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EXPONENTIAL ATTRACTOR FOR THE THREE-DIMENSIONAL NAVIER-STOKES EQUATION WITH NONLINEAR DAMPING

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Abstract

In this paper, we consider the asymptotic behaviour of solutions for three-dimensional incompressible Navier-Stokes equations with nonlinear damping. We show the squeezing property and the existence of exponential attractor for this equation.

1. Introduction

In this paper, we investigate the following Navier-Stokes equation with nonlinear damping:

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$$\begin{cases} u_t - \mu\Delta u + (u \cdot \nabla)u + \alpha|u|^{\beta-1}u + \nabla p = f, & (x, t) \in \Omega \times (0, T), \\ \operatorname{div} u = 0, & (x, t) \in \Omega \times [0, T), \\ u|_{t=0} = u_0, & x \in \Omega, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times [0, T), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^3$ be an open bounded domain with the boundary $\partial\Omega$ smooth enough. $\mu > 0$ is the kinematic viscosity of the fluid and $f = f(x)$ is the external body force. The unknown functions here are $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ and $p = p(x, t)$, which stand for the velocity field and the pressure of the flow, respectively. In dampness term, $\beta \geq 1$ and $\alpha > 0$ are two constants. The given function $u_0 = u_0(x)$ is the initial velocity.

We will use the following notations in this paper:

$$\mathcal{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\}, \quad H = cl_{(L^2(\Omega))^3}\mathcal{V}, \quad V = cl_{(H_0^1(\Omega))^3}\mathcal{V},$$

where cl_X denotes the closure in the space X. It is well known that H, V are separable Hilbert spaces and identifying H and its dual H', we have $V \hookrightarrow H \hookrightarrow V'$ with dense and continuous injections, and $V \hookrightarrow H$ is compact. H and V endowed, respectively, with the inner products

$$(u, v) = \int_\Omega u \cdot v\, dx, \quad \forall u, v \in H,$$

and

$$((u, v)) = \sum_{i=1}^{3} \int_\Omega \nabla u_i \cdot \nabla v_i dx, \quad \forall u, v \in V,$$

and norms $|\cdot|_2 = (\cdot, \cdot)^{1/2}$, $\|\cdot\| = ((\cdot, \cdot))^{1/2}$.

In this paper, $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, and we use $|\cdot|_p$ denotes the norm in $\mathbf{L}^p(\Omega)$. The letter c or c_i is a generic positive constant, which may change its value from line to line.

We say that $u \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^{\beta+1}(0, T; \mathbf{L}^{\beta+1}(\Omega))$ is a weak solution of (1) on $[0, T]$, if

$$\begin{cases} \dfrac{d}{dt}(u, v) + \mu((u, v)) + b(u, u, v) + (\alpha|u|^{\beta-1}u, v) = (f, v), \quad \forall v \in V, \quad \forall t > 0, \\ u(0) = u_0. \end{cases} \tag{2}$$

The weak formulation (2) is equivalent to the function equation

$$\begin{cases} \dfrac{du}{dt} + \mu Au + B(u) + G(u) = f, \quad \forall t > 0, \\ u(0) = u_0, \end{cases} \tag{3}$$

where $\widetilde{P}$ be the orthogonal projection of $(L^2(\Omega))^3$ onto H and $f \in H$. $Au = -\widetilde{P}\Delta u$ is the Stokes operator defined by $\langle Au, v\rangle = ((u, v))$, and $F(u) = \alpha|u|^{\beta-1}u$, $G(u) = \widetilde{P}F(u)$. $B : V \times V \to V'$ is a bilinear operator defined by $\langle B(u, v), w\rangle = b(u, v, w)$. $B(u) = B(u, u)$, where

$$b(u, v, w) = \sum_{i, j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

and $\langle \cdot, \cdot \rangle$ is the duality product between V and V'.

We call the function pair $(u(x, t), p(x, t))$ the strong solution of problem (1) if it is a weak solution of (1) satisfying

$$u \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; \mathbf{L}^{\beta+1}(\Omega)).$$

Let us recall some relevant researches in this area.

In [1], Cai and Jiu investigated the existence and uniqueness of strong solution of system (1) when $\frac{7}{2} \le \beta \le 5$, $u_0 \in V$. In [2], we showed the strong solutions has global attractors in V and $\mathbf{H}^2(\Omega)$ when initial data $u_0 \in V$. And in [3], we proved the strong solutions has uniform attractors in V and $H^2(\Omega)$ when initial data $u_\tau \in V$. As far as we know, there are no other results about attractors of system (1).

In this paper, we will discuss the existence of finite dimensional exponential attractor for system (1). In Section 2, we state some basic results on the existence of exponential attractors and recall some known results on the existence and uniqueness of solutions. In Section 3, we establish the Lipschitz continuity of the dynamical system $S(t)$ associated with system (1), and prove the semigroup $S(t)$ satisfies the squeezing property and obtain the existence of the exponential attractor.

2. Preliminaries

In this section, we state some basic results on the existence of exponential attractors and recall some known results of strong solutions.

Definition 2.1 ([4]). Assume $\{S(t)\}_{t\ge 0}$ be semigroup defined on E. A compact set $M \subset E$ is called an exponential attractor of $(\{S(t)\}_{t\ge 0}, E)$ if

(1) $S(t)M \subseteq M, \quad \forall t \ge 0,$

(2) M has finite fractal dimension, $d_F(M) < +\infty,$

(3) for any bounded set B in E, there exist positive constants c_0, c_1 such that

$$\mathrm{dist}_V(S(t)B, M) \le c_0 e^{-c_1 t}, \quad \forall t > 0,$$

where $\mathrm{dist}_V(A, B) = \sup_{x\in A} \inf_{y\in B} \|x - y\|_V$, B is a positively invariant set for $S(t)$ in V.

Definition 2.2 ([4]). If for every $\delta \in (0, \frac{1}{8})$, there exists a time $t_* > 0$, an integer $N_0 \geq 1$, and an orthogonal projection P_{N_0} of rank equal to N_0 such that for every u and v in B, either

$$\|S(t_*)u - S(t_*)v\| \leq \delta\|u - v\|,$$

or

$$\|Q_{N_0}(S(t_*)u - S(t_*)v)\| \leq \|P_{N_0}(S(t_*)u - S(t_*)v)\|,$$

then we call $S(t)$ is squeezing in B, where $Q_{N_0} = I - P_{N_0}$.

Theorem 2.1 ([5]). *Suppose* (3) *satisfies the following conditions*:

(1) *there exists a compact global attractor* $\mathcal{A}$ *in* V,

(2) *there exists a compact set* B *in* V *which is positively invariant for* $S(t)$,

(3) $S(t)$ *is squeezing and Lipschitz continuous, that is there exists a bounded function* $l(t)$ *such that* $\|S(t)u - S(t)v\| \leq l(t)\|u - v\|$ *for every* $u, v \in B$.

Then (3) *admits a exponential attractor* M *for* $(S(t), B)$ *in* V *and*

$$M = \bigcup_{0 \leq t \leq t_*} S(t)M_*,$$

where

$$M_* = \mathcal{A}\bigcup(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(t_*)^j(E^{(k)})).$$

Moreover,

$$d_F(M) \leq 1 + N_0 \log(1 + \sqrt{2}l / \delta)\log\frac{1}{\theta},$$

$$\mathrm{dist}_V(S(t)B, M) \leq c_0 e^{-c_1 t},$$

where θ, N_0, $E^{(k)}$ *are defined as in* [4], l *is the Lipschitz constant for* $S(t_*)$ *in* B, *and* t_* *is a positive constant.*

In [1], the authors have established the following result:

Theorem 2.2. *Suppose that* $\beta \geq \frac{7}{2}$, $u_0 \in V \cap \mathbf{L}^{\beta+1}(\Omega)$ *and* $f \in H$. *Then there exists a strong solution* $(u(x, t), p(x, t))$ *to problem* (1) *satisfying*

$$u \in L^{\infty}(0, T; V) \cap L^{\infty}(0, T; \mathbf{L}^{\beta+1}(\Omega)) \cap L^2(0, T; \mathbf{H}^2(\Omega)),$$

$$\nabla u|u|^{\frac{\beta-1}{2}} \in L^2(0, T; H), u_t \in L^2(0, T; H).$$

Moreover when $\frac{7}{2} \leq \beta \leq 5$, *the strong solution is unique.*

Because $\Omega \subset \mathbb{R}^3$ is sufficiently regular, so $V \hookrightarrow \mathbf{L}^{\beta+1}(\Omega)(\frac{7}{2} \leq \beta \leq 5)$, so $V \cap \mathbf{L}^{\beta+1}(\Omega) = V$. In the following, we use $u_0 \in V$ to replace $u_0 \in V \cap \mathbf{L}^{\beta+1}(\Omega)$.

In [2], we have established the following results:

Proposition 2.1. *Suppose that* $\frac{7}{2} \leq \beta \leq 5$, $u_0 \in V$ *and* $f \in H$. *Then there exists closed absorbing sets* B_1, B_2 *in* V, $D(A)$, *respectively*, *and*

$$B_1 = \{u \in V : \|u\| \leq \rho_1\},$$

$$B_2 = \{u \in D(A) : |Au|_2 \leq \rho_2\}.$$

Letting $B = B_2$, then $B = B_2 \subset B_1 \subset V$, and $S(t)B \subset B$.

Theorem 2.3. *Suppose that* $\frac{7}{2} \leq \beta \leq 5$, $u_0 \in V$ *and* $f \in H$. *Then problem* (1) *has a global attractor in V, which is invariant and compact and attracts every bounded subset of V with respect to the norm topology of V.*

Proposition 2.2 ([6])**.**

$$|b(u, v, w)| \le \|u\|_\infty \|v\| \, |w|_2, \; u \in L^\infty, \; v \in V, \; w \in H;$$

$$|b(u, v, w)| \le k\|u\| \, \|v\|^{1/2} |Av|_2^{1/2} |w|_2, \; u \in V, \; v \in D(A), \; w \in H.$$

Now, let us recall the known Gagliardo-Nirenberg inequality as follows.

Lemma 2.1. *Let* $\Omega = \mathbb{R}^n$ *or* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with smooth boundary* $\partial\Omega$, *and* $u \in \mathbf{L}^q(\Omega)$, $D^m u \in \mathbf{L}^r(\Omega)$, $1 \le q, r \le \infty$. *Then there exists a constant c, such that*

$$|D^j u|_p \le c |D^m u|_r^a |u|_q^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1 - a)\frac{1}{q}, \; 1 \le p \le \infty, \; 0 \le j \le m, \; \frac{j}{m} \le a \le 1;$$

c depends only on (n, m, j, a, q, r).

Since the Stokes operator A is positive, self-adjoint operator with a compact inverse, there exists a set of eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \; \lambda_N \to \infty \text{ as } N \to +\infty,$$

and corresponding eigenvectors $w_1, w_2, w_3, \ldots,$ in $D(A)$, which form a basis of V and orthonormal in H, such that

$$Aw_n = \lambda_n w_n, \quad \forall n \in \mathbb{N}.$$

Let $V_N = span\{w_1, \ldots, w_N\}$, then V_N is a finite dimensional subspace of V. Denote by P_N the orthogonal projector from V into V_N and we obviously have $\|P_N\| \le 1$ for each $N \in \mathbb{N}$, and we write $Q_N = I - P_N$.

In order to establish the existence of the exponential attractor, according to Theorem 2.1, we need only to show the Lipschitz continuity and the squeezing property of the dynamical system $S(t)$ in B. That is what we proceed to do in the following section.

3. The Exponential Attractor

In this section, we first establish the Lipschitz continuity of the dynamical system $S(t)$ associated with the Equation (1) in B. Then we show the squeezing property for semigroup $S(t)$ and deduce the existence of finite fractal dimensional exponential attractor.

Lemma 3.1. *Suppose* $\frac{7}{2} \le \beta \le 5$, $f \in H$, $u_1(t)$, $u_2(t)$ *are two solutions of the problem* (1) *with initial values* u_{01}, $u_{02} \in B$. *Then we have*

$$\|u_1(t) - u_2(t)\| \le e^{c_1 t}\|u_{01} - u_{02}\|,$$

where c_1 *is a constant depending only on* β.

Proof. Let $w(t) = u_1(t) - u_2(t)$. Then by (3) we find that

$$\frac{dw}{dt} + \mu Aw + B(u_1) - B(u_2) + G(u_1) - G(u_2) = 0, \tag{4}$$

$$w(0) = u_{01} - u_{02}. \tag{5}$$

Taking the inner product of (4) with Aw in H, we infer that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2 + \mu|Aw|_2^2 &= -b(u_1, u_1, Aw) + b(u_2, u_2, Aw) - (F(u_1) - F(u_2), Aw) \\
&= -b(u_1, w, Aw) - b(w, u_2, Aw) - (F(u_1) - F(u_2), Aw) \\
&\le c\|u_1\|\|w\|^{1/2}|Aw|_2^{3/2} + c\|w\|\|u_2\|^{1/2}|Au_2|_2^{1/2}|Aw|_2 \\
&\quad + |F(u_1) - F(u_2)|_2|Aw|_2 \\
&\le \frac{3}{4}\mu|Aw|_2^2 + c(\|u_1\|^4 + \|u_2\|\|Au_2|_2)\|w\|^2 + c|F(u_1) - F(u_2)|_2^2,
\end{aligned} \tag{6}$$

and because

$$|F(u_1) - F(u_2)|_2^2 = \int_\Omega \left|\alpha|u_1|^{\beta-1}u_1 - \alpha|u_2|^{\beta-1}u_2\right|^2 dx$$

$$\le c\int_\Omega \left[|u_1|^{\beta-1}|w| + \left||u_1|^{\beta-1} - |u_2|^{\beta-1}\right| \cdot |u_2|\right]^2 dx$$

$$\le c\int_\Omega |u_1|^{2(\beta-1)}|w|^2 dx + c\int_\Omega \left[|u_1|^{\beta-2} + |u_2|^{\beta-2}\right]^2 |u_2|^2|w|^2 dx,$$

in the last inequality, we used the fact that

$$|x^p - y^p| \le cp(|x|^{p-1} + |y|^{p-1})|x - y|,$$

for any $x,\ y \ge 0,$ where c is an absolute constant. So

$$|F(u_1) - F(u_2)|_2^2 \le c|u_1|_{3(\beta-1)}^{2(\beta-1)}|w|_6^2 + c\left||u_1|^{\beta-2} + |u_2|^{\beta-2}\right|_6^2 |u_2|_6^2|w|_6^2$$

$$\le c\|w\|^2\left[|u_1|_{3(\beta-1)}^{2(\beta-1)} + \left||u_1| + |u_2|\right|_{6(\beta-2)}^{2(\beta-2)}\|u_2\|^2\right]$$

$$\le c\|w\|^2\left[|u_1|_{3(\beta-1)}^{2(\beta-1)} + (|u_1|_{6(\beta-2)}^{2(\beta-2)} + |u_2|_{6(\beta-2)}^{2(\beta-2)})\|u_2\|^2\right]. \quad (7)$$

Substituting (7) into (6), and applying Gronwall inequality, we have

$$\|w(t)\|^2 \le \|w(0)\|^2 \exp\left\{c\int_0^t \left[\|u_1\|^4 + \|u_2\||Au_2|_2 + |u_1|_{3(\beta-1)}^{2(\beta-1)}\right.\right.$$

$$\left.\left. + \|u_2\|^2(|u_1|_{6(\beta-2)}^{2(\beta-2)} + |u_2|_{6(\beta-2)}^{2(\beta-2)})\right]ds\right\}. \quad (8)$$

Applying Gagliardo-Nirenberg inequality, we obtain

$$|u|_{3(\beta-1)}^{2(\beta-1)} \le c|\Delta u|_2^{\frac{8(\beta-2)}{\beta+7}}|u|_{\beta+1}^{\frac{2(\beta+1)^2}{\beta+7}},\ |u|_{6(\beta-2)}^{2(\beta-2)} \le c|\Delta u|_2^{\frac{2(5\beta-13)}{\beta+7}}|u|_{\beta+1}^{\frac{2(\beta^2-1)}{\beta+7}}. \quad (9)$$

Because $V \hookrightarrow L^{\beta+1}(\Omega),$ so there exists a positive constant $\rho_3,$ such that $|u|_{\beta+1} \le \rho_3$ when $u \in B.$

Because u_1, $u_2 \in B$, and thanks to (9), there must exists a positive constant c_1, such that

$$\|w(t)\|^2 \le \|w(0)\|^2 e^{2c_1 t} = e^{2c_1 t}\|u_{01} - u_{02}\|^2.$$

□

Lemma 3.2. *Assume that $u_1(t)$ and $u_2(t)$ are two solutions of problem* (1) *with initial values u_{01}, $u_{02} \in B$. Then $q = Q_N(u_1 - u_2)$ satisfies that*

$$\|q(t)\|^2 \le (e^{-\mu\lambda_{N+1}t} + \frac{c_2}{2c_1 + \mu\lambda_{N+1}} e^{2c_1 t})\|w(0)\|^2.$$

Proof. Let $w(t) = u_1(t) - u_2(t)$, $q(t) = Q_N w(t)$. Applying Q_N to

$$\frac{dw}{dt} + \mu Aw = -B(u_1) + B(u_2) - \widetilde{P}F(u_1) + \widetilde{P}F(u_2),$$

we obtain

$$\frac{\partial q(t)}{\partial t} + \mu Aq = Q_N(B(u_2) - B(u_1)) + Q_N(\widetilde{P}F(u_2) - \widetilde{P}F(u_1)). \tag{10}$$

Taking the inner product of (10) with Aq in H, we find that

$$\frac{1}{2}\frac{d}{dt}\|q\|^2 + \mu|Aq|_2^2 = \int_\Omega Q_N(B(u_2) - B(u_1))Aqdx + \int_\Omega Q_N(\widetilde{P}F(u_2) - \widetilde{P}F(u_1))Aqdx. \tag{11}$$

Because

$$\left|\int_\Omega Q_N(\widetilde{P}F(u_2) - \widetilde{P}F(u_1))Aqdx\right| \le |F(u_1) - F(u_2)|_2 \cdot |Aq|_2$$

$$\le \frac{1}{\mu}|F(u_1) - F(u_2)|_2^2 + \frac{\mu}{4}|Aq|_2^2, \tag{12}$$

and

$$\left|\int_\Omega Q_N(B(u_2) - B(u_1))Aqdx\right| \le |B(u_2) - B(u_1)|_2 \cdot |Aq|_2$$

$$\le \frac{1}{\mu}|B(u_2) - B(u_1)|_2^2 + \frac{\mu}{4}|Aq|_2^2, \tag{13}$$

$$|B(u_2)-B(u_1)|_2^2 = \int_\Omega \big[B(u_2,\, u_2-u_1)+B(u_2-u_1,\, u_1)\big]^2 dx$$

$$\le 2\Big(|B(u_2,\, u_2-u_1)|_2^2 + |B(u_2-u_1,\, u_1)|_2^2\Big) \tag{14}$$

$$\le 2\|u_2\|_\infty^2\|w\|^2 + c\|u_1\|\,|Au_1|_2\|w\|^2,$$

and since $\lambda_{N+1}\|q\|^2 \le |Aq|_2^2$, combining (12)-(14) with (11), we have

$$\frac{d}{dt}\|q\|^2 + \mu\lambda_{N+1}\|q\|^2 \le c\Big\{|u_1|_{3(\beta-1)}^{2(\beta-1)} + \big[|u_1|_{6(\beta-2)}^{2(\beta-2)} + |u_2|_{6(\beta-2)}^{2(\beta-2)}\big]\|u_2\|^2$$

$$+ \|u_2\|_\infty^2 + \|u_1\|\,|Au_1|_2\Big\}\|w\|^2. \tag{15}$$

Because $u_2 \in B$, so $|Au_2|_2 \le \rho_2$. Applying Agmon's inequality, there exists a positive constant ρ_4 such that

$$\|u_2\|_\infty \le \rho_4.$$

So when $u_{01},\, u_{02} \in B$,

$$\frac{d}{dt}\|q\|^2 + \mu\lambda_{N+1}\|q\|^2 \le c_2\|w\|^2 \le c_2 e^{2c_1 t}\|w(0)\|^2, \tag{16}$$

for some positive constant c_2.

Then it follows from Gronwall lemma that

$$\|q\|^2 \le \|q(0)\|^2 e^{-\mu\lambda_{N+1}t} + \frac{c_2}{2c_1+\mu\lambda_{N+1}}\, e^{2c_1 t}\|w(0)\|^2$$

$$\le \Big(e^{-\mu\lambda_{N+1}t} + \frac{c_2}{2c_1+\mu\lambda_{N+1}}\, e^{2c_1 t}\Big)\|w(0)\|^2, \tag{17}$$

which concludes the proof of Lemma 3.2. □

We now show the squeezing property.

Lemma 3.3. *Suppose* $\frac{7}{2} \le \beta \le 5$ *and* $f \in H$, *then the semigroup* $S(t)$ *associated with problem* (1) *is squeezing in* B.

Proof. Let $t_* > 0$ fixed, and $w(t) = u_1(t) - u_2(t)$. We now assume that

$$\|P_N w(t_*)\| \le \|Q_N w(t_*)\|.$$

Then we can deduce

$$\begin{aligned}
\|w(t_*)\|^2 &= \|P_N w(t_*)\|^2 + \|Q_N w(t_*)\|^2 \\
&\le 2\|Q_N w(t_*)\|^2 \\
&\le 2(e^{-\mu\lambda_{N+1}t_*} + \frac{c_2}{2c_1 + \mu\lambda_{N+1}} e^{2c_1 t_*})\|w(0)\|^2 \\
&\le 2(e^{-\mu\lambda_1 t_*} + \frac{c_2}{2c_1 + \mu\lambda_{N+1}} e^{2c_1 t_*})\|w(0)\|^2. \qquad (18)
\end{aligned}$$

Let t_* be large enough so that

$$e^{-\mu\lambda_1 t_*} \le \frac{1}{256}. \qquad (19)$$

Next we choose N_0 large enough so that

$$\frac{c_2}{2c_1 + \mu\lambda_{N+1}} e^{2c_1 t_*} < \frac{c_2}{\mu\lambda_{N+1}} e^{2c_1 t_*} \le \frac{1}{256}. \qquad (20)$$

From (18)-(20), we obtain that

$$\|w(t_*)\|^2 \le \frac{1}{64}\|w(0)\|^2, \qquad (21)$$

(21) completes the proof of Lemma 3.3. □

Theorem 3.1. *Suppose* $\frac{7}{2} \le \beta \le 5$ *and* $f \in H$, *then there exists* N_0 *large enough such that*

$$\lambda_{N_0+1} \ge \frac{256c_2}{\mu} 2^{16c_1/(\mu\lambda_1)}.$$

For the semigroup $S(t)$, $(\{S(t)\}_{t\geq 0}, B)$ *admits an exponential attractor* M, *and*

$$d_F(M) \leq 1 + 16c_3\sqrt{c_2/\mu}\,2^{8c_1/(\mu\lambda_1)}. \tag{22}$$

Proof. From Lemmas 3.1 and 3.3, we have known that $\{S(t)\}_{t\geq 0}$ is Lipschitz continuous and squeezing in B, by Theorem 2.1, we know $(\{S(t)\}_{t\geq 0}, B)$ admits an exponential attractor M. We note that, $\lambda_N \sim N^2$, and let δ, θ be fixed, thus obtain the estimate (22) of fractal dimension. Theorem 3.1 therefore is proved. □

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