]$ and $[m]$ i.e. responsible for strain energy.

$$\begin{aligned}
\Phi &= \Phi\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\eta &= \eta\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
{}_{s}\boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma}\left([{}^{d}_{s}J],\{g\},\theta\right)\\
\boldsymbol{m} &= \boldsymbol{m}\left([{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\boldsymbol{q} &= \boldsymbol{q}\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)
\end{aligned} \tag{5.2}$$

If the stress and strain fields, moment and rotation fields associated with mechanical work are assumed to be independent of $\boldsymbol{g}$, then $\boldsymbol{g}$ can be eliminated from the arguments of ${}_{s}\boldsymbol{\sigma}$ and $\boldsymbol{m}$. Thus, (5.2) reduce to

$$\begin{aligned}
\Phi &= \Phi\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
\eta &= \eta\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)\\
{}_{s}\boldsymbol{\sigma} &= {}_{s}\boldsymbol{\sigma}\left([{}^{d}_{s}J],\theta\right)\\
\boldsymbol{m} &= \boldsymbol{m}\left([{}^{\Theta}_{s}J],\theta\right)\\
\boldsymbol{q} &= \boldsymbol{q}\left([{}^{d}_{s}J],[{}^{\Theta}_{s}J],\{g\},\theta\right)
\end{aligned} \tag{5.3}$$

5.1.2. Entropy inequality: further considerations

It is more convenient to use $[\varepsilon]$ instead of $[{}_s^d J]$, $[\varepsilon]$ being the linear strain tensor as $[\varepsilon] = [{}_s^d J]$. Hence, (5.3) can be written as

$$\begin{aligned}
\Phi &= \Phi\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right) \\
\eta &= \eta\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right) \\
{}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}\left([\varepsilon], \theta\right) \\
\boldsymbol{m} &= \boldsymbol{m}\left([{}_s^\Theta J], \theta\right) \\
\boldsymbol{q} &= \boldsymbol{q}\left([\varepsilon], [{}_s^\Theta J], \{g\}, \theta\right)
\end{aligned} \tag{5.4}$$

Using Φ in (5.4) we can obtain the material derivative of Φ needed in (2.6)

$$\dot{\Phi} = \frac{\partial \Phi}{\partial \varepsilon_{ki}} \dot{\varepsilon}_{ik} + \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} ({}_s^\Theta \dot{J}_{ik}) + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta} \tag{5.5}$$

Substituting $\dot{\Phi}$ from (5.5) in (2.6) and using $[\dot{\varepsilon}] = [{}_s^d \dot{J}]$ we obtain

$$\rho_0 \left(\frac{\partial \Phi}{\partial \varepsilon_{ki}} \dot{\varepsilon}_{ik} + \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} ({}_s^\Theta \dot{J}_{ik}) + \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \frac{\partial \Phi}{\partial \theta} \dot{\theta} + \eta \frac{D\theta}{Dt} \right) + \frac{q_i g_i}{\theta} - {}_s\sigma_{ki} \dot{\varepsilon}_{ik} - m_{ki} {}_s^\Theta \dot{J}_{ik} \leq 0 \tag{5.6}$$

Regrouping terms in (5.6)

$$\left(\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} \right) \dot{\varepsilon}_{ik} + \left(\rho_0 \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} - m_{ki} \right) ({}_s^\Theta \dot{J}_{ik}) + \rho_0 \frac{\partial \Phi}{\partial g_i} \dot{g}_i + \rho_0 \left(\frac{\partial \Phi}{\partial \theta} + \eta \right) \dot{\theta} + \frac{q_i g_i}{\theta} \leq 0 \tag{5.7}$$

For the entropy inequality (5.7) to hold for arbitrary but admissible choices of $[\dot{\varepsilon}]$, $[{}_s^\Theta \dot{J}]$, $\dot{\boldsymbol{g}}$ and $\dot{\theta}$, the following must hold.

$$\rho_0 \frac{\partial \Phi}{\partial \varepsilon_{ki}} - {}_s\sigma_{ki} = 0, \qquad \frac{\partial \Phi}{\partial ({}_s^\Theta J_{ki})} - m_{ki} = 0 \tag{5.8}$$

$$\rho_0 \frac{\partial \Phi}{\partial g_i} = 0 \tag{5.9}$$

$$\rho_0\left(\frac{\partial\Phi}{\partial\theta}+\eta\right)=0 \quad \text{or} \quad \frac{\partial\Phi}{\partial\theta}+\eta=0 \tag{5.10}$$

From (5.9) we conclude that Φ is not a function of $\boldsymbol{g}$ and (5.10) implies that η is deterministic from $\frac{\partial\Phi}{\partial\theta}$, hence η is not a dependent variable in the constitutive theory. From (5.8) we obtain

$$[{}_s\sigma]=\rho_0\frac{\partial\Phi}{\partial[\varepsilon]} \tag{5.11}$$

$$[m]=\rho_0\frac{\partial\Phi}{\partial[{}_s^{\Theta}J]} \tag{5.12}$$

Thus, if Φ is known as a function of $[\varepsilon]$, then the constitutive theory for $[{}_s\sigma]$ can be derived using (5.11) and if Φ is known as a function of $[{}_s^{\Theta}J]$, then the constitutive theory for $[m]$ can be derived using (5.12). In view of (5.8)–(5.10), the entropy inequality (5.7) reduces to

$$\frac{q_i g_i}{\theta}\le 0 \quad \text{or} \quad q_i g_i \le 0 \tag{5.13}$$

Inequality (5.13) forms the basis for deriving constitutive theory for the heat vector $\{q\}$. We note there is no mechanism (other than physical reasoning) to remove $[\varepsilon]$ and $[{}_s^{\Theta}J]$ from the argument list of $\{q\}$ as in (5.4), hence we must maintain the arguments of $\{q\}$ in (5.4).

5.1.3. Constitutive theory for ${}_s\boldsymbol{\sigma}$

5.1.3.1. Assuming Φ is a function of the invariants of $\boldsymbol{\varepsilon}$ and θ

Consider

$$[{}_s\sigma]=\rho_0\frac{\partial\Phi}{\partial[\varepsilon]} \tag{5.14}$$

in which $\Phi=\Phi([\varepsilon],\theta)$. Due to the frame invariance requirement, Φ cannot be a function of $[\varepsilon]$, but instead we must consider Φ as a function of the invariants of $[\varepsilon]$. If we choose the principal invariants of $[\varepsilon]$ i.e. I_ε, II_ε and III_ε [3], then

$$\Phi=\Phi(I_\varepsilon,II_\varepsilon,III_\varepsilon,\theta) \tag{5.15}$$

Using (5.15) in (5.14) it is straightforward to derive [3]

$$ {}_s\boldsymbol{\sigma} = {}^{\sigma}\alpha^0 \boldsymbol{I} + {}^{\sigma}\alpha^1 \boldsymbol{\varepsilon} + {}^{\sigma}\alpha^2 (\boldsymbol{\varepsilon})^{-1} \tag{5.16} $$

In which

$$ \begin{aligned} {}^{\sigma}\alpha^0 &= \rho_0 \left(\frac{\partial \Phi}{\partial I_\varepsilon} + \frac{\partial \Phi}{\partial II_\varepsilon} I_\varepsilon \right) \\ {}^{\sigma}\alpha^1 &= \left(-\rho_0 \frac{\partial \Phi}{\partial II_\varepsilon} \right) \\ {}^{\sigma}\alpha^{-1} &= \left(\rho_0 \frac{\partial \Phi}{\partial III_\varepsilon} \right) \end{aligned} $$

Using Hamilton-Cayley theorem [3], (5.15) can be written as

$$ {}_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2 (\boldsymbol{\varepsilon})^2 \tag{5.17} $$

In which ${}^{\sigma}\tilde{\alpha}^i$; $i = 0,1,2$ are functions of ${}^{\sigma}\alpha^i$; $i = 0,1,2$ and the invariants I_ε, II_ε, III_ε. Form (5.17) is preferred over (5.16) due to obvious reasons, the absence of $(\boldsymbol{\varepsilon})^{-1}$. This constitutive theory is not usable yet due to the fact that ${}^{\sigma}\tilde{\alpha}^i$; $i = 0,1,2$ are functions of unknown deformation in the current configuration due to the fact that I_ε, II_ε, III_ε and θ are in the current configuration. We postpone further details of determining the material coefficients using (5.17) until a later section. However, (5.17) is a fundamental form of the constitutive theory for ${}_s\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$.

5.1.3.2. Using theory of generators and invariants

Consider

$$ {}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}([\varepsilon], \theta) \tag{5.18} $$

${}_s\boldsymbol{\sigma}$ is a symmetric tensor of rank two whose argument tensors are $\boldsymbol{\varepsilon}$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on the theory of generators and invariants [5–22], ${}_s\boldsymbol{\sigma}$ can be expressed as a linear combination of

$\boldsymbol{I}$, and the combined generators of its arguments, which in this case are generators of $\boldsymbol{\varepsilon}$ that are symmetric tensors of rank two. Between the argument tensors $\boldsymbol{\varepsilon}$ and θ, the combined generators that are symmetric tensors of rank two are $\boldsymbol{\varepsilon}$ and $(\boldsymbol{\varepsilon})^2$. Using the same coefficients in the linear combination as appears in (5.17), we can write:

$$_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2 (\boldsymbol{\varepsilon})^2 \tag{5.19}$$

in which the coefficients ${}^{\sigma}\tilde{\alpha}^i$; $i = 0,1,2$ are functions of I_ε, II_ε, III_ε and θ in the current configuration, i.e.

$${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i (I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta);\ i = 0,1,2 \tag{5.20}$$

We note that (5.19) is the same as (5.17) derived in section 5.1.3.1 with the same definition of the coefficients. Thus, the remarks made in section 5.1.3.1 regarding the coefficients hold here as well. When using the theory of generators and invariants, we can also use the invariants i_ε, ii_ε iii_ε instead of the principal invariants I_ε, II_ε, and III_ε in (5.20). Since the two sets of invariants are related [3], the final outcome remains the same as in section 5.1.3.1.

5.1.3.3. Definition of material coefficients using ${}^{\sigma}\tilde{\alpha}^i$; $i = 0,1,2$ in (5.17) or (5.19)

Consider

$$_s\boldsymbol{\sigma} = {}^{\sigma}\tilde{\alpha}^0 \boldsymbol{I} + {}^{\sigma}\tilde{\alpha}^1 \boldsymbol{\varepsilon} + {}^{\sigma}\tilde{\alpha}^2 (\boldsymbol{\varepsilon})^2 \tag{5.21}$$

We consider ${}^{\sigma}\tilde{\alpha}^i$; $i = 0,1,2$ to be functions of I_ε, II_ε, III_ε and temperature θ.

$${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i (I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta);\ i = 0,1,2 \tag{5.22}$$

We can expand ${}^{\sigma}\tilde{\alpha}^i$ in Taylor series in I_ε, II_ε, III_ε, and θ about a known configuration $\underline{\Omega}$. We retain only up to linear terms in the invariants of $\boldsymbol{\varepsilon}$ and temperature θ in the Taylor series expansion. We introduce the following notation to make the presentation compact:

$$ {}^{\sigma}\underset{\sim}{I}^1 = I_{\varepsilon}; \quad {}^{\sigma}\underset{\sim}{I}^2 = II_{\varepsilon}; \quad {}^{\sigma}\underset{\sim}{I}^3 = III_{\varepsilon} \tag{5.23} $$

Using the notation in (5.23), we can write

$$ {}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial \theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \tag{5.24} $$

Substituting from (5.24) into (5.21):

$$ \begin{aligned} {}_{s}\boldsymbol{\sigma} = & \left({}^{\sigma}\tilde{\alpha}^0\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial \theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \right) \boldsymbol{I} \\ & + \left({}^{\sigma}\tilde{\alpha}^1\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial \theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \right) \boldsymbol{\varepsilon} \\ & + \left({}^{\sigma}\tilde{\alpha}^2\big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j - \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial \theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}} \right) \right) (\boldsymbol{\varepsilon})^2 \end{aligned} \tag{5.25} $$

Collecting coefficients (defined in configuration $\underline{\Omega}$) of $\boldsymbol{I}$, $\boldsymbol{\varepsilon}$, ${}^{\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}$; $j = 1,2,3$, ${}^{\sigma}\underset{\sim}{I}^j\boldsymbol{\varepsilon}^2$; $j = 1,2,3$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{I}$, $(\theta - \theta_{\underline{\Omega}})\boldsymbol{\varepsilon}$, and $(\theta - \theta_{\underline{\Omega}})(\boldsymbol{\varepsilon})^2$, we can write the following using (5.25)

$$ \begin{aligned} {}_{s}\boldsymbol{\sigma} = & \left({}^{\sigma}\tilde{\alpha}^0_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) \boldsymbol{I} + \left({}^{\sigma}\tilde{\alpha}^1_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) \boldsymbol{\varepsilon} \\ & + \left({}^{\sigma}\tilde{\alpha}^2_{\underline{\Omega}} - \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j \right)_{\underline{\Omega}} \right) (\boldsymbol{\varepsilon})^2 + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j \boldsymbol{I} \right) \\ & + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j \boldsymbol{\varepsilon} \right) + \sum_{j=1}^{3} \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial {}^{\sigma}\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^{\sigma}\underset{\sim}{I}^j (\boldsymbol{\varepsilon})^2 \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial \theta}\Bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{I} \right) \\ & + \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial \theta}\Bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) \boldsymbol{\varepsilon} \right) + \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial \theta}\Bigg|_{\underline{\Omega}} \left((\theta - \theta_{\underline{\Omega}}) (\boldsymbol{\varepsilon})^2 \right) \end{aligned} \tag{5.26} $$

Let us define

$$
\begin{aligned}
{}^0\bar{\sigma}\big|_{\underline{\Omega}} &= {}^\sigma\underline{b}_0 & {}^\sigma\underline{a}_j &= \left.\frac{\partial\,{}^\sigma\tilde{\alpha}^0}{\partial\,{}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 \\
{}^\sigma\underline{b}_i &= {}^\sigma\tilde{\alpha}^i\big|_{\underline{\Omega}} - \sum_{j=1}^{3}\left.\frac{\partial\,{}^\sigma\tilde{\alpha}^i}{\partial\,{}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; i=0,1,2 & {}^\sigma\underline{c}_{1j} &= \left.\frac{\partial\,{}^\sigma\tilde{\alpha}^1}{\partial\,{}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 \\
{}^\sigma\underline{c}_{2j} &= \left.\frac{\partial\,{}^\sigma\tilde{\alpha}^2}{\partial\,{}^\sigma\underset{\sim}{I}^j}\right|_{\underline{\Omega}} \; ; \; j=1,2,3 & {}^\sigma\underline{d}_1 &= \left.\frac{\partial\,{}^\sigma\tilde{\alpha}^1}{\partial\theta}\right|_{\underline{\Omega}} \\
{}^\sigma\underline{d}_2 &= \left.\frac{\partial\,{}^\sigma\tilde{\alpha}^2}{\partial\theta}\right|_{\underline{\Omega}} & \underline{\alpha}_{\text{tm}} &= -\left.\frac{\partial\,{}^\sigma\tilde{\alpha}^0}{\partial\theta}\right|_{\underline{\Omega}}
\end{aligned}
\tag{5.27}
$$

Substituting (5.27) into (5.26)

$$
\begin{aligned}
{}_s\boldsymbol{\sigma} = {}&{}^0\bar{\sigma}\big|_{\underline{\Omega}}\boldsymbol{I} + {}^\sigma\underline{b}_1\boldsymbol{\varepsilon} + {}^\sigma\underline{b}_2\boldsymbol{\varepsilon}^2 + \sum_{j=1}^{3}{}^\sigma\underline{a}_j\left({}^\sigma\underset{\sim}{I}^j\boldsymbol{I}\right) + \sum_{j=1}^{3}{}^\sigma\underline{c}_{1j}\left({}^\sigma\underset{\sim}{I}^j\boldsymbol{\varepsilon}\right) \\
&+ \sum_{j=1}^{3}{}^\sigma\underline{c}_{2j}\left({}^\sigma\underset{\sim}{I}^j\boldsymbol{\varepsilon}^2\right) + {}^\sigma\underline{d}_1\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}\right) + {}^\sigma\underline{d}_2\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}^2\right) \\
&+ \underline{\alpha}_{\text{tm}}\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{I}\right)
\end{aligned}
\tag{5.28}
$$

${}^0\bar{\sigma}\big|_{\underline{\Omega}}$ is the initial stress in the configuration $\underline{\Omega}$. This constitutive theory requires determination of 14 material coefficients as defined in (5.27) (excluding ${}^0\bar{\sigma}\big|_{\underline{\Omega}}$), all evaluated in a known configuration $\underline{\Omega}$. The constitutive theory (5.28) for ${}_s\boldsymbol{\sigma}$ is the most general form of the constitutive theory for ${}_s\boldsymbol{\sigma}$ as a function of $\boldsymbol{\varepsilon}$ and temperature θ resulting from the entropy inequality or the theory of generators and invariants. This theory is based on integrity, hence complete, but it contains too many material coefficients to be determined, experimentally or otherwise.

Simplified theory

Here we consider simplifications of the constitutive theory for ${}_s\boldsymbol{\sigma}$ given by (5.28). If we only consider a constitutive theory for ${}_s\boldsymbol{\sigma}$ that is linear in the components of $\boldsymbol{\varepsilon}$ and if we further neglect the $(\theta-\theta_{\underline{\Omega}})\boldsymbol{\varepsilon}$ terms, then (5.28) reduces to

$$
{}_s\boldsymbol{\sigma} = {}^0\bar{\sigma}\big|_{\underline{\Omega}}\boldsymbol{I} + {}^\sigma\underline{b}_1\boldsymbol{\varepsilon} + {}^\sigma\underline{a}_1\,\text{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + (\underline{\alpha}_{\text{tm}})_{\underline{\Omega}}\left((\theta-\theta_{\underline{\Omega}})\boldsymbol{I}\right)
\tag{5.29}
$$

This constitutive theory only requires three material coefficients ${}^{\sigma}\underline{a}_1$, ${}^{\sigma}\underline{b}_1$, and $\underline{\alpha}_{\text{tm}}$, in a known configuration $\underline{\Omega}$.

5.1.4. Constitutive theory for moment tensor m

5.1.4.1. Assuming Φ is a function of the invariants of $[{}_s^{\Theta}J]$ and θ

Consider

$$[m] = \rho_0 \frac{\partial \Phi}{\partial [{}_s^{\Theta}J]} \tag{5.30}$$

in which $\Phi = \Phi([{}_s^{\Theta}J], \theta)$. Due to frame invariance requirements, Φ can not be a function of $[{}_s^{\Theta}J]$ but instead we must consider Φ as a function of the invariants of $[{}_s^{\Theta}J]$. If we choose the principle invariants of $[{}_s^{\Theta}J]$ i.e. I_{Θ}, II_{Θ}, III_{Θ} [3], then

$$\Phi = \Phi(I_{\Theta}, II_{\Theta}, III_{\Theta}, \theta) \tag{5.31}$$

using (5.31) in (5.30) it is straightforward to derive

$$[m] = {}^{m}\alpha^0 [I] + {}^{m}\alpha^1 [{}_s^{\Theta}J] + {}^{m}\alpha^2 [{}_s^{\Theta}J]^{-1} \tag{5.32}$$

in which

$$\begin{gathered} {}^{m}\alpha^0 = \rho_0 \left(\frac{\partial \Phi}{\partial I_{\Theta}} + \frac{\partial \Phi}{\partial II_{\Theta}} I_{\Theta} \right) \\ {}^{m}\alpha^1 = -\rho_0 \frac{\partial \Phi}{\partial II_{\Theta}} \quad ; \quad {}^{m}\alpha^{-1} = \rho_0 \frac{\partial \Phi}{\partial III_{\Theta}} \end{gathered} \tag{5.33}$$

Using Hamilton-Cayley theorem [3], (5.33) can be written as

$$[m] = {}^{m}\tilde{\alpha}^0 [I] + {}^{m}\tilde{\alpha}^1 [{}_s^{\Theta}J] + {}^{m}\tilde{\alpha}^2 [{}_s^{\Theta}J]^2 \tag{5.34}$$

in which ${}^{m}\tilde{\alpha}^i; i = 0,1,2$ are functions of ${}^{m}\alpha^i; i = 0,1,2$ and the invariants I_{Θ}, II_{Θ}, III_{θ} of $[{}_s^{\Theta}J]$. Form (5.34) is preferred over (5.32) due to obvious reasons, the absence of $[{}_s^{\Theta}J]^{-1}$. We note that ${}^{m}\tilde{\alpha}^i; i = 0,1,2$ are in the current configuration, hence (5.34) is not usable until the material coefficients are determined using ${}^{m}\tilde{\alpha}^i; i = 0,1,2$. None the less, (5.34) is a fundamental form for the constitutive theory for $[m]$.

5.1.4.2. Using the theory of generators and invariants

Consider

$$[m] = \left[m\left([{}_s^\Theta J],\theta\right)\right] \tag{5.35}$$

$[m]$ is a symmetric tensor of rank two whose argument tensors are $[{}_s^\Theta J]$, a symmetric tensor of rank two, and θ, a tensor of rank zero. Based on the theory of generators and invariants [5–22], $[m]$ can be expressed as a linear combination of $[I]$ and the combined generators of its argument tensors, which in this case are generators of $[{}_s^\Theta J]$ that are symmetric tensors of rank two. Between the argument tensors $[{}_s^\Theta J]$ and θ, the combined generators that are symmetric tensors of rank two are $[{}_s^\Theta J]$ and $[{}_s^\Theta J]^2$. Using the same coefficients in the linear combination as those used in (5.34), we can write

$$[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}_s^\Theta J] + {}^m\tilde{\alpha}^2[{}_s^\Theta J]^2 \tag{5.36}$$

In which the coefficients of ${}^m\tilde{\alpha}^i;\, i = 0,1,2$ are functions of I_Θ, II_Θ, III_θ, and θ in the current configuration. We note that (5.36) is the same as (5.34) derived in section 5.1.4.1 with the definition of coefficients.

5.1.4.3. Determination of material coefficients using ${}^m\tilde{\alpha}^i;\, i = 0,1,2$ in (5.34) or (5.36)

Consider

$$[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}_s^\Theta J] + {}^m\tilde{\alpha}^2[{}_s^\Theta J]^2 \tag{5.37}$$

in which

$${}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\,(I_\Theta, II_\Theta, III_\theta, \theta);\; i = 0,1,2 \tag{5.38}$$

we expand ${}^m\tilde{\alpha}^i;\, i = 0,1,2$ in Taylor series in I_Θ, II_Θ, III_θ, and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in the invariants and the temperature θ. We introduce the notation

$${}^m\underset{\sim}{I}^1 = I_\Theta, \quad {}^m\underset{\sim}{I}^2 = II_\Theta, \quad \text{and} \quad {}^m\underset{\sim}{I}^3 = III_\Theta \tag{5.39}$$

Using the notation (5.39), the Taylor series expansion yields

$$ {}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{3} \frac{\partial {}^m\tilde{\alpha}^i}{\partial {}^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}} \left({}^m\underset{\sim}{I}^j - ({}^m\underset{\sim}{I}^j)_{\underline{\Omega}}\right) + \frac{\partial {}^m\alpha^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta-\theta_{\underline{\Omega}}\right);\ i=0,1,2 \tag{5.40}$$

Substituting (5.40) into (5.37) and collecting coefficients (those defined in $\underline{\Omega}$) of $[I]$, ${}^m\underset{\sim}{I}^j[I]; j=1,2,3$, ${}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]; j=1,2,3$, ${}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]^2; j=1,2,3$, $(\theta-\theta_{\underline{\Omega}})[I]$, $(\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]$ and $(\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]^2$ and defining

$$\begin{aligned}
{}^0\bar{m}\big|_{\underline{\Omega}} &= {}^m\underline{b}_0 & {}^m\underline{a}_j &= \frac{\partial {}^m\tilde{\alpha}^0}{\partial {}^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}};\ j=1,2,3 \\
{}^m\underline{b}_i &= {}^m\tilde{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{3}\frac{\partial {}^m\tilde{\alpha}^i}{\partial {}^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}};\ i=0,1,2 & {}^m\underline{c}_{1j} &= \frac{\partial {}^m\tilde{\alpha}^1}{\partial {}^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}};\ j=1,2,3 \\
{}^m\underline{c}_{2j} &= \frac{\partial {}^m\tilde{\alpha}^2}{\partial {}^m\underset{\sim}{I}^j}\Bigg|_{\underline{\Omega}};\ j=1,2,3 & {}^m\underline{d}_1 &= \frac{\partial {}^m\tilde{\alpha}^1}{\partial\theta}\Bigg|_{\underline{\Omega}} \\
{}^m\underline{d}_2 &= \frac{\partial {}^m\tilde{\alpha}^2}{\partial\theta}\Bigg|_{\underline{\Omega}} & \underline{\alpha}_{\text{tm}} &= -\frac{\partial {}^m\tilde{\alpha}^0}{\partial\theta}\Bigg|_{\underline{\Omega}}
\end{aligned}\tag{5.41}$$

we can write the following for $[m]$ in (5.37)

$$\begin{aligned}
[m] = {}&{}^0\bar{m}\big|_{\underline{\Omega}}[I] + {}^m\underline{b}_1[{}^{\Theta}_{s}J] + {}^m\underline{b}_2[{}^{\Theta}_{s}J]^2 + \sum_{j=1}^{3}{}^m\underline{a}_j\left({}^m\underset{\sim}{I}^j[I]\right) + \sum_{j=1}^{3}{}^m\underline{c}_{1j}\left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]\right) \\
&+ \sum_{j=1}^{3}{}^m\underline{c}_{2j}\left({}^m\underset{\sim}{I}^j[{}^{\Theta}_{s}J]^2\right) + {}^m\underline{d}_1\left((\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]\right) + {}^m\underline{d}_2\left((\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]^2\right) \\
&+ \underline{\alpha}_{\text{tm}}\left((\theta-\theta_{\underline{\Omega}})[I]\right)
\end{aligned}\tag{5.42}$$

This constitutive theory requires determination of 14 material coefficients defined in (5.41), all evaluated in the known configuration $\underline{\Omega}$. Constitutive theory (5.42) is the most general and complete constitutive theory for $[m]$ as it is based on integrity.

A much more simplified constitutive theory for $[m]$ is possible if we only consider a constitutive theory for $[m]$ that is linear in $[{}^{\Theta}_{s}J]$ and if we further neglect the $(\theta-\theta_{\underline{\Omega}})[{}^{\Theta}_{s}J]$ term, then (5.42) reduces to

$$[m] = {}^0\bar{m}\big|_{\underline{\Omega}}[I] + {}^m\underline{b}_1[{}^\Theta_s J] + {}^m\underline{a}_1 \mathrm{tr}([{}^\Theta_s J])[I] + \underline{\alpha}_{\mathrm{tm}}\big((\theta - \theta_{\underline{\Omega}})[I]\big) \tag{5.43}$$

This constitutive theory only requires three material coefficients

5.2. Approach II

We consider conservation and balance laws (2.1)–(2.4) and (3.3), (3.4) in this derivation. The important aspect of this derivation is that the entropy inequality (3.4) does not contain rate of work due to the fact that for internal polar thermoelastic solids the rate of work does not contribute to rate of entropy production. As a consequence, the entropy inequality (3.4) provides no mechanism for deriving constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$. That is the constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$ have no thermodynamic restriction as long as they are derived for isotropic and homogeneous internal polar thermoelastic solid continua as the conservation and balance laws are only valid for this case. Thus, we have complete freedom of deriving the constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$, a direct consequence of the form of the entropy inequality which is purely a statement of rate of entropies.

5.3. Dependent variables in the constitutive theories and their argument tensors

As in the case of approach I, here also it is straightforward to conclude that $\underset{\sim}{\Phi}$, η, $[{}_s\sigma]$, $[m]$ and $\{g\}$ is a possible choice of dependent variables in the constitutive theories. Choice of $\{g\}$ and θ as argument tensors is rather obvious. $[{}^d_s J]$ or $[\varepsilon]$ and $[{}^\Theta_s J]$ must be considered as argument tensors as well due to the fact that these are conjugate with $[{}_s\sigma]$ and $[m]$. Thus we have $[\varepsilon]$, $[{}^\Theta_s J]$, $\{g\}$, and θ as argument tensors. The entropy inequality (3.4) does not contain rate of work, that is the rate of strain energy as for internal polar thermoelastic solid continua the rate of mechanical work can not result in rate of entropy production. Thus $[\varepsilon]$ and $[{}^\Theta_s J]$ can not be argument tensors of $\underset{\sim}{\Phi}$ and η. Furthermore, since $[{}_s\sigma]$, $[\varepsilon]$ and $[m]$, $[{}^\Theta_s J]$ are conjugate pairs, $[{}^\Theta_s J]$ can not be an argument tensor of $[{}_s\sigma]$ and $[\varepsilon]$ can not be an argument tensor of $[m]$. If the stress and strain fields, moment and rotation fields are assumed to be independent of $\{g\}$, then we can arrive at the following for the argument tensors of the dependent variables $\underset{\sim}{\Phi}$, η, $[{}_s\sigma]$, $[m]$ and $\{g\}$ in the constitutive theories. $[\varepsilon]$, $[{}^\Theta_s J]$, $\{g\}$, and θ must be maintained as argument tensors of $\{q\}$ as at this stage there is no mechanism to do otherwise.

$$
\begin{aligned}
\underset{\sim}{\Phi} &= \underset{\sim}{\Phi}(\{g\},\theta) \\
\eta &= \eta(\{g\},\theta) \\
[_s\sigma] &= [_s\sigma([\varepsilon],\theta)] \\
[m] &= \left[m \left([{}_s^{\Theta}J],\theta \right) \right] \\
\{q\} &= \left\{ q \left([\varepsilon],[{}_s^{\Theta}J],\{g\},\theta \right) \right\}
\end{aligned}
\tag{5.44}
$$

5.3.1. Entropy inequality, further considerations

We already know that for internal polar thermoelastic solid continua the entropy inequality (3.4) does not contain either of the two conjugate pairs responsible for rate of entropy production (rate of strain energy in this case). Thus in this case the entropy inequality provides no mechanism for deriving constitutive theories of these conjugate quantities for internal polar thermoelastic solid continua. From (5.44) with $\underset{\sim}{\Phi} = \underset{\sim}{\Phi}(\boldsymbol{g},\theta)$ we can obtain material derivatives of $\underset{\sim}{\Phi}$ needed in (3.4).

$$
\dot{\underset{\sim}{\Phi}} = \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{\partial \underset{\sim}{\Phi}}{\partial \theta} \dot{\theta}
\tag{5.45}
$$

Substituting (5.45) in the entropy inequality (3.4)

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{\partial \underset{\sim}{\Phi}}{\partial \theta} \dot{\theta} + \eta \dot{\theta} \right) + \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{5.46}
$$

or

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta \right) \dot{\theta} + \rho_0 \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} \dot{\boldsymbol{g}} + \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{5.47}
$$

For (5.47) to hold for arbitrary but admissible $\dot{\theta}$ and $\dot{\boldsymbol{g}}$ the following must hold.

$$
\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta \right) = 0 \quad ; \quad \rho_0 \frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} = 0 \quad ; \quad \frac{1}{\theta} \boldsymbol{q} \cdot \boldsymbol{g} \leq 0
\tag{5.48}
$$

Since ρ_0 is constant and $\theta \geq 0$ we can write

$$\frac{\partial \underset{\sim}{\Phi}}{\partial \theta} + \eta = 0 \tag{5.49}$$

$$\frac{\partial \underset{\sim}{\Phi}}{\partial \boldsymbol{g}} = 0 \tag{5.50}$$

$$\boldsymbol{q} \cdot \boldsymbol{g} \leq 0 \tag{5.51}$$

Equation (5.49) implies that $\eta = -\frac{\partial \underset{\sim}{\Phi}}{\partial \theta}$ i.e. if $\underset{\sim}{\Phi}$ is known as a function of θ then η is deterministic, thus η can not be a dependent variable in the constitutive theories. Equation (5.50) implies that $\underset{\sim}{\Phi}$ can not be a function of $\boldsymbol{g}$. We note that (5.49)–(5.51) as expected do not provide any mechanism for deriving constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$, however (5.51) can be used to derive constitutive theory for $\boldsymbol{q}$ (shown later). The importance of this derivation is that based on this derivation η is ruled out as a dependent variable in the constitutive theories and it is established that $\underset{\sim}{\Phi}$ is not a function of $\boldsymbol{g}$. Thus, finally (5.44) reduce to the following

$$\begin{aligned} \underset{\sim}{\Phi} &= \underset{\sim}{\Phi}(\theta) \\ {}_s\boldsymbol{\sigma} &= {}_s\boldsymbol{\sigma}\left([{}^{d}_{s}J], \theta\right) \\ \boldsymbol{m} &= \boldsymbol{m}\left([{}^{\Theta}_{s}J], \theta\right) \\ \boldsymbol{q} &= \boldsymbol{q}\left([{}^{d}_{s}J], [{}^{\Theta}_{s}J], \{g\}, \theta\right) \end{aligned} \tag{5.52}$$

We note that in (5.52) the argument tensors of $\boldsymbol{q}$ remain the same as in (5.44) as there is no additional information or restrictions to do otherwise.

6. Constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$: general considerations (approach II)

In this section we consider derivations of the constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$.

Constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$ are derived using the following:

1. Theory of generators and invariants [5–22]

2. Strain energy density function [3]

3. Complementary strain energy density function [3]

4. Using Taylor series expansions [3]

Constitutive theories for $\boldsymbol{q}$ are derived using:

1. Theory of generators and invariants

2. Conditions resulting from the entropy inequality

We consider details of each method in the following sections. First, we make some remarks.

a. We note that both conjugate pairs $([{}_s\sigma], [{}^d_sJ])$ and $([m], [{}^\Theta_sJ])$ result in strain energy, hence contribute to the strain energy density function. However, their contributions are independent of each other as established earlier.

b. Remark (a) clearly suggests that the derivations of the constitutive theories for $[{}_s\sigma]$ and $[m]$ are independent of each other.

c. If π is the total strain energy density function, then

$$\pi = {}^{s\sigma}\pi + {}^m\pi$$

in which ${}^{s\sigma}\pi$ and ${}^m\pi$ are strain energy density functions due to conjugate pairs $([{}_s\sigma], [{}^d_sJ])$ and $([m], [{}^\Theta_sJ])$. Derivations of the constitutive theories for $[{}_s\sigma]$ and $[m]$ can proceed individually by using ${}^{s\sigma}\pi$ and ${}^m\pi$ respectively. The same holds for complementary strain energy density functions (in a later section).

6.1. Constitutive theory for ${}_s\boldsymbol{\sigma}$ based on the theory of generators and invariants (Approach II)

In this approach [5–22] we consider ${}_s\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma}([{}^d_sJ], \theta) = {}_s\boldsymbol{\sigma}([\varepsilon], \theta)$ in which $[\varepsilon] = [{}^d_sJ]$ is the strain tensor for infinitesimal deformation and use the theory of generators and invariants [5–22] to derive constitutive theory for ${}_s\boldsymbol{\sigma}$. In the subsequent derivation it is more convenient to use $[\varepsilon]$ in place of $[{}^d_sJ]$ for the sake of simplicity of notation. Let ${}^\sigma\underset{\sim}{\boldsymbol{G}}^i; i = 1, 2, \ldots, N$ be the combined generators of the argument tensors $[\varepsilon]$ and θ that are symmetric tensors of rank two and

${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M$ be the combined invariants of the same argument tensors. The generators ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\dots,N$ form an integrity i.e. complete basis. We can now represent $[{}_s\sigma]$ as a linear combination of $[I]$ and ${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i; i = 1,2,\dots,N$.

$$[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0[I] + \sum_{i=1}^{N} {}^{\sigma}\tilde{\alpha}^i\, {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^i \tag{6.1}$$

in which the coefficients of ${}^{\sigma}\tilde{\alpha}^i; i = 0,1,\dots,N$ are functions of the invariants ${}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M$ and θ i.e.

$${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i \left({}^{\sigma}\underset{\sim}{I}^j; j = 1,2,\dots,M,\theta\right) \tag{6.2}$$

In this particular case we only have two generators and three invariants (i.e. $N = 2$ and $M = 3$)

$${}^{\sigma}\underset{\sim}{\boldsymbol{G}}^1 = [\varepsilon], \qquad {}^{\sigma}\underset{\sim}{\boldsymbol{G}}^2 = [\varepsilon]^2 \tag{6.3}$$

and

$${}^{\sigma}\underset{\sim}{I}^1 = i_{\varepsilon}\,(\text{or}\, I_{\varepsilon}); \quad {}^{\sigma}\underset{\sim}{I}^2 = ii_{\varepsilon}\,(\text{or}\, II_{\varepsilon}); \quad {}^{\sigma}\underset{\sim}{I}^3 = iii_{\epsilon}\,(\text{or}\, III_{\varepsilon}) \tag{6.4}$$

Choice of i_{ε}, ii_{ε}, iii_{ε} or I_{ε}, II_{ε}, III_{ε} (from characteristic equation of $[\varepsilon]$ i.e. principal invariants) does not matter as the two sets of invariants are related. Using (6.3) we can write (6.1) explicitly as follows.

$$[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0[I] + {}^{\sigma}\tilde{\alpha}^1[\varepsilon] + {}^{\sigma}\tilde{\alpha}^2[\varepsilon]^2 \tag{6.5}$$

in which

$${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\,(I_{\varepsilon}, II_{\varepsilon}, III_{\varepsilon}, \theta) \tag{6.6}$$

Equation (6.5) and (6.6) hold in the current configuration, hence ${}^{\sigma}\tilde{\alpha}^i; i = 1,2$ are unknown as these are functions of the deformation which is not known (yet). Thus, ${}^{\sigma}\tilde{\alpha}^i; i = 0,1,2$ are not material coefficients. We consider Taylor series expansion of each ${}^{\sigma}\tilde{\alpha}^i$ in $I_{\varepsilon}, II_{\varepsilon}, III_{\varepsilon}$ and θ about a known configuration $\underline{\Omega}$ and only retain up to linear terms (for simplicity) in the invariants and the temperature. This is valid based on the principle of smooth neighborhood (assuming ${}^{\sigma}\tilde{\alpha}^i$ are analytic functions of their arguments).

$$
\begin{aligned}
{}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial I_\varepsilon}\Big|_{\underline{\Omega}}\Big(I_\varepsilon - (I_\varepsilon)_{\underline{\Omega}}\Big) + \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial II_\varepsilon}\Big|_{\underline{\Omega}}\Big(II_\varepsilon - (II_\varepsilon)_{\underline{\Omega}}\Big) \\
+ \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial III_\varepsilon}\Big|_{\underline{\Omega}}\Big(III_\varepsilon - (III_\varepsilon)_{\underline{\Omega}}\Big) + \frac{\partial {}^{\sigma}\tilde{\alpha}^i}{\partial \theta}\Big|_{\underline{\Omega}}\big(\theta - \theta_{\underline{\Omega}}\big); \quad i = 0,1,2
\end{aligned}
\tag{6.7}
$$

Substituting from (6.7) into (6.6) and collecting coefficients of $[I]$, $[\varepsilon]$, $[\varepsilon]^2$, $I_\varepsilon[I]$, $I_\varepsilon[\varepsilon]$, $I_\varepsilon[\varepsilon]^2$, $II_\varepsilon[I]$, $II_\varepsilon[\varepsilon]$, $II_\varepsilon[\varepsilon]^2$, $III_\varepsilon[I]$, $III_\varepsilon[\varepsilon]$, $III_\varepsilon[\varepsilon]^2$, $(\theta - \theta|_{\underline{\Omega}})[I]$, $(\theta - \theta|_{\underline{\Omega}})[\varepsilon]$, $(\theta - \theta|_{\underline{\Omega}})[\varepsilon]^2$ and defining the following coefficients

$$
\begin{aligned}
b_0 &= {}^{\sigma}\tilde{\alpha}^0\big|_{\underline{\Omega}} - b_{01}(I_\varepsilon)_{\underline{\Omega}} - b_{02}(II_\varepsilon)_{\underline{\Omega}} - b_{03}(III_\varepsilon)_{\underline{\Omega}} \\
b_1 &= {}^{\sigma}\tilde{\alpha}^1\big|_{\underline{\Omega}} - b_{11}(I_\varepsilon)_{\underline{\Omega}} - b_{12}(II_\varepsilon)_{\underline{\Omega}} - b_{13}(III_\varepsilon)_{\underline{\Omega}} \\
b_2 &= {}^{\sigma}\tilde{\alpha}^2\big|_{\underline{\Omega}} - b_{21}(I_\varepsilon)_{\underline{\Omega}} - b_{22}(II_\varepsilon)_{\underline{\Omega}} - b_{23}(III_\varepsilon)_{\underline{\Omega}} \\
b_{01} &= \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial I_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{02} = \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial II_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{03} = \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial III_\varepsilon}\Big|_{\underline{\Omega}} \\
b_{11} &= \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial I_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{12} = \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial II_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{13} = \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial III_\varepsilon}\Big|_{\underline{\Omega}} \\
b_{21} &= \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial I_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{22} = \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial II_\varepsilon}\Big|_{\underline{\Omega}}, \quad b_{23} = \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial III_\varepsilon}\Big|_{\underline{\Omega}} \\
b_{31} &= \frac{\partial {}^{\sigma}\tilde{\alpha}^0}{\partial \theta}\Big|_{\underline{\Omega}}, \quad b_{32} = \frac{\partial {}^{\sigma}\tilde{\alpha}^1}{\partial \theta}\Big|_{\underline{\Omega}}, \quad b_{33} = \frac{\partial {}^{\sigma}\tilde{\alpha}^2}{\partial \theta}\Big|_{\underline{\Omega}}
\end{aligned}
\tag{6.8}
$$

We can write (6.5) as

$$
\begin{aligned}
[{}_s\sigma] = b_0[I] + b_1[\varepsilon] + b_2[\varepsilon]^2 \\
+ b_{01}I_\varepsilon[I] + b_{02}II_\varepsilon[I] + b_{03}III_\varepsilon[I] \\
+ b_{11}I_\varepsilon[\varepsilon] + b_{12}II_\varepsilon[\varepsilon] + b_{13}III_\varepsilon[\varepsilon] \\
+ b_{21}I_\varepsilon[\varepsilon]^2 + b_{22}II_\varepsilon[\varepsilon]^2 + b_{23}III_\varepsilon[\varepsilon]^2 \\
+ b_{31}\big(\theta - \theta_{\underline{\Omega}}\big)[I] + b_{32}\big(\theta - \theta_{\underline{\Omega}}\big)[\varepsilon] + b_{33}\big(\theta - \theta_{\underline{\Omega}}\big)[\varepsilon]^2
\end{aligned}
\tag{6.9}
$$

b_0, b_1, b_2; b_{ij}; $i = 0,1,2,3$; $j = 1,2,3$ are material coefficients defined in the known configuration $\underline{\Omega}$. These are functions of the invariants of $[\varepsilon]$ and θ in $\underline{\Omega}$.

The constitutive theory for $[_s\sigma]$ defined by (6.9) is based on integrity, hence is complete. It requires fifteen material coefficients and $[_s\sigma]$ in (6.9) is up to fifth degree polynomial in the components of $[\varepsilon]$ or displacement gradients, but is linear in temperature θ. Simplified forms of the constitutive theory (6.9) will be considered in a later section.

6.2. Constitutive theory for $_s\boldsymbol{\sigma}$ using strain energy density function $^{s\boldsymbol{\sigma}}\pi$ (Approach II)

Consider the rate of strain energy density function $^{s\sigma}\dot{\pi} \equiv \frac{D}{Dt}(^{s\sigma}\pi)$.

If $^{s\sigma}\pi$ is the strain energy density function (strain energy per unit mass) due to the conjugate pair $[_s\sigma]$ and $[\varepsilon]$, then it's rate $^{s\sigma}\dot{\pi} \equiv \frac{D}{Dt}(^{s\sigma}\pi)$ is given by

$$^{s\sigma}\dot{\pi} = \frac{D}{Dt}\int_V {^{s\sigma}\pi}\rho_0 dV = \frac{D}{Dt}\int_{\bar{V}(t)} {^{s\boldsymbol{\sigma}}\bar{\pi}}\bar{\rho}d\bar{V} \tag{6.10}$$

or

$$^{s\sigma}\dot{\pi} = \int_V \frac{D}{Dt}\left(^{s\sigma}\pi\right)\rho_0 dV \tag{6.11}$$

We recall that $[_s\sigma]$ and $[\varepsilon]$ are energy conjugate and $[_s\sigma]$, $[\dot{\varepsilon}]$ are conjugate in rate of energy, hence

$$^{s\sigma}\dot{\pi} = \int_V {_s\sigma_{ij}}\dot{\varepsilon}_{ij} dV \tag{6.12}$$

Using (6.11) and (6.12)

$$\int_V \left(\rho_0 \frac{D}{Dt}\left(^{s\sigma}\pi\right) - {_s\sigma_{ij}}\dot{\varepsilon}_{ij}\right) dV = 0 \tag{6.13}$$

Since V is arbitrary, we have

$$\rho_0 \frac{D}{Dt}\left(^{s\sigma}\pi\right) - {_s\sigma_{ij}}\dot{\varepsilon}_{ij} = 0 \tag{6.14}$$

or

$$\rho_0 \frac{\partial}{\partial t}\left({}^{s\sigma}\pi\right) - {}_s\sigma_{ij}\dot{\varepsilon}_{ij} = 0 \tag{6.15}$$

Assuming ${}^{s\sigma}\pi = {}^{s\sigma}\pi([\varepsilon], t)$ we can write (6.15) as

$$\rho_0 \frac{\partial^{s\sigma}\pi}{\partial \varepsilon_{ij}}\dot{\varepsilon}_{ij} - {}_s\sigma_{ij}\dot{\varepsilon}_{ij} = 0 \tag{6.16}$$

or

$$\left(\rho_0 \frac{\partial^{s\sigma}\pi}{\partial \varepsilon_{ij}} - {}_s\sigma_{ij}\right)\dot{\varepsilon}_{ij} = 0 \tag{6.17}$$

For (6.17) to hold for arbitrary but admissible $[\dot{\varepsilon}]$ the following must hold

$${}_s\sigma_{ij} = \rho_0 \frac{\partial({}^{s\sigma}\pi)}{\partial \varepsilon_{ij}} \tag{6.18}$$

or

$$[{}_s\sigma] = [{}_s\sigma]^T = \rho_0 \frac{\partial({}^{s\sigma}\pi)}{\partial[\varepsilon]} \tag{6.19}$$

Equation (6.19) can also be derived directly using ${}_s\boldsymbol{\sigma}$ versus $[\varepsilon]$ and constructing the strain energy density function ${}^{s\sigma}\pi$ as

$${}^{s\sigma}\pi = \frac{1}{\rho_0}\int_0^{[\varepsilon]} {}_s\sigma_{ij}\, d\varepsilon_{ij} \tag{6.20}$$

From (6.20) we can obtain (fundamental theorem of calculus)

$$[{}_s\sigma] = [{}_s\sigma]^T = \rho_0 \frac{\partial({}^{s\sigma}\pi)}{\partial[\varepsilon]} \tag{6.21}$$

In the following we derive a constitutive theory for $[{}_s\sigma]$ using (6.21). We consider ${}^{s\sigma}\pi$ as a function of $[\varepsilon]$ and θ, however the principle of frame invariance requires that instead of $[\varepsilon]$ and θ, ${}^{s\sigma}\pi$ must be a function of the invariants of $[\varepsilon]$ and θ. Consider

$${}^{s\sigma}\pi = {}^{s\sigma}\pi(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta) \tag{6.22}$$

where

$$
\begin{aligned}
I_\varepsilon &= \operatorname{tr}[\varepsilon] = \varepsilon_{ii} \\
II_\varepsilon &= \frac{1}{2}\left((\operatorname{tr}[\varepsilon])^2 - \operatorname{tr}\left([\varepsilon]^2\right)\right) = \frac{1}{2}\varepsilon_{ii}\varepsilon_{kk} - \varepsilon_{ii}\varepsilon_{kk} \\
III_\varepsilon &= \det[\varepsilon]
\end{aligned}
\tag{6.23}
$$

Using (6.22) and (6.19) we can write

$$
[_s\sigma] = \rho_0 \left(\frac{\partial({}^{s\sigma}\pi)}{\partial I_\varepsilon}\frac{\partial I_\varepsilon}{\partial[\varepsilon]} + \frac{\partial({}^{s\sigma}\pi)}{\partial II_\varepsilon}\frac{\partial II_\varepsilon}{\partial[\varepsilon]} + \frac{\partial({}^{s\sigma}\pi)}{\partial III_\varepsilon}\frac{\partial III_\varepsilon}{\partial[\varepsilon]} \right) \tag{6.24}
$$

In the following we determine $\frac{\partial I_\varepsilon}{\partial[\varepsilon]}$, $\frac{\partial II_\varepsilon}{\partial[\varepsilon]}$ and $\frac{\partial III_\varepsilon}{\partial[\varepsilon]}$.

Consider $\frac{\partial I_\varepsilon}{\partial[\varepsilon]}$:

$$
\frac{\partial I_\varepsilon}{\partial \varepsilon_{ij}} = \frac{\partial \varepsilon_{ll}}{\partial \varepsilon_{ij}} = \delta_{ij} \tag{6.25}
$$

or

$$
\frac{\partial I_\varepsilon}{\partial[\varepsilon]} = [I] \tag{6.26}
$$

Consider $\frac{\partial II_\varepsilon}{\partial[\varepsilon]}$: Using (6.23) we can write

$$
\begin{aligned}
\frac{\partial II_\varepsilon}{\partial \varepsilon_{ij}} &= \frac{1}{2}\left(-\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{ij}}\varepsilon_{lk} - \varepsilon_{kl}\frac{\partial \varepsilon_{lk}}{\partial \varepsilon_{ij}} + \frac{\partial \varepsilon_{ll}}{\partial \varepsilon_{ij}}\varepsilon_{kk} + \varepsilon_{ll}\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{ij}} \right) \\
&= \frac{1}{2}\left(-\varepsilon_{ij} - \varepsilon_{ij} + \varepsilon_{kk}\delta_{ij} + \varepsilon_{ll}\delta_{ij} \right) \\
&= -\varepsilon_{ij} + \varepsilon_{kk}\delta_{ij}
\end{aligned}
\tag{6.27}
$$

Consider $\frac{\partial III_\varepsilon}{\partial[\varepsilon]}$:

$$
\frac{\partial III_\varepsilon}{\partial \varepsilon_{ij}} = \frac{\partial(\det[\varepsilon])}{\partial[\varepsilon]} = (\det[\varepsilon])\left[[\varepsilon]^{-1}\right]^T = (\det[\varepsilon])[\varepsilon]^{-1} = III_\varepsilon[\varepsilon]^{-1} \tag{6.28}
$$

Substituting from (6.26), (6.27) and (6.28) into (6.24)

$$[_s\sigma] = \rho_0 \left(\frac{\partial({}_s^\sigma\pi)}{\partial I_\varepsilon}[I] + \frac{\partial({}_s^\sigma\pi)}{\partial II_\varepsilon}(-[\varepsilon] - I_\varepsilon[I]) + \frac{\partial({}_s^\sigma\pi)}{\partial III_\varepsilon} III_\varepsilon[\varepsilon]^{-1} \right) \tag{6.29}$$

or

$$[_s\sigma] = \rho_0 \left(\frac{\partial({}_s^\sigma\pi)}{\partial I_\varepsilon} + \frac{\partial({}_s^\sigma\pi)}{\partial II_\varepsilon} I_\varepsilon \right)[I] + \left(-\rho_0 \frac{\partial({}_s^\sigma\pi)}{\partial II_\varepsilon} \right)[\varepsilon] + \left(\rho_0 \frac{\partial({}_s^\sigma\pi)}{\partial III_\varepsilon} III_\varepsilon \right)[\varepsilon]^{-1} \tag{6.30}$$

Let

$$\begin{aligned} {}^\sigma\alpha^0 &= \rho_0 \left(\frac{\partial({}_s^\sigma\pi)}{\partial I_\varepsilon} + \frac{\partial({}_s^\sigma\pi)}{\partial II_\varepsilon} I_\varepsilon \right) \\ {}^\sigma\alpha^1 &= -\rho_0 \frac{\partial({}_s^\sigma\pi)}{\partial II_\varepsilon} \\ {}^\sigma\alpha^{-1} &= \rho_0 \frac{\partial({}_s^\sigma\pi)}{\partial III_\varepsilon} III_\varepsilon \end{aligned} \tag{6.31}$$

Using (6.31) in (6.30), we can write

$$[_s\sigma] = {}^\sigma\alpha^0[I] + {}^\sigma\alpha^1[\varepsilon] + {}^\sigma\alpha^{-1}[\varepsilon]^{-1} \tag{6.32}$$

Recall the Hamilton-Cayley theorem [3]

$$[\varepsilon]^3 - I_\varepsilon[\varepsilon]^2 + II_\varepsilon[\varepsilon] - III[I] = 0 \tag{6.33}$$

For non-singular $[\varepsilon]$ i.e. $III_\varepsilon \neq 0$, we can solve (6.33) for $[\varepsilon]^{-1}$ to obtain

$$[\varepsilon]^{-1} = \frac{1}{III_\varepsilon}\left([\varepsilon]^2 - I_\varepsilon[\varepsilon] + II_\varepsilon[I]\right) \tag{6.34}$$

Substituting from (6.34) into (6.32)

$$[_s\sigma] = {}^\sigma\alpha^0[I] + {}^\sigma\alpha^1[\varepsilon] + \frac{{}^\sigma\alpha^{-1}}{III_\varepsilon}\left([\varepsilon]^2 - I_\varepsilon[\varepsilon] + II_\varepsilon[I]\right) \tag{6.35}$$

Collecting coefficients of $[I]$, $[\varepsilon]$, and $[\varepsilon]^2$ and defining

$$
\begin{aligned}
{}^{\sigma}\tilde{\alpha}^0 &= {}^{\sigma}\alpha^0 + \frac{{}^{\sigma}\alpha^{-1} II_\varepsilon}{III_\varepsilon} \\
{}^{\sigma}\tilde{\alpha}^1 &= {}^{\sigma}\alpha^1 - \frac{{}^{\sigma}\alpha^{-1} I_\varepsilon}{III_\varepsilon} \\
{}^{\sigma}\tilde{\alpha}^2 &= \frac{{}^{\sigma}\alpha^{-1}}{III_\varepsilon}
\end{aligned}
\tag{6.36}
$$

We can write (6.35) as

$$
[{}_s\sigma] = {}^{\sigma}\tilde{\alpha}^0[I] + {}^{\sigma}\tilde{\alpha}^1[\varepsilon] + {}^{\sigma}\tilde{\alpha}^2[\varepsilon]^2 \tag{6.37}
$$

Since ${}^{\sigma}\alpha^i = {}^{\sigma}\alpha^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta);\ i = 0,1,2$ we can conclude from (6.37) that ${}^{\sigma}\tilde{\alpha}^i = {}^{\sigma}\tilde{\alpha}^i(I_\varepsilon, II_\varepsilon, III_\varepsilon, \theta);\ i = 0,1,2$. The constitutive theory (6.37) is the same as the one derived using the theory of generators and invariants, thus determination of the material coefficients follows the same procedure as used in section 6.1 and finally we obtain the same constitutive theory as in section 6.1 (equation 6.9) with the same definition of material coefficients.

6.3. Constitutive theory for $[\varepsilon]$ in terms of $[{}_s\sigma]$ based on complementary strain energy density function ${}^{s\sigma}\pi^c$ (Approach II)

Similar to the material in section 6.2 we begin with the integral defined by (using work conjugate pair $[{}_s\sigma], [\varepsilon]$)

$$
{}^{s\sigma}\pi^c = \frac{1}{\rho_0}\int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij}\, d({}_s\sigma_{ij}) \tag{6.38}
$$

in which ${}^{s\sigma}\pi^c$ is the complementary strain energy density function. From (6.38) we can obtain (fundamental theorem of calculus)

$$
[\varepsilon]^T = [\varepsilon] = \rho_0 \frac{\partial({}^{s\sigma}\pi^c({}_s\boldsymbol{\sigma}))}{\partial[{}_s\sigma]} \tag{6.39}
$$

The complementary strain energy density function ${}^{s\sigma}\pi^c$ and the strain energy density function ${}^{s\sigma}\pi$ are obviously related. Consider

$$
\frac{1}{\rho_0}{}_s\sigma_{ij}\varepsilon_{ij} = \frac{1}{\rho_0}\int_0^{\varepsilon} {}_s\sigma_{ij}\, d\varepsilon_{ij} + \frac{1}{\rho_0}\int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij}\, d({}_s\sigma_{ij}) \tag{6.40}
$$

or

$$\frac{1}{\rho_0}{}_s\sigma_{ij}\varepsilon_{ij} = {}^{s\sigma}\pi([\varepsilon]) + {}^{s\sigma}\pi^c({}_s\boldsymbol{\sigma}) \tag{6.41}$$

In the case of linear elasticity

$$^{s\sigma}\pi = \frac{1}{\rho_0}\int_0^{\boldsymbol{\varepsilon}} {}_s\sigma_{ij} d\varepsilon_{ij} = \frac{1}{2\rho_0}{}_s\sigma_{ij}\varepsilon_{ij} \tag{6.42}$$

$$^{s\sigma}\pi^c = \frac{1}{\rho_0}\int_0^{{}_s\boldsymbol{\sigma}} \varepsilon_{ij} d_s\sigma_{ij} = \frac{1}{2\rho_0}{}_s\sigma_{ij}\varepsilon_{ij} \tag{6.43}$$

$$\therefore \quad {}^{s\sigma}\pi = {}^{s\sigma}\pi^c \tag{6.44}$$

Using (6.39) we can derive a constitutive theory for $[\varepsilon]$ if we know $^{s\sigma}\pi^c$ as a function of ${}_s\boldsymbol{\sigma}$. Consider

$$[\varepsilon] = [\varepsilon]\left(I_{{}_s\boldsymbol{\sigma}}, II_{{}_s\boldsymbol{\sigma}}, III_{{}_s\boldsymbol{\sigma}}, \theta\right) \tag{6.45}$$

In which

$$\begin{aligned} I_{{}_s\sigma} &= \text{tr}[{}_s\sigma] = {}_s\sigma_{ii} \\ II_{{}_s\sigma} &= \frac{1}{2}\left((\text{tr}[{}_s\sigma])^2 - \text{tr}\left([{}_s\sigma]^2\right)\right) \\ III_{{}_s\sigma} &= \det[{}_s\sigma] \end{aligned} \tag{6.46}$$

Using (6.45) and (6.39) we can write

$$[{}_s\sigma] = \rho_0\left(\frac{\partial({}^{s\sigma}\pi)}{\partial I_{{}_s\sigma}}\frac{\partial I_{{}_s\sigma}}{\partial[{}_s\sigma]} + \frac{\partial({}^{s\sigma}\pi)}{\partial II_{{}_s\sigma}}\frac{\partial II_{{}_s\sigma}}{\partial[{}_s\sigma]} + \frac{\partial({}^{s\sigma}\pi)}{\partial III_{{}_s\sigma}}\frac{\partial III_{{}_s\sigma}}{\partial[{}_s\sigma]}\right) \tag{6.47}$$

Using (6.46) we can obtain

$$\frac{\partial I_{{}_s\sigma}}{\partial[{}_s\sigma]} = [I] \tag{6.48}$$

$$\frac{\partial II_{{}_s\sigma}}{\partial[{}_s\sigma]} = -[{}_s\sigma] + I_{{}_s\sigma}[I] \tag{6.49}$$

$$\frac{\partial III_{s\sigma}}{\partial[_s\sigma]} = III_{s\sigma}[_s\sigma]^{-1} \tag{6.50}$$

Substituting (6.48)–(6.50) into (6.47), defining

$$\begin{aligned} {}^{\varepsilon}\alpha^0 &= \rho_0\left(\frac{\partial({}^{s\sigma}\pi^c)}{\partial I_{s\sigma}} + \frac{\partial({}^{s\sigma}\pi^c)}{\partial II_{s\sigma}} I_{s\sigma}\right) \\ {}^{\varepsilon}\alpha^1 &= -\rho_0\frac{\partial({}^{s\sigma}\pi^c)}{\partial II_{s\sigma}} \\ {}^{\varepsilon}\alpha^{-1} &= \rho_0\frac{\partial({}^{s\sigma}\pi^c)}{\partial III_{s\sigma}} III_{s\sigma} \end{aligned} \tag{6.51}$$

Collecting coefficients of $[I]$, $[_s\sigma]$, $[_s\sigma]^{-1}$ and using (6.51) we can write

$$[\varepsilon] = {}^{\varepsilon}\alpha^0[I] + {}^{\varepsilon}\alpha^1[_s\sigma] + {}^{\varepsilon}\alpha^{-1}[_s\sigma]^{-1} \tag{6.52}$$

$[_s\sigma]^{-1}$ in (6.52) can be obtained in terms of $[I]$, $[_s\sigma]$, and $[_s\sigma]^2$ and the invariants of $[_s\sigma]$ using the Hamilton-Cayley theorem to obtain

$$[\varepsilon] = {}^{\varepsilon}\tilde{\alpha}^0[I] + {}^{\varepsilon}\tilde{\alpha}^1[_s\sigma] + {}^{\varepsilon}\tilde{\alpha}^2[_s\sigma]^2 \tag{6.53}$$

in which

$${}^{\varepsilon}\tilde{\alpha}^i = {}^{\varepsilon}\tilde{\alpha}^i\left(I_{s\sigma}, II_{s\sigma}, III_{s\sigma}, \theta\right);\ i = 0,1,2 \tag{6.54}$$

Since ${}^{\varepsilon}\tilde{\alpha}^i; i = 0,1,2$ are functions of ${}^{\varepsilon}\alpha^i; i = -1,0,1$ and ${}^{\varepsilon}\alpha^i$ are functions of $I_{s\sigma}, II_{s\sigma}, III_{s\sigma}$, and θ, (6.54) holds.

Material coefficients in (6.53) are derived using exactly the same approach as used in section 6.1, which would lead to a constitutive theory for $[\varepsilon]$ similar to that for $[_s\sigma]$ in equation (6.9). Derivation is straight forward.

6.4. Constitutive theory for $[_s\sigma]$ using strain energy density function ${}^{s\sigma}\pi([\varepsilon],\theta)$ and expanding it in Taylor series about a known configuration $\underline{\Omega}$ (Approach II)

Consider ${}^{s\sigma}\pi = {}^{s\sigma}\pi([\varepsilon],\theta)$ and expand ${}^{s\sigma}\pi$ in $[\varepsilon]$ about a known configuration $\underline{\Omega}$ using Taylor series [3].

$$\begin{aligned}{}^{s\sigma}\pi = {}^{s\sigma}\pi\big|_{\underline{\Omega}} + \frac{\partial({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right) + \frac{1}{2!}\left.\frac{\partial^2({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}}\right|_{\underline{\Omega}}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right) \\ + \frac{1}{3!}\left.\frac{\partial^3({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{pq}}\right|_{\underline{\Omega}}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq}-(\varepsilon_{pq})_{\underline{\Omega}}\right) + \cdots\end{aligned} \tag{6.55}$$

Let

$$\begin{aligned}{}^{s\sigma}\pi\big|_{\underline{\Omega}} &= C \\ \left.\frac{\partial({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}}\right|_{\underline{\Omega}} &= C_{ij} \\ \left.\frac{\partial^2({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}}\right|_{\underline{\Omega}} &= \hat{C}_{ijkl} \\ \left.\frac{\partial^3({}^{s\sigma}\pi)}{\partial\varepsilon_{ij}\varepsilon_{kl}\varepsilon_{pq}}\right|_{\underline{\Omega}} &= \tilde{C}_{ijklpq}\end{aligned} \tag{6.56}$$

Substituting from (6.56) into (6.55)

$$\begin{aligned}{}^{s\sigma}\pi = C + C_{ij}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right) + \hat{C}_{ijkl}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right) \\ + \tilde{C}_{ijklpq}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq}-(\varepsilon_{pq})_{\underline{\Omega}}\right) + \cdots\end{aligned} \tag{6.57}$$

Substituting ${}^{s\sigma}\pi$ from (6.57) into (6.21) and differentiating ${}^{s\sigma}\pi$ with respect to $[\varepsilon]$ and noting that partial derivatives of (6.56) with respect to $[\varepsilon]$ are zero and that

$$\begin{aligned}&\frac{\partial}{\partial\varepsilon_{mn}}\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right) = \delta_{im}\delta_{jn} \\ &\frac{\partial}{\partial\varepsilon_{mn}}\left(\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\right) \\ &\qquad = \delta_{im}\delta_{jn}\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right) + \left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln} \\ &\frac{\partial}{\partial\varepsilon_{mn}}\left(\left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq}-(\varepsilon_{pq})_{\underline{\Omega}}\right)\right) \\ &\qquad = \delta_{im}\delta_{jn}\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\left(\varepsilon_{pq}-(\varepsilon_{pq})_{\underline{\Omega}}\right) \\ &\qquad + \left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln}\left(\varepsilon_{pq}-(\varepsilon_{pq})_{\underline{\Omega}}\right) \\ &\qquad + \left(\varepsilon_{ij}-(\varepsilon_{ij})_{\underline{\Omega}}\right)\left(\varepsilon_{kl}-(\varepsilon_{kl})_{\underline{\Omega}}\right)\delta_{pm}\delta_{qn}\end{aligned} \tag{6.58}$$

We obtain the following (note that ρ_0 is absorbed in the coefficients in (6.59)).

$$ {}_s\sigma_{mn} = ({}_s\sigma_{mn})_{\underline{\Omega}} + \underset{=}{C}_{mnij}\varepsilon_{ij} + \bar{\bar{C}}_{mnijkl}\varepsilon_{ij}\varepsilon_{kl} + \cdots \tag{6.59} $$

In obtaining (6.59) we collect those terms that are defined in the known configuration $\underline{\Omega}$ to define coefficients of $[\varepsilon]$ and $[\varepsilon]^2$ and we use symmetry of the coefficients [3] i.e. $\hat{C}_{mnij} = \hat{C}_{ijmn} \ldots$ etc.

6.5. Constitutive theory of $[\varepsilon]$ using complementary strain energy density function ${}^{s\sigma}\pi^c([{}_s\sigma],\theta)$ and expanding it in Taylor series about a known configuration (Approach II)

Considering ${}^{s\sigma}\pi^c = {}^{s\sigma}\pi^c([{}_s\sigma],\theta)$ and expanding this in Taylor series in $[{}_s\sigma]$ about a known configuration $\underline{\Omega}$, then using (6.39) and following exactly the same procedure as in section 6.4 we can derive a constitutive theory for $[\varepsilon]$ as a function of $[{}_s\sigma]$ that is exactly parallel to (6.59). Details are straight forward and hence omitted for the sake of brevity.

6.6. Constitutive theory for $[m]$ based on the theory of generators and invariants (Approach II)

Following section 6.1 we consider $[m] = [m([{}_s^\Theta J],\theta)]$ in which $[{}_s^\Theta J]$ is the symmetric part of the rotation gradient tensor [1]. $[m]$ and $[{}_s^\Theta J]$ are symmetric tensors of rank tow and θ is a tensor of rank zero. The combined generators of $[{}_s^\Theta J]$ and θ that are symmetric tensors of ranks two are $[{}_s^\Theta J]$ and $[{}_s^\Theta J]^2$. Thus, we can express $[m]$ as a linear combination of $[I]$, $[{}_s^\Theta J]$, and $[{}_s^\Theta J]^2$.

$$ [m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}_s^\Theta J] + {}^m\tilde{\alpha}^2[{}_s^\Theta J]^2 \tag{6.60} $$

in which ${}^m\tilde{\alpha}^i$; $i = 0,1,2$ are functions of $I_\Theta, II_\Theta, III_\Theta$ and temperature θ, where $I_\Theta, II_\Theta, III_\Theta$ are the principal invariants of $[{}_s^\Theta J]$ based on the characteristic equation of $[{}_s^\Theta J]$ i.e.

$$ {}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i(I_\Theta, II_\Theta, III_\Theta, \theta) \tag{6.61} $$

Equations (6.60) and (6.61) hold in the current configuration. Using (6.60) and (6.61) we define material coefficients. We expand ${}^m\tilde{\alpha}^i; i = 0,1,2$ in Taylor series in $I_\Theta, II_\Theta, III_\Theta$ and θ about a known configuration $\underline{\Omega}$ and only retain up to linear terms in the invariants and the temperature.

$$
\begin{aligned}
{}^m\tilde{\alpha}^i = {}^m\tilde{\alpha}^i\Big|_{\underline{\Omega}} + \frac{\partial {}^m\tilde{\alpha}^i}{\partial I_\Theta}\Big|_{\underline{\Omega}}\Big(I_\Theta - (I_\Theta)_{\underline{\Omega}}\Big) + \frac{\partial {}^m\tilde{\alpha}^i}{\partial II_\Theta}\Big|_{\underline{\Omega}}\Big(II_\Theta - (II_\Theta)_{\underline{\Omega}}\Big) \\
+ \frac{\partial {}^m\tilde{\alpha}^i}{\partial III_\Theta}\Big|_{\underline{\Omega}}\Big(III_\Theta - (III_\Theta)_{\underline{\Omega}}\Big) + \frac{\partial {}^m\tilde{\alpha}^i}{\partial \theta}\Big|_{\underline{\Omega}}\big(\theta - \theta_{\underline{\Omega}}\big); \quad i = 0,1,2
\end{aligned}
\tag{6.62}
$$

Substituting from (6.62) into (6.61) and collecting coefficients of $[I]$, $[{}_s^\Theta J]$, $[{}_s^\Theta J]^2$, $I_\Theta[I]$, $I_\Theta[{}_s^\Theta J]$, $I_\Theta[{}_s^\Theta J]^2$, $II_\Theta[I]$, $II_\Theta[{}_s^\Theta J]$, $II_\Theta[{}_s^\Theta J]^2$, $III_\Theta[I]$, $III_\Theta[{}_s^\Theta J]$, $III_\Theta[{}_s^\Theta J]^2$, $(\theta - \theta|_{\underline{\Omega}})[I]$, $(\theta - \theta|_{\underline{\Omega}})[{}_s^\Theta J]$, $(\theta - \theta|_{\underline{\Omega}})[{}_s^\Theta J]^2$ and defining the following coefficients

$$
\begin{aligned}
\tilde{b}_0 &= {}^m\tilde{\alpha}^0|_{\underline{\Omega}} - \tilde{b}_{01}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{02}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{03}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_1 &= {}^m\tilde{\alpha}^1|_{\underline{\Omega}} - \tilde{b}_{11}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{12}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{13}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_2 &= {}^m\tilde{\alpha}^2|_{\underline{\Omega}} - \tilde{b}_{21}(I_\Theta)_{\underline{\Omega}} - \tilde{b}_{22}(II_\Theta)_{\underline{\Omega}} - \tilde{b}_{23}(III_\Theta)_{\underline{\Omega}} \\
\tilde{b}_{01} &= \frac{\partial {}^m\tilde{\alpha}^0}{\partial I_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{02} = \frac{\partial {}^m\tilde{\alpha}^0}{\partial II_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{03} = \frac{\partial {}^m\tilde{\alpha}^0}{\partial III_\Theta}\Big|_{\underline{\Omega}} \\
\tilde{b}_{11} &= \frac{\partial {}^m\tilde{\alpha}^1}{\partial I_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{12} = \frac{\partial {}^m\tilde{\alpha}^1}{\partial II_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{13} = \frac{\partial {}^m\tilde{\alpha}^1}{\partial III_\Theta}\Big|_{\underline{\Omega}} \\
\tilde{b}_{21} &= \frac{\partial {}^m\tilde{\alpha}^2}{\partial I_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{22} = \frac{\partial {}^m\tilde{\alpha}^2}{\partial II_\Theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{23} = \frac{\partial {}^m\tilde{\alpha}^2}{\partial III_\Theta}\Big|_{\underline{\Omega}} \\
\tilde{b}_{31} &= \frac{\partial {}^m\tilde{\alpha}^0}{\partial \theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{32} = \frac{\partial {}^m\tilde{\alpha}^1}{\partial \theta}\Big|_{\underline{\Omega}}, \quad \tilde{b}_{33} = \frac{\partial {}^m\tilde{\alpha}^2}{\partial \theta}\Big|_{\underline{\Omega}}
\end{aligned}
\tag{6.63}
$$

We can write (6.60) as

$$
\begin{aligned}
[m] = {} & \tilde{b}_0[I] + \tilde{b}_1[{}_s^\Theta J] + \tilde{b}_2[{}_s^\Theta J]^2 \\
& + \tilde{b}_{01}I_\Theta[I] + \tilde{b}_{02}II_\Theta[I] + \tilde{b}_{03}III_\Theta[I] \\
& + \tilde{b}_{11}I_\Theta[{}_s^\Theta J] + \tilde{b}_{12}II_\Theta[{}_s^\Theta J] + \tilde{b}_{13}III_\Theta[{}_s^\Theta J] \\
& + \tilde{b}_{21}I_\Theta[{}_s^\Theta J]^2 + \tilde{b}_{22}II_\Theta[{}_s^\Theta J]^2 + \tilde{b}_{23}III_\Theta[{}_s^\Theta J]^2 \\
& + \tilde{b}_{31}\big(\theta - \theta_{\underline{\Omega}}\big)[I] + \tilde{b}_{32}\big(\theta - \theta_{\underline{\Omega}}\big)[{}_s^\Theta J] + \tilde{b}_{33}\big(\theta - \theta_{\underline{\Omega}}\big)[{}_s^\Theta J]^2
\end{aligned}
\tag{6.64}
$$

$\tilde{b}_0$, $\tilde{b}_1$, $\tilde{b}_2$; $\tilde{b}_{ij}$; $i = 0,1,2,3$; $j = 1,2,3$ are material coefficients defined in the known configuration $\underline{\Omega}$. These are functions of the invariants of $[{}_s^\Theta J]$ and θ in $\underline{\Omega}$.

The constitutive theory for $[m]$ defined by (6.64) is based on integrity, hence is complete. It requires fifteen material coefficients and $[m]$ in (6.64) is up to a fifth degree polynomial in the components of $[{}_s^\Theta J]$ or rotation gradients, but is linear in temperature θ. Simplified forms of the constitutive theory (6.64) will be considered in a later section.

6.7. Constitutive theory for $[m]$ using strain energy density function ${}^m\pi$ (Approach II)

If ${}^m\pi$ is the strain energy density function due to the conjugate pair ($[m]$, $[{}_s^\Theta J]$), then following the details in section 6.2 for the work conjugate pair ($[{}_s\sigma], [\varepsilon]$) we can derive the following (similar to equation (6.19)) for the moment tensor $[m]$. or

$$[m] = [m]^T = \rho_0 \frac{\partial({}^m\pi)}{\partial[{}_s^\Theta J]} \tag{6.65}$$

Choosing

$${}^m\pi = {}^m\pi(I_\Theta, II_\Theta, III_\Theta, \theta) \tag{6.66}$$

and using in (6.65) and following derivation parallel to section 6.2 we can derive

$$[m] = {}^m\alpha^0[I] + {}^m\alpha^1[{}_s^\Theta J] + {}^m\alpha^{-1}[{}_s^\Theta J]^{-1} \tag{6.67}$$

Using Hamilton-Cayley theorem [3] to substitute for $[{}_s^\Theta J]^{-1}$ we obtain

$$[m] = {}^m\tilde{\alpha}^0[I] + {}^m\tilde{\alpha}^1[{}_s^\Theta J] + {}^m\tilde{\alpha}^2[{}_s^\Theta J]^2 \tag{6.68}$$

Since ${}^\sigma\alpha^i = {}^\sigma\alpha^i(I_\Theta, II_\Theta, III_\Theta, \theta)$; $i = 0,1,2$ we can conclude from (6.37) that

$${}^\sigma\tilde{\alpha}^i = {}^\sigma\tilde{\alpha}^i(I_\Theta, II_\Theta, III_\Theta, \theta);\ i = 0,1,2 \tag{6.69}$$

This constitutive theory is the same as the one derived using the theory of generators and invariants (section 6.6), thus determination of the material coefficients follows the same procedure as used in section 6.6 and finally we obtain exactly the same constitutive theory with the same definition of material coefficients (equation 6.64).

6.8. Constitutive theory for $[{}^{\Theta}_{s}J]$ in terms of $[m]$ based on complementary strain energy density function ${}^{m}\pi^{c}$

If ${}^{m}\pi^{c}$ is the complementary strain energy density function due to conjugate pair $([m],[{}^{\Theta}_{s}J])$, then following the details in section 6.3 for conjugate pair $([{}_{s}\sigma],[\varepsilon])$ we can derive the following (similar to equation (6.39)) for $[{}^{\Theta}_{s}J]$ (Also see reference [3] for more details on the method).

$$[{}^{\Theta}_{s}J] = \rho_0 \frac{\partial({}^{m}\pi^{c})}{\partial[m]} \tag{6.70}$$

In which we assume

$${}^{m}\pi^{c} = {}^{m}\pi^{c}(I_m, II_m, III_m, \theta) \tag{6.71}$$

I_m, II_m, III_m are the principal invariants of the moment tensor $[m]$. Substituting (6.71) in (6.70) and following the derivation parallel to section 6.3 we can derive

$$[{}^{\Theta}_{s}J] = {}^{J}\alpha^{0}[I] + {}^{J}\alpha^{1}[m] + {}^{J}\alpha^{-1}[m]^{-1} \tag{6.72}$$

Using the Hamilton-Cayley theorem to substitute for $[m]^{-1}$ we obtain

$$[{}^{\Theta}_{s}J] = {}^{J}\tilde{\alpha}^{0}[I] + {}^{J}\tilde{\alpha}^{1}[m] + {}^{J}\tilde{\alpha}^{2}[m]^{2} \tag{6.73}$$

in which

$${}^{J}\tilde{\alpha}^{i} = {}^{J}\tilde{\alpha}^{i}(I_m, II_m, III_m, \theta);\ i = 0,1,2 \tag{6.74}$$

I_m, II_m, III_m are the principal invariants of the tensor $[m]$. Clearly ${}^{J}\tilde{\alpha}^{i}; i = 0,1,2$ are functions of ${}^{J}\alpha^{i}; i = -1,0,1$ and ${}^{J}\alpha^{i}$ are functions of I_m, II_m, III_m, and θ, (6.74) is valid. Material coefficients in (6.73) are derived using exactly the same approach based on Taylor series as used in section 6.1, hence is not repeated here for the sake of brevity.

6.9. Constitutive theory for $[m]$ using strain energy density function ${}^m\pi([{}^{\Theta}_{s}J],\theta)$ and expanding it in Taylor series about a known configuration $\underline{\Omega}$

Consider ${}^m\pi = {}^m\pi([{}^{\Theta}_{s}J],\theta)$ and expand ${}^m\pi$ in $[{}^{\Theta}_{s}J]$ about a known configuration $\underline{\Omega}$ using Taylor series.

$$
\begin{aligned}
{}^m\pi = {}^m\pi\big|_{\underline{\Omega}} &+ \frac{\partial({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right) + \frac{1}{2!}\left.\frac{\partial^2({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})\partial({}^{\Theta}_{s}J_{kl})}\right|_{\underline{\Omega}}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl} - ({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right) \\
&+ \frac{1}{3!}\left.\frac{\partial^3({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})\partial({}^{\Theta}_{s}J_{kl})\partial({}^{\Theta}_{s}J_{pq})}\right|_{\underline{\Omega}}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl} - ({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{pq} - ({}^{\Theta}_{s}J_{pq})_{\underline{\Omega}}\right) + \cdots
\end{aligned}
\tag{6.75}
$$

Let

$$
\begin{aligned}
{}^m\pi\big|_{\underline{\Omega}} &= {}^mC \\
\left.\frac{\partial({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})}\right|_{\underline{\Omega}} &= {}^mC_{ij} \\
\left.\frac{\partial^2({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})\partial({}^{\Theta}_{s}J_{kl})}\right|_{\underline{\Omega}} &= {}^m\hat{C}_{ijkl} \\
\left.\frac{\partial^3({}^m\pi)}{\partial({}^{\Theta}_{s}J_{ij})\partial({}^{\Theta}_{s}J_{kl})\partial({}^{\Theta}_{s}J_{pq})}\right|_{\underline{\Omega}} &= {}^m\tilde{C}_{ijklpq}
\end{aligned}
\tag{6.76}
$$

Substituting from (6.76) into (6.75)

$$
\begin{aligned}
{}^m\pi = {}^mC &+ {}^mC_{ij}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right) + {}^m\hat{C}_{ijkl}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl} - ({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right) \\
&+ {}^m\tilde{C}_{ijklpq}\left({}^{\Theta}_{s}J_{ij} - ({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl} - ({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{pq} - ({}^{\Theta}_{s}J_{pq})_{\underline{\Omega}}\right) + \cdots
\end{aligned}
\tag{6.77}
$$

Substituting ${}^m\pi$ from (6.77) into (6.65) and differentiating ${}^m\pi$ with respect to $[{}^{\Theta}_{s}J]$ and noting that partial derivatives of (6.76) with respect to $[{}^{\Theta}_{s}J]$ are zero and that

$$
\begin{aligned}
&\frac{\partial}{\partial({}^{\Theta}_{s}J_{mn})}\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)=\delta_{im}\delta_{jn}\\
&\frac{\partial}{\partial({}^{\Theta}_{s}J_{mn})}\left(\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl}-({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\right)\\
&\qquad=\delta_{im}\delta_{jn}\left({}^{\Theta}_{s}J_{kl}-({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)+\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln}\\
&\frac{\partial}{\partial({}^{\Theta}_{s}J_{mn})}\left(\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl}-({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{pq}-({}^{\Theta}_{s}J_{pq})_{\underline{\Omega}}\right)\right)\\
&\qquad=\delta_{im}\delta_{jn}\left({}^{\Theta}_{s}J_{kl}-({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{pq}-({}^{\Theta}_{s}J_{pq})_{\underline{\Omega}}\right)\\
&\qquad+\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\delta_{km}\delta_{ln}\left({}^{\Theta}_{s}J_{pq}-({}^{\Theta}_{s}J_{pq})_{\underline{\Omega}}\right)\\
&\qquad+\left({}^{\Theta}_{s}J_{ij}-({}^{\Theta}_{s}J_{ij})_{\underline{\Omega}}\right)\left({}^{\Theta}_{s}J_{kl}-({}^{\Theta}_{s}J_{kl})_{\underline{\Omega}}\right)\delta_{pm}\delta_{qn}
\end{aligned}
\tag{6.78}
$$

We obtain the following (note that ρ_0 is absorbed in the coefficients in (6.79)).

$$
m_{mn}=(m_{mn})_{\underline{\Omega}}+\underset{=}{C}_{mnij}{}^{\Theta}_{s}J_{ij}+\bar{\bar{C}}_{mnijkl}{}^{\Theta}_{s}J_{ij}{}^{\Theta}_{s}J_{kl}+\cdots
\tag{6.79}
$$

In obtaining (6.79) we collect those terms that are defined in the known configuration $\underline{\Omega}$ to define coefficients of $[{}^{\Theta}_{s}J]$ and $[{}^{\Theta}_{s}J]^2$ and we use symmetry of the coefficients [3] i.e. $\hat{C}_{mnij}=\hat{C}_{ijmn}\dots$ etc.

6.10. Constitutive theory of $[{}^{\Theta}_{s}J]$ using complementary strain energy density function ${}^{m}\pi^{c}([m],\theta)$ and expanding it in Taylor series about a known configuration (Approach II)

Considering ${}^{m}\pi^{c}={}^{m}\pi^{c}([m],\theta)$ and expanding this in Taylor series in $[m]$ about a known configuration $\underline{\Omega}$, then using (6.70) and following exactly the same procedure as in section 6.9 we can derive a constitutive theory for $[{}^{\Theta}_{s}J]$ as a function of $[m]$ that is exactly parallel to (6.59). Details are straight forward and hence omitted for the sake of brevity.

7. Remarks on the constitutive theories (Approach II)

In section 6.1 through 6.10 the most general constitutive theories have been derived using approach II for $[{}_{s}\sigma]$ and $[m]$ using the theory of generators and invariants and strain energy density functions ${}^{s\sigma}\pi$ and ${}^{m}\pi$. Additionally, the constitutive theories for $[{}_{s}\sigma]$ and $[m]$ are also presented using Taylor series expansions and the strain energy density functions. Constitutive theories for $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ in terms of $[{}_s\sigma]$ and $[m]$ have also been derived using the theory of generators and invariants and the complementary strain energy density functions ${}^{s\sigma}\pi^c$ and ${}^{m}\pi^c$ including the constitutive theories based on Taylor series expansions and the complementary strain energy density functions. In the following we make some specific remarks pertaining to the specific constitutive theories presented so far.

1. The constitutive theory for $[{}_s\sigma]$ resulting from the theory of generators and invariants and the strain energy density function ${}^{s\sigma}\pi$ are the same. Thus, when considering simplified theories we can consider either one. The same is true for the constitutive theories for $[m]$ derived using the theory of generators and invariants and the strain energy density function ${}^{m}\pi$.

2. The constitutive theories derived for $[{}_s\sigma]$ and $[m]$ using Taylor series expansions violate the frame invariance principle as the material coefficients are functions of the argument tensors, not their invariants (in a known configuration). Unfortunately this is a common and serious drawback of the approaches for deriving constitutive theories that are based on Taylor series expansions.

3. The constitutive theory for $[\varepsilon]$ in terms of $[{}_s\sigma]$ resulting from the theory of generators and invariants and the complementary strain energy density functions ${}^{s\sigma}\pi^c$ are also the same. The same is true for the constitutive theory for $[{}^{\Theta}_{s}J]$ in terms of $[m]$ derived using the theory of generators and invariants and the complementary strain energy density function ${}^{s\sigma}\pi^c$.

4. The constitutive theories for $[\varepsilon]$ and $[{}^{\Theta}_{s}J]$ in terms of $[{}_s\sigma]$ and $[m]$ derived using the complementary strain energy density functions ${}^{s\sigma}\pi^c$ and ${}^{m}\pi^c$ and the Taylor series expansions also violate the frame invariance principle as the material coefficients in these theories are functions of the argument tensors and not of their invariants as required by the axioms of the constitutive theory.

8. Simplified form of the constitutive theories for $[{}_s\sigma]$ and $[m]$

The constitutive theories derived in sections 6.1 through 6.10 when based on integrity contain too many material coefficients. Simplified forms of these constitutive theories containing fewer material coefficients are necessary for determination of material coefficients as well as for their use in practical applications. In this section we consider some simplified forms of these theories. Based on the remarks in section 7, we only consider the constitutive theories derived using the theory of generators and invariants.

8.1. Simplified constitutive theory for $[{}_s\sigma]$

Using the most general form of the constitutive theory for $[{}_s\sigma]$ given by (6.9), we can derive various simplified constitutive theories for $[{}_s\sigma]$. For example, if we limit the constitutive theory for $[{}_s\sigma]$ to only up to second degree terms in the components of $[\varepsilon]$, then we obtain the following.

$$\begin{aligned}[{}_s\sigma] &= b_0[I] + b_1[\varepsilon] + b_2[\varepsilon]^2 \\ &+ b_{01} I_\varepsilon [I] + b_{02} II_\varepsilon [I] + b_{11} I_\varepsilon [\varepsilon] \\ &+ b_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + b_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] + b_{33}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon]^2\end{aligned} \tag{8.1}$$

A further simplification of (8.1) would be a constitutive theory for $[{}_s\sigma]$ that is linear in the components of $[\varepsilon]$.

$$[{}_s\sigma] = b_0[I] + b_1[\varepsilon] + b_{01} I_\varepsilon [I] + b_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + b_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] \tag{8.2}$$

If we redefine material coefficients in (8.2), we can write

$$[{}_s\sigma] = (\sigma_0)_{\underline{\Omega}}[I] + 2\tilde{\mu}_{\underline{\Omega}}[\varepsilon] + \tilde{\lambda}_{\underline{\Omega}}(\text{tr}[\varepsilon])[I] - ({}^1\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + ({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon] \tag{8.3}$$

If we neglect $\left(\theta - \theta_{\underline{\Omega}}\right)[\varepsilon]$ terms in (8.3), then we obtain

$$[{}_s\sigma] = (\sigma_0)_{\underline{\Omega}}[I] + 2\tilde{\mu}_{\underline{\Omega}}[\varepsilon] + \tilde{\lambda}_{\underline{\Omega}}(\text{tr}[\varepsilon])[I] - (\tilde{\alpha}_{tm})_{\underline{\Omega}}\left(\theta - \theta_{\underline{\Omega}}\right)[I] \tag{8.4}$$

This is the simplest possible constitutive theory for $[{}_s\sigma]$. $[{}_s\sigma]$ in (8.4) can also be represented in matrix and vector notation (Voigt's notation). See reference [3] for details. In (8.3) $(\sigma_0)_{\underline{\Omega}} = b_0$, $2\tilde{\mu}_{\underline{\Omega}} = b_1$, $\tilde{\lambda}_{\underline{\Omega}} = b_{01}$, $({}^1\tilde{\alpha}_{tm})_{\underline{\Omega}} = b_{31}$ and $({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}} = b_{32}$.

8.2. Simplified constitutive theory for $[m]$

Using the most general form of the constitutive theory for $[m]$ given by (6.64), we can derive various simplified constitutive theories for $[m]$. For example, if we limit the constitutive theory for $[m]$ to only up to second degree terms in the components of $[{}_s^\Theta J]$, then we obtain the following.

$$\begin{aligned}[m] &= \tilde{b}_0[I] + \tilde{b}_1[{}_s^\Theta J] + \tilde{b}_2[{}_s^\Theta J]^2 \\ &+ \tilde{b}_{01} I_J [I] + \tilde{b}_{02} II_J [I] + \tilde{b}_{11} I_J [{}_s^\Theta J] \\ &+ \tilde{b}_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + \tilde{b}_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[{}_s^\Theta J] + \tilde{b}_{33}\left(\theta - \theta_{\underline{\Omega}}\right)[{}_s^\Theta J]^2\end{aligned} \tag{8.5}$$

A further simplification of (8.5) would be a constitutive theory for $[m]$ that is linear in the components of $[{}_s^\Theta J]$.

$$[m] = \tilde{b}_0[I] + \tilde{b}_1[{}_s^\Theta J] + \tilde{b}_{01} I_J [I] + \tilde{b}_{31}\left(\theta - \theta_{\underline{\Omega}}\right)[I] + \tilde{b}_{32}\left(\theta - \theta_{\underline{\Omega}}\right)[{}_s^\Theta J] \tag{8.6}$$

If we neglect $\left(\theta - \theta_{\underline{\Omega}}\right)[{}_s^\Theta J]$ terms in (8.6), then we obtain

$$[m] = (m_0)_{\underline{\Omega}}[I] + 2{}^m\tilde{\mu}_{\underline{\Omega}}[{}_s^\Theta J] + {}^m\tilde{\lambda}_{\underline{\Omega}}(\mathrm{tr}[{}_s^\Theta J])[I] - {}^m\tilde{\alpha}_{tm}\left(\theta - \theta_{\underline{\Omega}}\right)[I] \tag{8.7}$$

This is the simplest possible constitutive theory for $[m]$. In (8.7) $(m_0)_{\underline{\Omega}} = \tilde{b}_0$, $2{}^m\tilde{\mu}_{\underline{\Omega}} = \tilde{b}_1$, ${}^m\tilde{\lambda}_{\underline{\Omega}} = \tilde{b}_{01}$, ${}^m\tilde{\alpha}_{tm} = \tilde{b}_{31}$ and $({}^2\tilde{\alpha}_{tm})_{\underline{\Omega}} = \tilde{b}_{32}$. $[{}_s\sigma]$ in (8.7) can also be represented in matrix and vector notation (Voigt's notation). See reference [3] for details.

9. Constitutive theory for heat vector $\boldsymbol{q}$

The constitutive theory for $\boldsymbol{q}$ can be derived: (i) using $\boldsymbol{q} = \boldsymbol{q}([{}_s^d J], [{}_s^\Theta J], \{g\}, \theta)$ in (5.3) and by using the theory of generators and invariants (ii) or using $\boldsymbol{q} \cdot \boldsymbol{g} \le 0$ (inequality (5.9)) resulting from the entropy inequality.

9.1. Constitutive theory for $\boldsymbol{q}$ using the theory of generators and invariants

Consider $\boldsymbol{q}$ in (5.3)

$$\boldsymbol{q} = \boldsymbol{q}\left({}_s^d\boldsymbol{J}, {}_s^\Theta\boldsymbol{J}, \boldsymbol{g}, \theta\right) \tag{9.1}$$

Let $\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$; $i = 1,2,\dots,\tilde{N}$ be the combined generators of the argument tensors $[{}_s^dJ]$, $[{}_s^\Theta J]$, and $\{g\}$ that are tensors of rank one. Let ${}^q\underset{\sim}{I}^j$; $j = 1,2,\dots,\tilde{M}$ be the combined invariants of the same argument tensors. Then, we can express $\{q\}$ as a linear combination of $\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$; $i = 1,2,\dots,\tilde{N}$.

$$\{q\} = -\sum_{i=1}^{\tilde{N}} {}^q\underset{\sim}{\alpha}^i \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{9.2}$$

The absence of a unit vector in (9.2) is due to the fact that a uniform temperature field does not contribute to $\{q\}$. The negative sign in (9.2) is because a positive $\{q\}$ in the direction of the exterior unit normal to the surface of the volume of matter results in heat removal from the volume of matter. The coefficients ${}^q\underset{\sim}{\alpha}^i$; $i = 1,2,\dots,\tilde{N}$ are functions of ${}^q\underset{\sim}{I}^j$; $j = 1,2,\dots,\tilde{M}$ and θ in the current configuration. To determine the material coefficients from ${}^q\underset{\sim}{\alpha}^i$; $i = 1,2,\dots,\tilde{N}$ (in the current configuration), we consider Taylor series expansion of each ${}^q\underset{\sim}{\alpha}^i$; $i = 1,2,\dots,\tilde{N}$ about a known configuration $\underline{\Omega}$ in θ and ${}^q\underset{\sim}{I}^j$; $j = 1,2,\dots,\tilde{M}$ and retain only up to linear terms in θ and the invariants.

$${}^q\underset{\sim}{\alpha}^i = {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right);\ i = 1,2,\dots,\tilde{N} \tag{9.3}$$

${}^q\underset{\sim}{\alpha}^i\big|_{\underline{\Omega}}$, $\left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\right|_{\underline{\Omega}}$; $j = 1,2,\dots,\tilde{M}$, and $\left.\frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\right|_{\underline{\Omega}}$ are functions of $\theta|_{\underline{\Omega}}$ and ${}^q\underset{\sim}{I}^j\big|_{\underline{\Omega}}$; $j = 1,2,\dots,\tilde{M}$ whereas ${}^q\underset{\sim}{\alpha}^i$ are functions of the same quantities in the current configuration. By substituting (9.3) in (9.2) we obtain the most general form of the constitutive theory for $\boldsymbol{q}$ that is based on integrity i.e. complete basis. Details are given in the following.

$$\{q\} = -\sum_{i=1}^{\tilde{N}} \left({}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} + \sum_{j=1}^{\tilde{M}} \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j - \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}}\right) + \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}} \left(\theta - \theta_{\underline{\Omega}}\right)\right)\left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} \tag{9.4}$$

Collecting coefficients (quantities defined in $\underline{\Omega}$) of the terms in (9.4) that are defined in the current configuration i.e. coefficients of $\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$, ${}^q\underset{\sim}{I}^j\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$ and $(\theta - \theta_{\underline{\Omega}})\{{}^q\underset{\sim}{\boldsymbol{G}}^i\}$ and defining the following

$$
\begin{aligned}
{}^q\underset{-}{b}_i &= {}^q\underset{\sim}{\alpha}^i\Big|_{\underline{\Omega}} - \sum_{j=1}^{\tilde{M}} \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \left({}^q\underset{\sim}{I}^j\right)_{\underline{\Omega}} \\
{}^q\underset{-}{c}_{ij} &= \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\left({}^q\underset{\sim}{I}^j\right)}\Bigg|_{\underline{\Omega}} \\
{}^q\underset{-}{d}_i &= \frac{\partial\, {}^q\underset{\sim}{\alpha}^i}{\partial\theta}\Bigg|_{\underline{\Omega}}
\end{aligned}
\tag{9.5}
$$

for $i = 1,2,\dots,\tilde{N}$ and $j = 1,2,\dots,\tilde{M}$ and using these in (9.4) we can write

$$
\{q\} = -\sum_{i=1}^{\tilde{N}} {}^q\underset{-}{b}_i \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}}\sum_{i=1}^{\tilde{M}} {}^q\underset{-}{c}_{ij}\, {}^q\underset{\sim}{I}^j \left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\} - \sum_{i=1}^{\tilde{N}} {}^q\underset{-}{d}_i(\theta - \theta_{\underline{\Omega}})\left\{{}^q\underset{\sim}{\boldsymbol{G}}^i\right\}
\tag{9.6}
$$

${}^q\underset{-}{b}_i$, ${}^q\underset{-}{c}_{ij}$, and ${}^q\underset{-}{d}_i$ are material coefficients defined in a known configuration $\underline{\Omega}$. The constitutive theory for $\boldsymbol{q}$ defined by(9.6) requires $(\tilde{N} + \tilde{N}\tilde{M} + \tilde{N})$ material coefficients. The material coefficients are functions of $\theta|_{\underline{\Omega}}$ and $({}^q\underset{\sim}{I}^j)_{\underline{\Omega}}$; $j = 1,2,\dots,\tilde{M}$. This constitutive theory for $\boldsymbol{q}$ is based on integrity, hence is complete.

9.2. Simplified constitutive theory for heat vector $\boldsymbol{q}$

Much simpler (but with limitations) constitutive theories for $\boldsymbol{q}$ can be derived if we limit its argument tensors. Consider

$$
\boldsymbol{q} = \boldsymbol{q}\,(\boldsymbol{g},\theta)
\tag{9.7}
$$

In this case, we have only one generator and one invariant (i.e. $\tilde{N} = 1$ and $\tilde{M} = 1$).

$$
\left\{{}^q\underset{\sim}{\boldsymbol{G}}^1\right\} = \{q\}; \qquad {}^q\underset{\sim}{I}^1 = \{g\}^T\{g\}
\tag{9.8}
$$

Following the general derivation in section 9.1 we can write the following for $\tilde{N} = 1, \tilde{M} = 1$

$$\{q\} = -{}^{q}\underset{\sim}{b}_1\{g\} - {}^{q}\underset{\sim}{c}_{11}\left(\{g\}^T\{g\}\right)\{g\} - {}^{q}\underset{\sim}{d}_1(\theta - \theta_{\underline{\Omega}})\{g\} \tag{9.9}$$

Material coefficients in (9.9) are defined by (9.5). This constitutive theory is cubic in $\{g\}$, requires only three material coefficients and is the most general constitutive theory based on (9.7). If we denote ${}^{q}\underset{\sim}{b}_1 = k_1|_{\underline{\Omega}}$ and ${}^{q}\underset{\sim}{c}_{11} = k_2|_{\underline{\Omega}}$, then (9.9) can be written as

$$\{q\} = -\,k_1|_{\underline{\Omega}}\{g\} - k_2|_{\underline{\Omega}}\left(\{g\}^T\{g\}\right)\{g\} - {}^{q}\underset{\sim}{d}_1(\theta - \theta_{\underline{\Omega}})\{g\} \tag{9.10}$$

If we neglect the $(\theta - \theta_{\underline{\Omega}})$ term in (9.10), then we obtain

$$\{q\} = -\,k_1|_{\underline{\Omega}}\{g\} - k_2|_{\underline{\Omega}}\left(\{g\}^T\{g\}\right)\{g\} \tag{9.11}$$

If we assume that $\{q\}$ is a linear function of $\{g\}$, then we have

$$\{q\} = -\,k_1|_{\underline{\Omega}}\{g\} \tag{9.12}$$

Equation (9.12) is the Fourier heat conduction law in which the thermal conductivity $k_1|_{\underline{\Omega}}$ can be a function of $\theta|_{\underline{\Omega}}$ and $\left(\{g\}^T\{g\}\right)_{\underline{\Omega}}$.

9.3. Constitutive theory for *q* based on conditions resulting from the entropy inequality

We recall that satisfying the entropy inequality requires that

$$\{q\}^T\{g\} \leq 0 \tag{9.13}$$

must hold. The derivation of the constitutive theory for $\{q\}$ based on (9.13) is standard and can be found in reference [3] and many others. The resulting constitutive theory for $\{q\}$ can be written as (9.12) except that in this case $k_1|_{\underline{\Omega}} = k_1(\theta_{\underline{\Omega}})$ i.e. the conductivity can only be a function of temperature $\theta|_{\underline{\Omega}}$ and not the temperature $\theta|_{\underline{\Omega}}$ and the invariant $\left(\{g\}^T\{g\}\right)_{\underline{\Omega}}$ as there is no basis for dependence of $k_i|_{\underline{\Omega}}$ on $\left(\{g\}^T\{g\}\right)_{\underline{\Omega}}$.

10. Model problems

In this section we consider simple model problems in which the non-internal polar physics is well understood so that the influence of internal polar physics on the deformation behavior can be clearly demonstrated. We consider conservation and balance laws in $\mathbb{R}^2$ for plane stress behavior. This mathematical model is used to study non-internal polar i.e. classical and internal polar physics and its influence on deformation of a clamped-clamped beam and a simply supported beam.

10.1. Mathematical model in $\mathbb{R}^3$

Following references [1, 2] the conservation and balance laws (conservation of mass, balance of linear momenta, balance of angular momenta, balance of moments of moments, first and second laws of thermodynamics) in $\mathbb{R}^3$ for internal polar thermoelastic solid continua with small deformation and small strain in Lagrangian description can be written (using modified Helmholtz free energy density $\underset{\sim}{\Phi}$ and modified specific internal energy $\underset{\sim}{e}$) as

$$
\begin{aligned}
\rho_0 &= |J|\rho(\boldsymbol{x},t) \\
\rho_0 \frac{D\boldsymbol{v}}{Dt} - \rho_0 \boldsymbol{F}^b - \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}) &= 0 \\
m_{mk,m} - \epsilon_{ijk}(\sigma_{ij}) &= 0 \\
\epsilon_{ijk} m_{ij} &= 0 \\
\rho_0 \frac{D\underset{\sim}{e}}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} &= 0 \\
\rho_0\left(\frac{D\underset{\sim}{\Phi}}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} &\leq 0 \\
\boldsymbol{v} &= \frac{D\boldsymbol{u}}{Dt}
\end{aligned}
\tag{10.1}
$$

If we consider the stress decomposition

$$
\boldsymbol{\sigma} = {}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma} \tag{10.2}
$$

in which ${}_s\boldsymbol{\sigma}$ and ${}_a\boldsymbol{\sigma}$ are symmetric and antisymmetric stress tensors, then using

$$\epsilon_{ijk}\sigma_{ij} = \epsilon_{ijk}\left({}_a\sigma_{ij}\right) \tag{10.3}$$

and (10.2) in (10.1), the conservation and balance laws can be written as (Substituting for $\boldsymbol{v} = \frac{D\boldsymbol{u}}{Dt}$)

$$\begin{aligned}
\rho_0 &= |J|\rho \\
\rho_0\frac{D^2\boldsymbol{u}}{Dt^2} - \rho_0\boldsymbol{F}^b - \boldsymbol{\nabla}\cdot({}_s\boldsymbol{\sigma} + {}_a\boldsymbol{\sigma}) &= 0 \\
m_{mk,m} - \epsilon_{ijk}({}_a\sigma_{ij}) &= 0 \\
\epsilon_{ijk}m_{ij} &= 0 \\
\rho_0\frac{D\underset{\sim}{e}}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} &= 0 \\
\rho_0\left(\frac{D\underset{\sim}{\Phi}}{Dt} + \eta\frac{D\theta}{Dt}\right) + \frac{q_i g_i}{\theta} &\leq 0
\end{aligned} \tag{10.4}$$

Using the constitutive theories derived in this paper, we consider the following for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$ and $\boldsymbol{q}$ (equations (8.4), (8.7), and (9.12) in the absence of $(\theta - \theta_{\underline{\Omega}})$ term)

$$[{}_s\sigma] = 2\mu[\varepsilon] + \lambda\operatorname{tr}[\varepsilon] \tag{10.5}$$

$$[m] = 2\mu_m[{}^{\Theta}_{s}J] + \lambda_m\operatorname{tr}[{}^{\Theta}_{s}J] \tag{10.6}$$

$$\{q\} = -k\{g\} \tag{10.7}$$

In which

$$[{}^{\Theta}_{s}J] = \frac{1}{2}\left([{}^{\Theta}J] + [{}^{\Theta}J]^T\right) \tag{10.8}$$

$$[{}^{\Theta}J] = \frac{\partial\{\Theta\}}{\partial\{x\}} \quad \text{or} \quad {}^{\Theta}J_{ij} = \frac{\partial\Theta_i}{\partial x_j} \tag{10.9}$$

$$\{\Theta\}^T = \left[\Theta_{x_1}, \Theta_{x_2}, \Theta_{x_3}\right] \tag{10.10}$$

$$\begin{aligned}
\Theta_{x_1} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\
\Theta_{x_2} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\
\Theta_{x_3} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)
\end{aligned} \tag{10.11}$$

$$[\varepsilon] = \frac{1}{2}\left([{}^dJ] + [{}^dJ]^T\right) \tag{10.12}$$

$$[{}^dJ] = \frac{\partial\{u\}}{\partial\{x\}} \quad \text{or} \quad {}^dJ_{ij} = \frac{\partial u_i}{\partial x_j} \tag{10.13}$$

$$g_i = \frac{\partial \theta}{\partial x_i} \tag{10.14}$$

μ, λ, μ_m, λ_m and k are material coefficients.

10.2. Mathematical model in $\mathbb{R}^2$

For the sake of convenience we choose x_1, x_2 as x, y; u_1, u_2 as u, v, and express the material derivative of $\boldsymbol{v}$ in the linear momentum equations in terms of displacements (i.e. use balance of linear momenta in (10.4)). For small deformation, $|J| \approx 1$, hence $\rho_0 = \rho$ i.e. the solid continua is not compressible. The mathematical model in section 10.1 in $\mathbb{R}^3$ can be reduced to (using $\frac{D}{Dt} = \frac{\partial}{\partial t}$ in Lagrangian description) $\mathbb{R}^2$, keeping in mind that ${}_a\sigma_{xy} = -{}_a\sigma_{yx}$ and for boundary value problems the inertial terms in the linear momentum equations are absent. We further assume the body forces to be absent.

Conservation and balance laws:

$$
\begin{aligned}
&\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial({}_s\sigma_{yx})}{\partial y} + \frac{\partial({}_a\sigma_{yx})}{\partial y} = 0 \\
&\frac{\partial({}_s\sigma_{xy})}{\partial y} + \frac{\partial({}_s\sigma_{yy})}{\partial y} - \frac{\partial({}_a\sigma_{yx})}{\partial x} = 0 \\
&\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) = 0 \\
&\rho_0 \frac{\partial \underset{\sim}{e}}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \\
&\rho_0 \left(\frac{\partial \underset{\sim}{\Phi}}{\partial t} + \eta \frac{\partial \theta}{\partial t} \right) + q_x \frac{\partial \theta}{\partial x} + q_y \frac{\partial \theta}{\partial y} \leq 0
\end{aligned}
\tag{10.15}
$$

Constitutive theories:

$$
\begin{aligned}
{}_s\sigma_{xx} &= D_{11} \frac{\partial u}{\partial x} + D_{12} \frac{\partial v}{\partial y} \\
{}_s\sigma_{yy} &= D_{21} \frac{\partial u}{\partial x} + D_{22} \frac{\partial v}{\partial y}; \quad D_{21} = D_{12} \\
{}_s\sigma_{xy} &= D_{33} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
m_{xz} &= E_m \frac{\partial}{\partial x} (\Theta_z) \\
m_{yz} &= E_m \frac{\partial}{\partial y} (\Theta_z) \\
\Theta_z &= \frac{1}{2} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)
\end{aligned}
\tag{10.16}
$$

For plane stress, the coefficients D_{ij} are given by

$$
\begin{aligned}
D_{11} = D_{22} &= \frac{E}{1-\nu^2} \\
D_{12} = D_{21} &= \frac{\nu E}{1-\nu^2} \\
D_{33} = G &= \frac{E}{2(1+\nu)}
\end{aligned}
\tag{10.17}
$$

in which E, ν are modulus of elasticity and Poisson's ratio and E_m is the modulus related to the internal polar physics.

For elastic solids with isothermal assumptions, the energy equation is eliminated. We can also eliminate the entropy inequality from the mathematical model, thus (10.15) reduce to

$$\begin{aligned}
&\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial\left({}_s\sigma_{yx}\right)}{\partial y} + \frac{\partial\left({}_a\sigma_{yx}\right)}{\partial y} = 0 \\
&\frac{\partial\left({}_s\sigma_{xy}\right)}{\partial y} + \frac{\partial\left({}_s\sigma_{yy}\right)}{\partial y} - \frac{\partial\left({}_a\sigma_{yx}\right)}{\partial x} = 0 \\
&\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) = 0
\end{aligned} \tag{10.18}$$

The final mathematical model for the plane stress case consists of (10.18), (10.16), and (10.17). These are nine first order partial differential equations in nine dependent variables: u, v ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, m_{xz}, m_{yz}, and Θ_z, hence the mathematical model has closure.

10.3. Dimensionless form of the mathematical model in $\mathbb{R}^2$ for plane stress

We nondimensionalize the mathematical model presented in $\mathbb{R}^2$ for the plane stress case ((10.18), (10.16), and (10.17)). We rewrite (10.18), (10.16) and (10.17) with a hat (ˆ) on all quantities indicating that the quantities have their usual dimensions in terms of length ($\hat{L}$), force ($\hat{F}$) and time ($\hat{t}$). If we choose L_0, F_0 and t_0 as reference values of length, force, and time then the dimensionless length, force, and time (L, F and t) are defined as

$$L = \frac{\hat{L}}{L_0}, \qquad F = \frac{\hat{F}}{F_0}, \qquad t = \frac{\hat{t}}{t_0}$$

If we choose L_0, $E_0 = \tau_0$, $m_0 = \frac{\tau_0}{L_0}$, hence $F_0 = \tau_0 L_0^2$ then the dimensionless form or the mathematical model (10.18), (10.16) and (10.17) becomes

$$
\begin{aligned}
\frac{\partial({}_s\sigma_{xx})}{\partial x} + \frac{\partial\left({}_s\sigma_{yx}\right)}{\partial y} + \frac{\partial\left({}_a\sigma_{yx}\right)}{\partial y} &= 0 \\
\frac{\partial\left({}_s\sigma_{xy}\right)}{\partial y} + \frac{\partial\left({}_s\sigma_{yy}\right)}{\partial y} - \frac{\partial\left({}_a\sigma_{yx}\right)}{\partial x} &= 0 \\
\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + 2\left({}_a\sigma_{yx}\right) &= 0 \\
{}_s\sigma_{xx} &= D_{11}\frac{\partial u}{\partial x} + D_{12}\frac{\partial v}{\partial y} \\
{}_s\sigma_{yy} &= D_{21}\frac{\partial u}{\partial x} + D_{22}\frac{\partial v}{\partial y} \\
{}_s\sigma_{xy} &= D_{33}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
m_{xz} &= \left(\frac{E_0}{m_0 L_0}\right) E_m \frac{\partial}{\partial x}(\Theta_z) \\
m_{yz} &= \left(\frac{E_0}{m_0 L_0}\right) E_m \frac{\partial}{\partial y}(\Theta_z) \\
D_{11} = D_{22} = \frac{E}{1-\nu^2} \quad ; \quad D_{12} = D_{21} &= \frac{\nu E}{1-\nu^2} \quad ; \quad D_{33} = G = \frac{E}{2(1+\nu)} \\
\Theta_z &= \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)
\end{aligned}
\tag{10.19}
$$

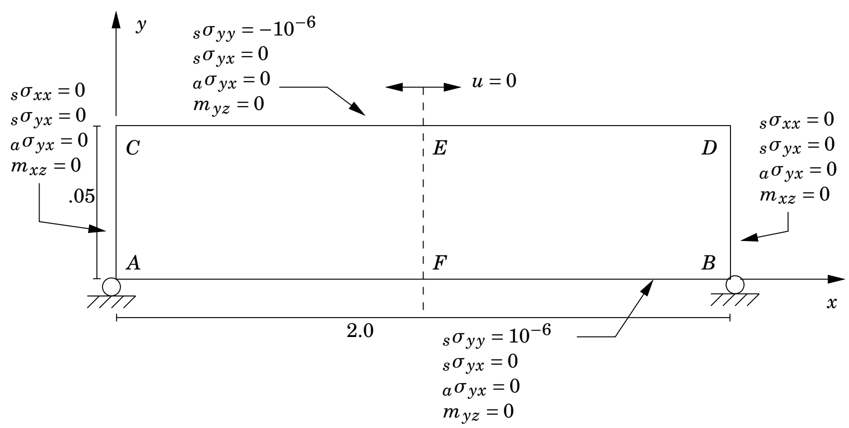
In (10.19) we have used $E_m = \frac{\hat{E}_m}{E_0}$ hence $\frac{E_0}{m_0 L_0}$ is in fact one, but it has been left in the constitutive theory for the moment tensor for the sake of clarity. Equations (10.19) are a system of nine first order linear coupled differential equations in nine dependent variables u, v ${}_s\sigma_{xx}$, ${}_s\sigma_{yy}$, ${}_s\sigma_{xy}$, ${}_a\sigma_{yx}$, m_{xz}, m_{yz}, and Θ_z.

10.4. Computational framework for solutions of the model problems

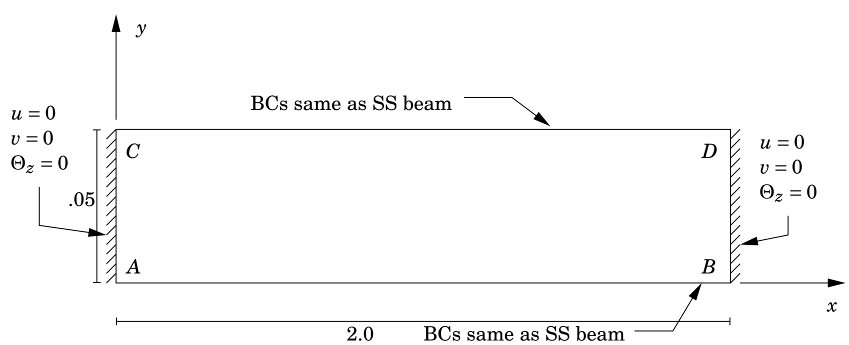
Even though theoretical or analytical solutions of (10.19) for some special simplified model boundary value problems may be possible, in the present work we consider numerical solutions of (10.19) for the two model problems considered here using finite element formulations based on the residual functional in which hierarchical local approximations are considered in higher order global differentiability scalar product spaces. Details are well documented in may references [23–30], hence are not repeated here.

10.5. Model problems

We consider a thin plate with length $\hat{l}$ of 20 inches, width $\hat{b}$ of 0.5 inches and thickness $\hat{t}$ of 0.1 inches. With $L_0 = 10$ inches the dimensionless plate is $2 \times 0.05 \times 0.01$. We consider loads applied in the plane of the plate. We choose $\hat{E} = E_0 = 30 \times 10^6$ psi, hence $E = 1$. Dimensionless $E_m = \frac{\hat{E}_m}{E_0}$ is increased starting with 0.0 and is chosen to be a fraction of the dimensionless modulus of elasticity (which is unity). Clearly for $E_m = 0$, the internal polar physics is absent i.e. the usual small strain approximation theory of elasticity applies for this case.



(a) Simply supported (SS) beam



(b) Clamped-clamped (CC) beam

Figure 1: Schematics and boundary conditions for simply supported and clamped-clamped beams (dimensionless)

Model problem 1

In this case we consider the plate to be simply supported as shown in figure 1 (a). Points A and B are constrained in the y direction, but are free to move in the x direction. On Face AB of the plate $\sigma_{yy} = 10^{-6}$ and on face CD, σ_{yy} of -10^{-6} is applied causing deflection of the plate in the negative y direction. At the center plane (EF) the x displacement is constrained (due to symmetry). Since $\hat{b}$ and $\hat{t}$ are much smaller than $\hat{l}$, the deformation behavior is like a simply supported 2D slender beam (shear deformation is not significant). The domain $(l \times b)$ 2×0.05 is modeled using a twenty element uniform discretization (ten elements along the length l and two elements along the width b) using nine node $p-$version hierarchical plane stress elements with higher order global differentiability local approximations in $H^{k,p}(\bar{\Omega}^e)$ scalar product spaces. Boundary conditions on the four boundaries of the domain $ABCD$ of the beam are also shown in figure $1(a)$. The nine node elements are mapped in a two unit square with the origin of the coordinate system ξ, η (natural coordinate system) at the center of the element. The element local approximation as well as all computations are performed using the natural coordinate system ξ, η. The degrees of local approximation in ξ and η (p_ξ, p_η) are chosen to be equal $p = p_\xi = p_\eta$ and are chosen to be the same for all dependent variables. Since the mathematical model is a system of first order partial differential equations, if the order of approximation space in x and y is chosen to be two i.e. local approximations of class C^1 in both x and y then the integrals over the discretization are Riemann. On the other hand, if we choose the order of the approximation space to be one, then the local approximations are of class C^0 implying that the integrals over the discretization are in Lebesgue sense. Due to the smoothness of the solution of the model problem, both choices work well, i.e. the C^0 solutions approach C^1 solutions upon convergence, but in the weak sense. In the results presented here we choose $k = 1$ i.e. local approximations of class C^0. A $p-$convergence study with $p = p_\xi = p_\eta = 3, 5, \dots$ shows that at $p = 9$ the integrated sum of squares of the residuals are of the order of $O(10^{-16})$, confirming that the equations in the mathematical model are satisfied accurately in the pointwise sense. This is confirmed by the similar studies with solutions of class C^1 and their comparison with C^0 studies. Thus, we present results for $p = p_\xi = p_\eta = 9$ with local approximations of class C^0 for all dependent variables using the 20 element uniform discretization described earlier.

Model problem 2

This model problem consists of the same plate as used in model problem 1 but is considered clamped at the two ends ($x = 0$ and $x = 2$ shown in figure 1 (b)). The boundary conditions on the boundaries AB and CD (excluding points A and B) remain the same as in model problem 1. Boundary conditions on AC and BD (clamped boundaries) are $u = v = \theta_z = 0$ as shown in figure 1 (b). The details of the discretization, choice of $p-$levels, choice of order of approximation space etc. are the same for this model problem as those described for model problem 1. In this case also, a $p-$convergence study for solutions of class C^0 yields integrated sum of squares of the residual of the order of $O(10^{-16})$ as in the case of model problem 1. Thus, for this model problem also, $p = p_\xi = p_\eta = 9$ and C^0 local approximations for all dependent variables yields very accurate solutions, hence are used to compute the results presented here.

Solution for model problems 1 and 2

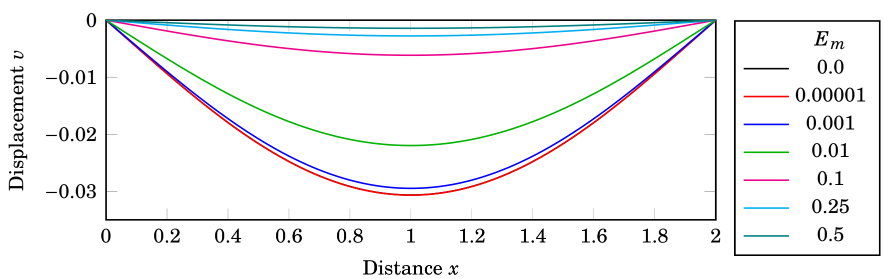


Figure 2: Displacement v versus distance x at $y = 0.025$ (simply supported beam)

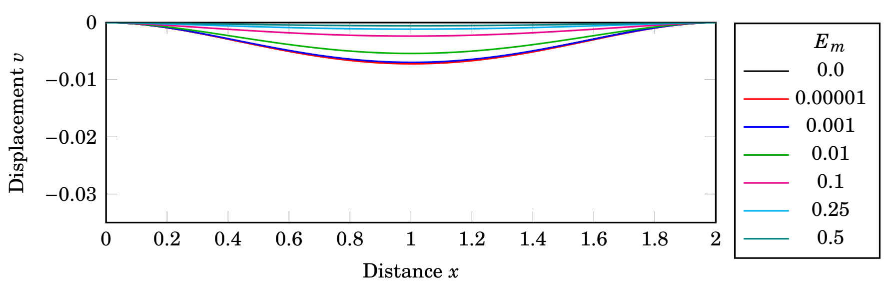


Figure 3: Displacement v versus distance x at $y = 0.025$ (clamped-clamped beam)

In the solutions presented here for model problems 1 and 2 the dimension-

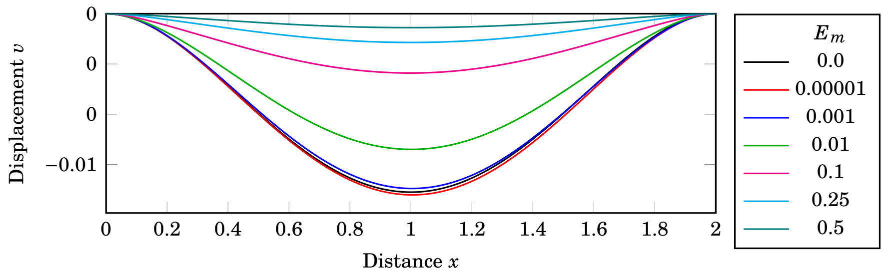


Figure 4: Displacement v versus distance x at $y = 0.025$ (clamped-clamped beam)

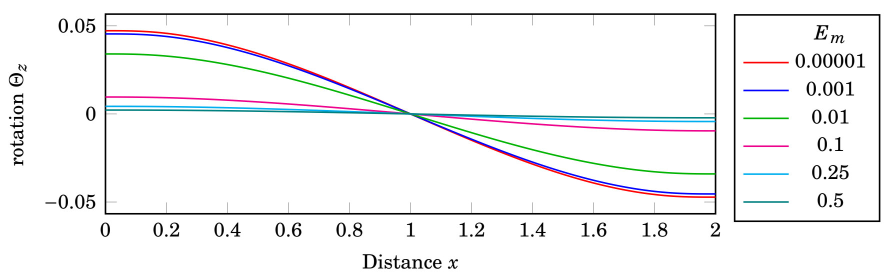


Figure 5: Rotation Θ_z versus distance x at $y = 0.025$ (simply supported beam)

less modulus of elasticity of 1 corresponds to $\hat{E} = 30 \times 10^6$ psi. When E_m, the dimensionless material coefficient related to the internal polar physics, is zero the internal polar physics is completely absent and we have standard equations in the mathematical model for plane stress behavior based on infinitesimal theory of elasticity. We choose Poisson's ratio $\nu = 0.3$. Values of E_m used here range from 0.00001–0.5. Progressively increasing values of E_m reflect progressively increasing influence of internal polar physics. Figure 2 shows plots of displacement v versus x at the centerline ($y = 0.025$) for the simply supported (SS) beam of (figure 1 (a)). For $E_m = 0$, the solution is in agreement with Timoshenko beam theory. $E_m = 0.00001$, representing extremely small influence of internal polar physics, hardly has any influence on the deflection (as expected). For $E_m = 0.001, 0.01, 0.1, 0.25$, and 0.5 we observe progressively reducing vertical displacement of v of the beam centerline due to progressively increased resistance to deformation due to progressively increased influence of internal polar physics. The displacements v of the bottom and the top faces of the beam ($y = 0.0$ and $y = 0.05$) are virtually the same as the displacement v of the center-

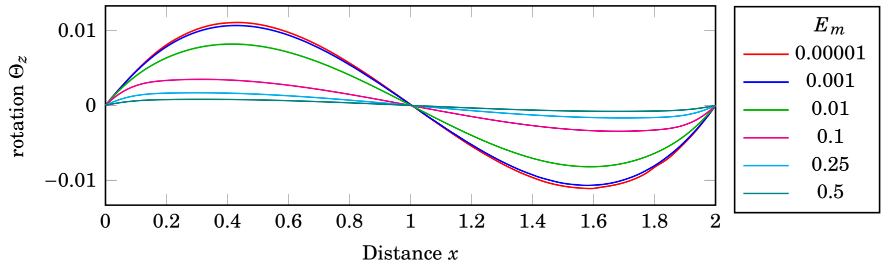


Figure 6: Rotation Θ_z versus distance x at $y = 0.025$ (clamped-clamped beam)

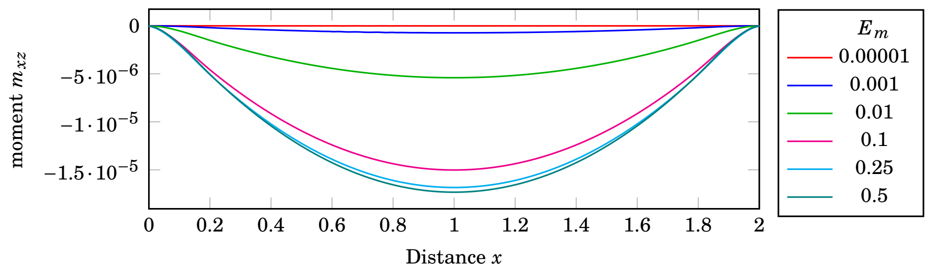


Figure 7: Moment m_{xz} versus distance x at $y = 0.025$ (simply supported beam)

line ($y = 0.025$) of the beam as expected for a slender beam like what is used here. Graphs of v versus x at the centerline of the clamped (CC) beam of figure 1 (b) for the same σ_{yy} and the same values of E_m as used for the SS beam are shown in figure 3. These are plotted using the same x, y scales as in figure 2. Substantially reduced displacement v values for all values of E_m compared to the SS beam are obvious. For $E_m = 0$, the deflection v is in agreement with Timoshenko beam theory. The purpose of the results in figure 3 is to compare directly with the results in figure 2 for the SS case so that the substantial reduction in v for the CC beam can be observed easily. Figure 4 shows the same results as in figure 3 but using an enlarged scale for the y axis for more clarity. Behavior is similar to the SS beam i.e. progressively increasing values of E_m result in progressively reduced displacement v due to progressively increasing resistance to deformation offered by the internal polar physics. In figures 2 and 3 the results are symmetric about $x = 1.0$ due to symmetry of geometry, loading, and boundary conditions.

Plots of rotation Θ_z versus x at $y = 0.025$ for SS and CC beams for $E_m = 0.00001, 0.001, 0.01, \dots, 0.5$ are shown in figures 5 and 6. We make some observa-

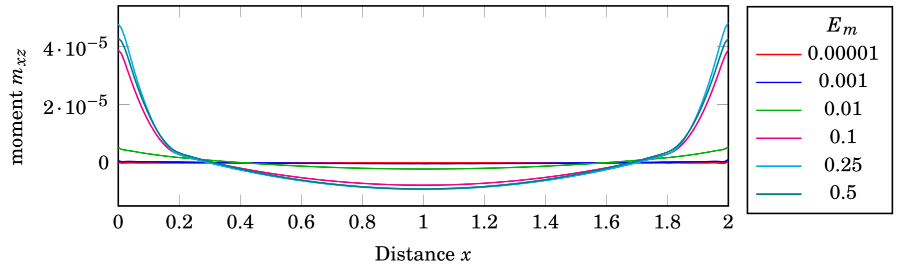


Figure 8: Moment m_{xz} versus distance x at $y = 0.025$ (clamped-clamped beam)

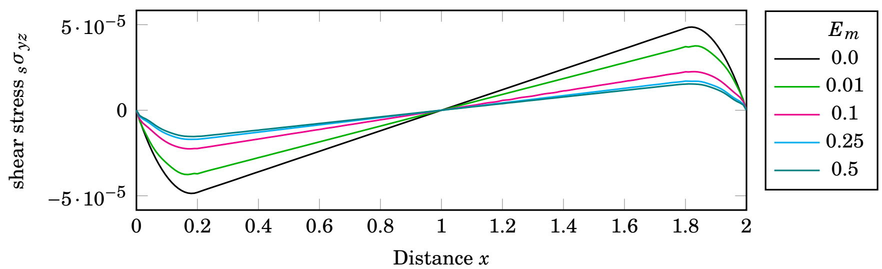


Figure 9: Shear stress ${}_s\sigma_{yx}$ versus distance x at $y = 0.025$ (simply supported beam)

tions and remarks.

(a) When $E_m = 0.00001$, the internal polar physics is virtually absent, hence Θ_z and its gradient in the $x-$direction have the largest magnitude for the SS beam as well as the CC beam compared to the higher values of E_m as for $E_m = 0.00001$ the internal polar resistance to deformation is minimal.

(b) As E_m increases Θ_z reduces due to progressively increasing resistance offered by the progressively increasing influence of internal polar physics.

(c) Even though Θ_z and its gradients are the highest for $E_m = 0.00001$, the internal resistance due to internal polar physics is smallest for this value of E_m compared to all other higher values used here.

(d) Antisymmetry of Θ_z about $x = 1.0$ (as expected) is quite obvious from the graphs.

Figures 7 and 8 show plots of moment m_{xz} versus x at $y = 0.025$ for SS and

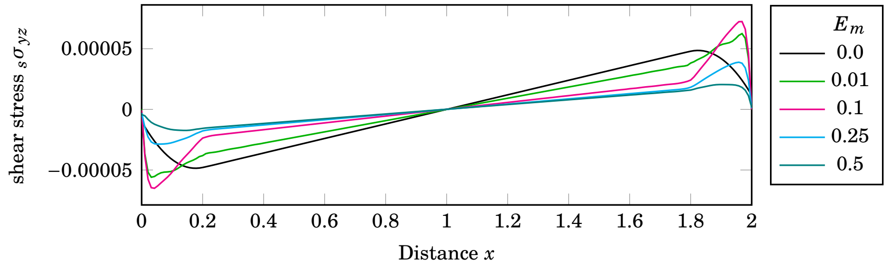


Figure 10: Shear stress $_s\sigma_{yx}$ versus distance x at $y = 0.025$ (clamped-clamped beam)

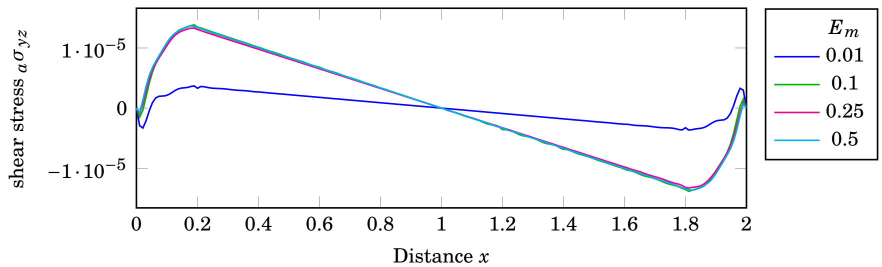


Figure 11: Shear stress $_a\sigma_{yx}$ versus distance x at $y = 0.025$ (simply supported beam)

CC beams for the same values of E_m used in figures 5 and 6. We observe that

(i) For $E_m = 0.00001$, m_{xz} has the lowest value for both SS and CC beams even though Θ_z and $\Theta_{z,x}$ have the largest values (figures 5 and 6). This is of course due to the fact that such a low value of E_m implies virtually no internal polar physics, hence virtually no resistance to rotations, thus resulting in extremely small values of moment m_{xz}.

(ii) As E_m increases, Θ_z reduces but m_{xz} increases due to increased contribution of internal polar physics, hence progressively increasing resistance to rotations.

(iii) We note that for $E_m = 0.5$, Θ_z and $\Theta_{z,x}$ are the lowest (figures 5 and 6) but the corresponding m_{xz} (figures 7 and 8) have the highest values due to increased resistance to deformation offered by the pronounced influence of internal polar physics.

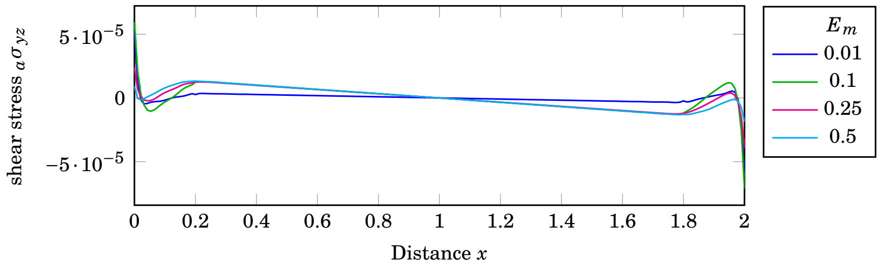


Figure 12: Shear stress ${}_a\sigma_{yx}$ versus distance x at $y = 0.025$ (clamped-clamped beam)

(iv) Symmetry of m_{xz} about $x = 1.0$ is clearly observed.

(v) In the absence of internal polar physics m_xz would be zero as evidenced by m_{xz} values for $E_m = 0.00001$ for which Θ_z and its gradient in the x direction are the largest.

(vi) The existence of the extent of internal polar physics is dependent on the constitution of the matter. In the simplified constitutive theory used in the model problems the material coefficient E_m is the measure of the extent of internal polar physics.

Graphs of ${}_s\sigma_{yx}$ in figures 9 and 10 for SS and CC and those of ${}_a\sigma_{yx}$ in figures 11 and 12 for SS and CC beams at the centerline are shown for progressively increasing values of E_m: $E_m = 0.0, 0.01, 0.1, 0.25$, and 0.5 for figures 9 and 10 and $E_m = 0.01, 0.1, 0.25$, and 0.5 for figures 11 and 12. Both ${}_s\sigma_{yx}$ and ${}_a\sigma_{yx}$ are antisymmetric about $x = 1.0$ as expected. Since internal polar physics influences displacements and their gradients, ${}_s\sigma_{yx}$ purely due to non-polar physics when $E_m = 0$ is influenced by the presence of internal polar physics as evident in figures 9 and 10. Of course, in the absence of internal polar physics, ${}_a\sigma_{yx}$ and m_{xz} would be zero. With progressively increasing influence of internal polar physics (progressively increasing values of E_m) ${}_a\sigma_{yx}$ values along the length of the beam increase in magnitude as expected (just like the moment m_{xz}).

11. Summary and conclusions

In this paper constitutive theories for homogeneous, isotropic internal polar thermoelastic solid continua [1, 2] are derived. In these solid continua, internally varying rotations and the conjugate moments are considered in addition to the usual physics (non-polar elasticity) in the derivations of the conservation and balance laws. For internal polar thermoelastic solid continua the deformation is reversible, hence the rate of work does not result in entropy production. As a result, the Clausius-Duhem inequality (strictly containing rates of entropy) does not contain any physics related to reversibility i.e. strain energy density is absent from the Clausius-Duhem inequality. Thus, the Clausius-Duhem inequality contains no mechanism for deriving constitutive theories for stress (or strain tensor) and the moment tensor as these require the presence of strain energy density in the entropy inequality. The entropy inequality expressed in Helmholtz free energy density Φ is derived by introducing strain energy density in the entropy inequality to offset its presence in Φ. The constitutive theories for the stress tensor and moment tensor can now be derived using the entropy inequality in which strain energy density has been introduced even though it is reversible. In this process in appearance it seems that the constitutive theories are supported by the entropy inequality but in reality their derivations are in fact due to the strain energy density that is forced in the entropy inequality through the introduction of Φ. In this paper constitutive theories are derived using the entropy inequality in Helmholtz free energy density Φ. It is shown that the entropy inequality expressed in modified Helmholtz free energy density $\underset{\sim}{\Phi}$ (free of strain energy density) contains no mechanism for deriving the constitutive theories for stress and the moment tensors. This is quite significant as now the constitutive theories for the stress and moment tensors are free of any thermodynamic restrictions. In this paper the constitutive theories for stress, moment, and strain tensors are also derived using: (i) strain energy density function (ii) conjugate strain energy density function (iii) theory of generators and invariants as well as Taylor series expansions of the strain energy density function and the conjugate strain energy density function. In each case the material coefficients are derived. The most simplified forms of the constitutive theories are also presented. The constitutive theories for heat vector are derived using the conditions resulting from the entropy inequality as well as using the theory of generators and invariants for different possible choices of argument tensors. For this case also the material coefficients are established and the simplified forms of the constitutive theories are presented.

The dependent variables Φ, η ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$, and $\boldsymbol{q}$ in the constitutive theories are established by examining the conservation and balance laws. Based on conjugate pairs, $\boldsymbol{q}$ and the principle of equipresence [3–6] we must choose $[{}_s^d J]$ or $[\varepsilon]$, $[{}_s^\Theta J]$, $\{g\}$ and θ as argument tensors of all dependent variables. As ${}_s\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$; $\boldsymbol{m}$, ${}_s^\Theta\boldsymbol{J}$ are conjugate, $[\varepsilon]$ and $[{}_s^\Theta J]$ are eliminated from the arguments of $\boldsymbol{m}$ and ${}_s\boldsymbol{\sigma}$, however these remain the argument tensors of $\boldsymbol{q}$ as there is no additional information that suggests otherwise. The entropy inequality in $\underset{\sim}{\Phi}$ neither has any mechanism for deriving the constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$ nor does it impose any restrictions on the constitutive theories for ${}_s\boldsymbol{\sigma}$ and $\boldsymbol{m}$. Thus, all approaches of deriving constitutive theories for ${}_s\boldsymbol{\sigma}$, $\boldsymbol{m}$ as well as $\boldsymbol{\varepsilon}$ are valid from the point of view of the thermodynamic framework. We remark that the constitutive theories derived using Taylor series expansion violate the frame invariance aspect of the material coefficient, a requirement that must be satisfied by the definitions of material coefficients. However, if the material coefficients are constant (not dependent on deformation), then these derivations are not in violation of the frame invariance principle.

Two model problems consisting of a slender plate (bending of a slender beam) subjected to uniformly distributed transverse loads with both simply supported and clamped boundary conditions are solved to illustrate the influence of internal polar physics on the transverse deflection of the beam. Due to the complexity of the mathematical model (explicit equations presented in the paper) numerical solutions are calculated using nine-node plane stress p-version hierarchical finite elements formulated based on the residual functional (least-squares finite element process). The integrated sum of squares of the residuals are $O(10^{-16})$ or lower for the whole discretization for all numerical studies, which ensures that the numerical solutions presented in the paper are accurate and satisfy the differential equations in the mathematical model in the pointwise sense. From the results presented in the paper for the two model problems we observe decreasing transverse deflection with progressively increasing internal polar physics due to progressively increasing resistance to deformation. The rotation and the gradients of rotation are largest in the absence of internal polar physics. Progressively increasing internal polar physics reduces rotations and their gradients, but with progressively increasing magnitudes of moment and antisymmetric shear stress as expected due to internal resistance offered by the presence of internal polar physics.

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