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ON OBJECTIVE AND STRONG OBJECTIVE CONSISTENT ESTIMATES OF UNKNOWN PARAMETERS FOR STATISTICAL STRUCTURES IN A POLISH GROUP ADMITTING AN INVARIANT METRIC

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Abstract

By using the notion of a Haar ambivalent set introduced by Balka et al. [1], essentially new classes of statistical structures having objective and strong objective estimates of unknown parameters are introduced in a Polish non-locally-compact group admitting an invariant metric and relations between them are studied in this paper. An example of such a weakly separated statistical structure is constructed for which a question asking "whether there exists a consistent estimate of an unknown parameter" is not solvable within the theory (*ZF*) & (*DC*). A question asking "whether there exists an objective consistent estimate of an unknown parameter for any statistical structure in a non-locally compact Polish group with an invariant metric when subjective one

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exists" is answered positively when there exists at least one such a parameter the pre-image of which under this subjective estimate is a prevalent. These results extend recent results of authors. Some examples of objective and strong objective consistent estimates in a compact Polish group $\{0; 1\}^N$ are considered in this paper.

1. Introduction

In order to explain a big gap between the theory of mathematical statistics and results of hypothesis testing, concepts of subjective and objective infinite sample consistent estimates of a useful signal in the linear one-dimensional stochastic model were introduced in [18]. This approach essentially uses the concept of Haar null sets in Polish topological vector spaces introduced by Christensen [2].

The Polish topological vector space $\mathbf{R}^N$ of all real-valued sequences (equivalently, of infinite samples) equipped with Tychonoff metric plays a central role in the theory of statistical decisions because a definition of any consistent estimate of an unknown parameter in various stochastic models without infinite samples is simply impossible.

Let explain from the point of view of the theory of Haar null sets in $\mathbf{R}^N$ some confusions which were described by Nunnally [14] and Cohen [4]:

Let $x_1, x_2, \cdots$ be an infinite sample obtained by observation on independent and normally distributed real-valued random variables with parameters $(\theta, 1)$, where θ is an unknown mean and the variance is equal to 1. Using this infinite sample, we want to estimate an unknown mean. If we denote by μ_θ a linear Gaussian measure on $\mathbf{R}$ with the probability density $\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}$, then the triplet

$$(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta^N)_{\theta \in R}, \tag{1.1}$$

stands a statistical structure described our experiment, where $\mathcal{B}(\mathbf{R}^N)$ denotes the σ-algebra of Borel subsets of $\mathbf{R}^N$. By virtue of the strong law of large numbers, we know that the condition

$$\mu_\theta^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \;\&\; \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = \theta\}) = 1 \tag{1.2}$$

holds true for each $\theta \in \mathbf{R}$.

Take into account the validity of (1.2), for construction of a consistent infinite sample estimation of an unknown parameter θ a mapping T defined by

$$T((x_k)_{k\in N}) = \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n}, \tag{1.3}$$

is used by statisticians. As usual, null hypothesis significance testing in the case $H_0 : \theta = \theta_0$ assumes the following procedure: If an infinite sample $(x_k)_{k\in N} \in T^{-1}(\theta_0)$, then H_0 hypothesis is accepted and H_0 hypothesis is rejected, otherwise. There naturally arises a question asking whether can be explained Cohen statement [4]: "... Dont look for a magic alternative to NHST [null hypothesis significance testing] ... It does not exist." Notice that a set S of all infinite samples $(x_k)_{k\in N}$ for which there exist finite limits of arithmetic means of their first n elements constitutes a proper Borel measurable vector subspace of $\mathbf{R}^N$. Following Christensen [2], each proper Borel measurable vector subspace of an arbitrary Polish topological vector space is Haar null set and since S is a Borel measurable proper vector subspace of $\mathbf{R}^N$ we claim that the mapping T is not defined for "almost every" (in the sense of Christensen[1]

[1]We say that a sentence $P(\cdot)$ formulated in term of an element of a Polish group G is true for "almost every" element of G if a set of all elements $g \in G$ for which $P(g)$ is false constitutes a Haar null set in G.

[2]) infinite sample. The latter relation means that for "almost every" infinite sample we reject null hypothesis H_0. This discussion can be used also to explain Nunnally's [14] following conjecture: "If the decisions are based on convention they are termed arbitrary or mindless while those not so based may be termed subjective. To minimize type II errors, large samples are recommended. In psychology, practically all null hypotheses are claimed to be false for sufficiently large samples so ... it is usually nonsensical to perform an experiment with the sole aim of rejecting the null hypothesis".

Now, let $T_1 : \mathbf{R}^N \to R$ be another infinite sample consistent estimate of an unknown parameter θ in the above mentioned model, i.e.,

$$\mu_\theta^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \ \&\ T_1((x_k)_{k\in N}) = \theta\}) = 1, \tag{1.4}$$

for each $\theta \in \mathbf{R}$. Here naturally arises a question asking what are those additional conditions imposed on the estimate T_1 under which the above-described confusions will be settled.

In this direction, first, notice that there must be no a parameter $\theta_0 \in R$ for which $T_1^{-1}(\theta_0)$ is Haar null set, because then for "almost every" infinite sample null hypothesis $H_0 : \theta = \theta_0$ will be rejected. Second, there must be no a parameter $\theta_1 \in \mathbf{R}$ for which $T_1^{-1}(\theta_1)$ is a prevalent set (equivalently, a complement of a Haar null set) because then for "almost every" infinite sample null hypothesis $H_0 : \theta = \theta_2$ will be rejected for each $\theta_2 \neq \theta_1$. This observations lead us to additional conditions imposed on the estimate T_1, which assumes that $T_1^{-1}(\theta)$ must be neither Haar null nor prevalent for each $\theta \in \mathbf{R}$. Following [1], a set which is neither Haar null nor prevalent is called a Haar ambivalent set. Such estimates firstly were adopted as objective infinite sample consistent estimates of a useful signal in the linear one-dimensional stochastic model (see [27], Theorem 4.1, p. 482).

It was proved in [27] that $T_n : \mathbf{R}^n \to \mathbf{R}(n \in N)$ defined by

$$T_n(x_1, \cdots, x_n) = -F^{-1}(n^{-1}\#(\{x_1, \cdots, x_n\} \cap (-\infty; 0])), \tag{1.5}$$

for $(x_1, \cdots, x_n) \in \mathbf{R}^n$, is a consistent estimator of a useful signal θ in one-dimensional linear stochastic model

$$\xi_k = \theta + \Delta_k (k \in N), \tag{1.6}$$

where $\#(\cdot)$ denotes a counting measure, Δ_k is a sequence of independent identically distributed random variables on $\mathbf{R}$ with strictly increasing continuous distribution function F and expectation of Δ_1 does not exist. In this direction, the following two examples of simulations of linear one-dimensional stochastic models have been considered.

Example 1.1 ([27], Example 4.1, p. 484)**.** Since a sequence of real numbers $(\pi \times n - [\pi \times n])_{n \in N}$, where $[\cdot]$ denotes an integer part of a real number, is uniformly distributed on $(0, 1)$ (see [10], Example 2.1, p.17), we claim that a simulation of a $\mu_{(\theta,1)}$-equidistributed sequence $(x_n)_{n \leq M}$ on R (M is a "sufficiently large" natural number and depends on a representation quality of the irrational number π), where $\mu_{(\theta,1)}$ denotes a θ-shift of the measure μ defined by distribution function F, can be obtained by the formula

$$x_n = F_\theta^{-1}(\pi \times n - [\pi \times n]), \tag{1.7}$$

for $n \leq M$ and $\theta \in R$, where F_θ denotes a distribution function corresponding to the measure μ_θ.

In this model, θ stands a "useful signal".

We set:

(i) n - the number of trials;

(ii) T_n - an estimator defined by the formula (1.5);

(iii) $\overline{X}_n$ - a sample average.

Table 1. Estimates of the useful signal $\theta = 1$ when the white noise is standard Gaussian random variable

n	T_n	$\overline{X}_n$	n	T_n	$\overline{X}_n$
50	0.994457883	1.146952654	550	1.04034032	1.034899747
100	1.036433389	1.010190601	600	1.036433389	1.043940988
150	1.022241387	1.064790041	650	1.03313984	1.036321771
200	1.036433389	1.037987511	700	1.030325691	1.037905202
250	1.027893346	1.045296447	750	1.033578332	1.03728633
300	1.036433389	1.044049728	800	1.03108705	1.032630945
350	1.030325691	1.034339407	850	1.033913784	1.037321098
400	1.036433389	1.045181911	900	1.031679632	1.026202323
450	1.031679632	1.023083495	950	1.034178696	1.036669278
500	1.036433389	1.044635371	1000	1.036433389	1.031131694

When $F(x)$ is a standard Gaussian distribution function, by using Microsoft Excel we have obtained numerical data placed in Table 1. Notice that results of computations presented in Table 1 show us that both statistics T_n and $\overline{X}_n$ give us a good estimates of the "useful signal" θ whenever a generalized "white noise" in that case has a finite absolute moment of the first order and its moment of the first order is equal to zero.

Now let F be a linear Cauchy distribution function on R, i.e.,

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1 + t^2)} dt (x \in R). \tag{1.8}$$

Numerical data placed in Table 2 were obtaining by using Microsoft Excel and Cauchy distribution calculator of the high accuracy [8]. On the one hand, the results of computations placed in Table 2 do not contradict to the above mentioned fact that T_n is a consistent estimator of the parameter $\theta = 1$. On the other hand, we know that a sample average $\overline{X}_n$ does not work in that case because the mean and variance of the "white noise” (i.e., Cauchy random variable) are not defined. By this reason, attempts to estimate the “useful signal” $\theta = 1$ by using the sample average will not be successful.

Table 2. Estimates of the useful signal $\theta = 1$ when the white noise is Cauchy random variable

n	T_n	$\overline{X}_n$	n	T_n	$\overline{X}_n$
50	1.20879235	2.555449288	550	1.017284476	41.08688757
100	0.939062506	1.331789564	600	1.042790358	41.30221291
150	1.06489184	71.87525566	650	1.014605804	38.1800532
200	1.00000000	54.09578271	700	1.027297114	38.03399768
250	1.06489184	64.59240343	750	1.012645994	35.57956117
300	1.021166379	54.03265563	800	1.015832638	35.25149408
350	1.027297114	56.39846672	850	1.018652839	33.28723503
400	1.031919949	49.58316089	900	1.0070058	31.4036155
450	1.0070058	44.00842613	950	1.023420701	31.27321466
500	1.038428014	45.14322051	1000	1.012645994	29.73405416

In [27] has been established that the estimators $\overline{\lim}\widetilde{T_n} := \inf_n \sup_{m \geq n} \widetilde{T_m}$ and $\underline{\lim}\widetilde{T_n} := \sup_n \inf_{m \geq n} \widetilde{T_m}$ are consistent infinite sample estimates of a useful signal θ in the model (1.6) (see [27], Theorem 4.2, p. 483). When we begin to study properties of these infinite sample estimators from the point of view of the theory of Haar null sets in $\mathbf{R}^N$, we observed a surprising and an unexpected fact for us that these both estimates are objective (see [20], Theorem 3.1).

As the described approach naturally divides a class of consistent infinite sample estimates into objective and subjective estimates should not seem excessively highly told our suggestion that each consistent infinite sample estimate must pass the theoretical test on the objectivity.

The present manuscript introduces the concepts of the theory of objective infinite sample consistent estimates in $\mathbf{R}^N$ and gives its extension to all non-locally-compact Polish groups admitting an invariant metric.

The rest of this note is the following:

In Section 2, we give some notions and facts from the theory of Haar null sets in complete metric linear spaces and equidistributed sequences on the real axis **R**. Concepts of objective and strong objective infinite sample consistent estimates for statistical structures are introduced also in this section. Section 3 presents a certain construction of the objective infinite sample consistent estimate of an unknown distribution function which generalises the recent results obtained in [27]. There is proved an existence of the infinite sample consistent estimate of an unknown distribution function $F(F \in \mathcal{F})$ for the family of Borel probability measures $\{p_F^N : F \in \mathcal{F}\}$, where $\mathcal{F}$ denotes the family of all strictly increasing and continuous distribution functions on **R** and p_F^N denotes an infinite power of the Borel probability measure p_F on **R** defined by *F*. Section 4 presents an effective construction of the strong objective infinite sample consistent estimate of the "useful signal" in a certain linear one-dimensional stochastic model. An infinite sample consistent estimate of an unknown probability density is constructed for the separated class of positive continuous probability densities and a problem about existence of an objective one is stated in Section 5. In Section 6, by using the notion of a Haar ambivalent set introduced in [1], essentially new classes of statistical structures having objective and strong objective estimates of an unknown parameter are introduced in a Polish non-locally-compact group admitting an invariant metric and relations between them are studied in this section. An example of such a weakly separated statistical structure is constructed for which a question asking whether there exists a consistent estimate of an unknown parameter is not solvable within the theory (*ZF*) & (*DC*). These results extend recent results obtained in [19]. In addition, we extend the concept of objective and subjective consistent estimates introduced for $\mathbf{R}^N$ to all Polish groups and consider a question asking whether there exists an objective consistent estimate of an unknown parameter for any statistical structure in a non-locally compact Polish group with an invariant metric when subjective one exists. We show that this question is answered positively when there exists at least one such a parameter the pre-image of which under this subjective estimate is a prevalent. In Section 7, we consider some examples of objective and strong objective consistent estimates in a compact Polish group $\{0; 1\}^N$.

2. Auxiliary Notions and Facts from Functional Analysis and Measure Theory

Let V be a complete metric linear space, by which we mean a vector space (real or complex) with a complete metric for which the operations of addition and scalar multiplication are continuous. When we speak of a measure on V we will always mean a nonnegative measure that is defined on the Borel sets of V and is not identically zero. We write $S + v$ for the translation of a set $S \subseteq V$ by a vector $v \in V$.

Definition 2.1 ([7], Definition 1, p. 221)**.** A measure μ is said to be transverse to a Borel set $S \subset V$ if the following two conditions hold:

(i) There exists a compact set $U \subset V$ for which $0 < \mu(U) < 1$;

(ii) $\mu(S + v) = 0$ for every $v \in V$.

Definition 2.2 ([7], Definition 2, p. 222; [1], p. 1579)**.** A Borel set $S \subset V$ is called shy if there exists a measure transverse to S. More generally, a subset of V is called shy if it is contained in a shy Borel set. The complement of a shy set is called a prevalent set. We say that a set is Haar ambivalent if it is neither shy nor prevalent.

Definition 2.3 ([7], p. 226)**.** We say that "almost every" element of V satisfies some given property, if the subset of V on which this property holds is prevalent.

Lemma 2.4 ([7], Fact 300, p. 223)**.** *The union of a countable collection of shy sets is shy.*

Lemma 2.5 ([7], Fact 8, p. 224). *If V is infinite dimensional, all compact subsets of V are shy.*

Lemma 2.6 ([9], Lemma 2, p. 58). *Let* μ *be a Borel probability measure defined in complete separable metric space V. Then there exists a countable family of compact sets* $(F_k)_{k \in N}$ *in V such that* $\mu(V \setminus \cup_{k \in N} F_k) = 0.$

Let $\mathbf{R}^N$ *be a topological vector space of all real valued sequences equipped with Tychonoff metric* ρ *defined by* $\rho((x_k)_{k \in N}, (y_k)_{k \in N}) = \sum_{k \in N} |x_k - y_k| / 2^k (1 + |x_k - y_k|)$ *for* $(x_k)_{k \in N}, (y_k)_{k \in N} \in \mathbf{R}^N.$

Lemma 2.7 ([16], Lemma 15.1.3, p. 202). *Let J be an arbitrary subset of N. We set*

$$A_J = \{(x_i)_{i \in N} : x_i \leq 0 \text{ for } i \in J \;\&\; x_i > 0 \text{ for } i \in N \setminus J\}. \tag{2.1}$$

Then the family of subsets $\Phi = \{A_J : J \subseteq N\}$ *has the following properties*:

(i) *Every element of* Φ *is Haar ambivalent.*

(ii) $A_{J_1} \cap A_{J_2} = \emptyset$ *for all different* $J_1, J_2 \subseteq N.$

(iii) Φ *is a partition of* $\mathbf{R}^N$ *such that* $card(\Phi) = 2^{\aleph_0}.$

Remark 2.8. The proof of the Lemma 2.7 employs an argument stated that each Borel subset of $\mathbf{R}^N$ which for each compact set contains it's any translate is non-shy set.

Definition 2.9 ([10]). A sequence $(x_k)_{k \in N}$ of real numbers from the interval (a, b) is said to be equidistributed or uniformly distributed on an interval (a, b) if for any subinterval $[c, d]$ of $(a, b),$ we have

$$\lim_{n \to \infty} n^{-1} \# (\{x_1, x_2, \cdots, x_n\} \cap [c, d]) = (b - a)^{-1}(d - c), \tag{2.2}$$

where $\#$ denotes a counting measure.

Now, let X be a compact Polish space and μ be a probability Borel measure on X. Let $\mathcal{R}(X)$ be a space of all bounded continuous functions defined on X.

Definition 2.10. A sequence $(x_k)_{k\in N}$ of elements of X is said to be μ-equidistributed or μ-uniformly distributed on the X if for every $f \in \mathcal{R}(X)$, we have

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} f(x_k) = \int_X f d\mu. \tag{2.3}$$

Lemma 2.11 ([10], Lemma 2.1, p. 199)**.** *Let $f \in \mathcal{R}(X)$. Then, for μ^N-almost every sequences $(x_k)_{k\in N} \in X^N$, we have*

$$\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} f(x_k) = \int_X f d\mu. \tag{2.4}$$

Lemma 2.12 ([10], pp. 199-201)**.** *Let S be a set of all μ-equidistributed sequences on X. Then we have $\mu^N(S) = 1$.*

Corollary 2.13 ([27], Corollary 2.3, p. 473)**.** *Let ℓ_1 be a Lebesgue measure on $(0, 1)$. Let D be a set of all ℓ_1-equidistributed sequences on $(0, 1)$, Then we have $\ell_1^N(D) = 1$.*

Definition 2.14. Let μ be a probability Borel measure on R and F be its corresponding distribution function. A sequence $(x_k)_{k\in N}$ of elements of R is said to be μ-equidistributed or μ-uniformly distributed on R if for every interval $[a, b](-\infty \le a < b \le +\infty)$, we have

$$\lim_{n\to\infty} n^{-1}\#([a, b]\cap\{x_1, \cdots x_n\}) = F(b) - F(a). \tag{2.5}$$

Lemma 2.15 ([27], Lemma 2.4, p. 473)**.** *Let $(x_k)_{k\in N}$ be ℓ_1-equidistributed sequence on $(0, 1)$, F be a strictly increasing continuous distribution function on R and p be a Borel probability measure on R defined by F. Then $(F^{-1}(x_k))_{k\in N}$ is p-equidistributed on R.*

Corollary 2.16 ([27], Corollary 2.4, p. 473). *Let F be a strictly increasing continuous distribution function on R and p_F be a Borel probability measure on R defined by F. Then for a set $D_F \subset R^N$ of all p-equidistributed sequences on R, we have*

(i) $D_F = \{(F^{-1}(x_k))_{k \in N} : (x_k)_{k \in N} \in D\}$;

(ii) $p_F^N(D_F) = 1$.

Lemma 2.17. *Let F_1 and F_2 be two different strictly increasing continuous distribution functions on R and p_1 and p_2 be Borel probability measures on R defined by F_1 and F_2, respectively. Then, there does not exist a sequence of real numbers $(x_k)_{k \in N}$ which simultaneously is p_1-equidistributed and p_2-equidistributed.*

Proof. Assume the contrary and let $(x_k)_{k \in N}$ be such a sequence. Since F_1 and F_2 are different there is a point $x_0 \in R$ such that $F_1(x_0) \neq F_2(x_0)$. The latter relation is not possible under our assumption, because $(x_k)_{k \in N}$ simultaneously is p_1-equidistributed and p_2-equidistributed, which implies

$$F_1(x_0) = \lim_{n \to \infty} n^{-1} \# ((-\infty, x_0] \cap \{x_1, \cdots, x_n\}) = F_2(x_0). \tag{2.6}$$

□

Theorem 2.18. *Let F_1 and F_2 be two different strictly increasing continuous distribution functions on R and p_1 and p_2 be Borel probability measures on R defined by F_1 and F_2, respectively. Then the measures p_1^N and p_2^N are orthogonal.*

Proof. Let D_{F_1} and D_{F_2} denote p_1-equidistributed and p_2-equidistributed sequences on R, respectively. By Lemma 2.17, we know that $D_{F_1} \cap D_{F_2} = \emptyset$. By Corollary 2.16, we know that $p_1^N(D_{F_1}) = 1$ and $p_2^N(D_{F_2}) = 1$.

This ends the proof of the theorem. □

Definition 2.19. Let $\{\mu_i : i \in I\}$ be a family of probability measures defined on a measure space (X, M). Let $S(X)$ be defined by

$$S(X) = \cap_{i \in I} dom(\overline{\mu}_i), \tag{2.7}$$

where $\overline{\mu}_i$ denotes a usual completion of the measure μ_i and $dom(\overline{\mu}_i)$ denotes the sigma-algebra of all $\overline{\mu}_i$-measurable subsets of X for each $i \in I$. We say that the family $\{\mu_i : i \in I\}$ is strong separable if there exists a partition $\{C_i : i \in I\}$ of the space X into elements of the σ-algebra $S(X)$ such that $\overline{\mu}_i(C_i) = 1$ for each $i \in I$.

Definition 2.20. Let $\{\mu_i : i \in I\}$ be a family of probability measures defined on a measure space (X, M). Let $L(I)$ denotes a minimal σ-algebra generated by all singletons of I and $S(X)$ be the σ-algebra of subsets of X defined by (2.7). We say that a $(S(X), L(I))$-measurable mapping $T : X \to I$ is a consistent (or well-founded) estimate of an unknown parameter $i(i \in I)$ for the family $\{\mu_i : i \in I\}$ if the following condition:

$$(\forall i)(i \in I \to \mu_i(T^{-1}(\{i\})) = 1) \tag{2.8}$$

holds true.

Lemma 2.21 ([27], Lemma 2.5, p. 474)**.** *Let $\{\mu_i : i \in I\}$ be a family of probability measures defined on a measure space (X, M). The following sentences are equivalent*:

(i) *The family of probability measures $\{\mu_i : i \in I\}$ is strong separable.*

(ii) *There exists a consistent estimate of an unknown parameter $i(i \in I)$ for the family $\{\mu_i : i \in I\}$.*

Now, let $X_1, X_2, \cdots$ be an infinite sampling of independent, equally distributed real-valued random variables with unknown distribution function F. Assume that we know only that F belongs to the family of distribution functions $\{F_\theta : \theta \in \Theta\}$, where Θ is a non-empty set. Using these infinite sampling, we want to estimate an unknown distribution function F. Let μ_θ denotes a Borel probability measure on the real axis $\mathbf{R}$ generated by F_θ for $\theta \in \Theta$. We denote by μ_θ^N an infinite power of the measure μ_θ, i.e., $\mu_\theta^N = \mu_\theta \times \mu_\theta \cdots$.

The triplet $(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta^N)_{\theta\in\Theta}$ is called a statistical structure described our infinite experiment.

Definition 2.22. A Borel measurable function $T_n : \mathbf{R}^n \to \mathbf{R}(n \in N)$ is called a consistent estimator of a parameter θ (in the sense of everywhere convergence) for the family $(\mu_\theta^N)_{\theta\in\Theta}$ if the following condition:

$$\mu_\theta^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \ \& \ \lim_{n\to\infty} T_n(x_1, \cdots, x_n) = \theta\}) = 1 \tag{2.9}$$

holds true for each $\theta \in \Theta$.

Definition 2.23. A Borel measurable function $T_n : \mathbf{R}^n \to \mathbf{R}(n \in N)$ is called a consistent estimator of a parameter θ (in the sense of convergence in probability) for the family $(\mu_\theta^N)_{\theta\in\Theta}$ if for every $\epsilon > 0$ and $\theta \in \Theta$ the following condition

$$\lim_{n\to\infty} \mu_\theta^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \ \& \ |T_n(x_1, \cdots, x_n) - \theta| > \epsilon\}) = 0 \tag{2.10}$$

holds true.

Definition 2.24. A Borel measurable function $T_n : \mathbf{R}^n \to \mathbf{R}(n \in N)$ is called a consistent estimator of a parameter θ (in the sense of convergence in distribution) for the family $(\mu_\theta^N)_{\theta \in \Theta}$ if for every continuous bounded real valued function *f* on **R** the following condition:

$$\lim_{n \to \infty} \int_{\mathbf{R}^N} f(T_n(x_1, \cdots, x_n)) d\mu_\theta^N((x_k)_{k \in N}) = f(\theta) \tag{2.11}$$

holds.

Remark 2.25. Following [23] (see Theorem 2, p. 272), for the family $(\mu_\theta^N)_{\theta \in R}$, we have

(a) An existence of a consistent estimator of a parameter θ in the sense of everywhere convergence implies an existence of a consistent estimator of a parameter θ in the sense of convergence in probability.

(b) An existence of a consistent estimator of a parameter θ in the sense of convergence in probability implies an existence of a consistent estimator of a parameter θ in the sense of convergence in distribution.

Now, let $L(\Theta)$ be a minimal σ-algebra of subsets generated by all singletons of the set Θ.

Definition 2.26. A $(\mathcal{B}(\mathbf{R}^N), L(\Theta))$-measurable function $T : \mathbf{R}^N \to \Theta$ is called a infinite sample consistent estimate (or estimator) of a parameter θ for the family $(\mu_\theta^N)_{\theta \in \Theta}$ if the condition

$$\mu_\theta^N(\{(x_k)_{k \in N} : (x_k)_{k \in N} \in \mathbf{R}^N \ \& \ T((x_k)_{k \in N}) = \theta\}) = 1 \tag{2.12}$$

holds true for each $\theta \in \Theta$.

Definition 2.27. An infinite sample consistent estimate $T : \mathbf{R}^N \to \Theta$ of a parameter θ for the family $(\mu_\theta^N)_{\theta \in \Theta}$ is called objective if $T^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$. Otherwise, *T* is called subjective.

Definition 2.28. An objective infinite sample consistent estimate $T : \mathbf{R}^N \to \Theta$ of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$ is called strong if each $\theta_1, \theta_2 \in \Theta$ there exists an isometric (with respect to Tychonoff metric) transformation $A_{(\theta_1, \theta_2)}$ of $\mathbf{R}^N$ such that $A_{(\theta_1, \theta_2)}(T^{-1}(\theta_1))\Delta T^{-1}$ (θ_2) is shy.

Definition 2.29. Following [26], the family $(\mu_\theta^N)_{\theta\in\Theta}$ is called strictly separated if there exists a family $(Z_\theta)_{\theta\in\Theta}$ of Borel subsets of $\mathbf{R}^N$ such that

(i) $\mu_\theta^N(Z_\theta) = 1$ for $\theta \in \Theta$.

(ii) $Z_{\theta_1} \cap Z_{\theta_2} = \emptyset$ for all different parameters θ_1 and θ_2 from Θ.

(iii) $\bigcup_{\theta\in\Theta} Z_\theta = \mathbf{R}^N$.

Remark 2.30. Notice that an existence of an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$ implies that the family $(\mu_\theta^N)_{\theta\in\Theta}$ is strictly separated. Indeed, if we set $Z_\theta = \{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \ \&\ T((x_k)_{k\in N}) = \theta\}$ for $\theta \in \Theta$, then all conditions participated in the Definition 2.29 will be satisfied.

3. An Objective Infinite Sample Consistent Estimate of an Unknown Distribution Function

Theorem 3.1. *Let $\mathcal{F}$ be a family of distribution functions on* $\mathbf{R}$ *satisfying the following properties*:

(i) *Each element of $\mathcal{F}$ is strictly increasing and continuous.*

(ii) *There exists a point x_* such that $F_1(x_*) \neq F_2(x_*)$ for each different $F_1, F_2 \in \mathcal{F}$.*

Setting $\Theta = \{\theta = F(x_*) : F \in \mathcal{F}\}$ and $F_\theta = F$ for $\theta = F(x_*)$, we get the following parametrization $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$. We denote by μ_θ a Borel probability measure in $\mathbf{R}$ defined by F_θ for $\theta \in \Theta$. Then a function $T_n : \mathbf{R}^n \to \mathbf{R}$, defined by

$$T_n(x_1, \cdots, x_n) = \frac{\#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*])}{n}, \tag{3.1}$$

for $(x_1, \cdots, x_n) \in \mathbf{R}^n (n \in N)$, is a consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta \in \Theta}$ in the sense of almost everywhere convergence.

Proof. It is clear that T_n is Borel measurable function for $n \in N$. For $\theta \in \mathbf{R}$, we set

$$A_\theta = \{(x_k)_{k \in N} : (x_k)_{k \in N} \textit{ is } \mu_\theta\text{-uniformly distributed on } \mathbf{R}\}. \tag{3.2}$$

Following Corollary 2.16, we have $\mu_\theta^N(A_0) = 1$ for $\theta \in \Theta$.

For $\theta \in \Theta$, we get

$$\mu_\theta^N(\{(x_k)_{k \in N} \in \mathbf{R}^n : \lim_{n \to \infty} T_n(x_1, \cdots, x_n) = \theta\}) = \mu_\theta^N(\{(x_k)_{k \in N} \in \mathbf{R}^n :$$

$$\lim_{n \to \infty} n^{-1} \#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*]) = F_\theta(x_*)\}) \geq \mu_\theta^N(A_\theta) = 1. \tag{3.3}$$

□

The following corollaries are simple consequences of Theorem 3.1 and Remark 2.25.

Corollary 3.2. *An estimator* T_n *defined by* (3.1) *is a consistent estimator of a parameter* θ *for the family* $(\mu_\theta^N)_{\theta \in \Theta}$ *in the sense of convergence in probability.*

Corollary 3.3. *An estimator* T_n *defined by* (3.1) *is a consistent estimator of a parameter* θ *for the family* $(\mu_\theta^N)_{\theta \in \Theta}$ *in the sense of convergence in distribution.*

Theorem 3.4. *Let* $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ *and* $(\mu_\theta^N)_{\theta\in\Theta}$ *come from Theorem* 3.1. *Let* $\theta_0 \in \Theta$ *and define an estimate* $T_{\theta_0}^{(1)} : \mathbf{R}^n \to \Theta$ *as follows*: $T_{\theta_0}^{(1)}((x_k)_{k\in N}) = \overline{\lim}\widetilde{T_n}((x_k)_{k\in N})$ *if* $\overline{\lim}\widetilde{T_n}((x_k)_{k\in N}) \in \Theta\setminus\{\theta_0\}$ *and* $T_{\theta_0}^{(1)}((x_k)_{k\in N}) = \theta_0$, *otherwise, where* $\overline{\lim}\widetilde{T_n} = \inf_n \sup_{m\ge n} \widetilde{T_m}$ *and*

$$\widetilde{T_n}((x_k)_{k\in N}) = n^{-1}\,\#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*)), \tag{3.4}$$

for $(x_k)_{k\in N} \in \mathbf{R}^N$. *Then* $T_{\theta_0}^{(1)}$ *is an objective infinite sample consistent estimator of a parameter* θ *for the family* $(\mu_\theta^N)_{\theta\in\Theta}$.

Proof. Following [23] (see p. 189), the function $\overline{\lim}\widetilde{T_n}$ is Borel measurable which implies that the function $\overline{\lim}\widetilde{T_n}$ is $(\mathcal{B}(\mathbf{R}^N), L(\Theta))$-measurable. Following Corollary 2.16, we have $\mu_\theta^N(A_\theta) = 1$ for $\theta \in \Theta$, where A_θ is defined by (3.2). Hence, we get

$$\begin{aligned}
&\mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : T_{\theta_0}^{(1)}(x_k)_{k\in N} = \theta\}) \\
&\quad \ge \mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n}(x_k)_{k\in N} = \theta\}) \\
&\quad \ge \mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n}(x_k)_{k\in N} \\
&\quad = \underline{\lim}\widetilde{T_n}(x_k)_{k\in N} = F_\theta(x_*)\}) \ge \mu_\theta^N(A_\theta) = 1,
\end{aligned} \tag{3.5}$$

for $\theta \in \Theta$. Thus, we have proved that the estimator $T_{\theta_0}^{(1)}$ is an infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$.

Now, let us show that $\mathbf{T}_{\theta_0}^{(1)}$ is an objective infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$.

Let us show that $B(\theta) := (\mathbf{T}^{(1)}_{\theta_0})^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$.

Let $(x_k)_{k\in N}$ be μ_θ-uniformly distributed sequence on $\mathbf{R}$. Then we get

$$\lim_{n\to\infty} n^{-1}\#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*]) = \theta. \tag{3.6}$$

Let us consider a set

$$C(\theta) = \{(y_k)_{k\in N} : y_k \le x_k \text{ if } x_k \le x_* \ \&\ y_k > x_k \text{ if } x_k > x_*\}. \tag{3.7}$$

Setting $J = \{k : x_k \le x_*\}$, we claim that $C(\theta) - (x_k)_{k\in N} = A_J$, where A_J comes from Lemma 2.7. Since any translate of Haar ambivalent set is again Haar ambivalent set, we claim that $C(\theta)$ is Haar ambivalent set. A set $B(\theta)$ which contains the Haar ambivalent set $C(\theta)$ is non-shy. Since $\theta \in \Theta$ was taken arbitrary we deduce that each B_θ is Haar ambivalent set. The latter relation means that the estimator $\mathbf{T}^{(1)}_{\theta_0}$ is an objective infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$. □

Theorem 3.5. *Let* $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ *and* $(\mu_\theta^N)_{\theta\in\Theta}$ *come from Theorem* 3.1. *Let fix* $\theta_0 \in \Theta$ *and define an estimate* $T^{(2)}_{\theta_0} : \mathbf{R}^N \to \Theta$ *as follows*: $\mathbf{T}^{(2)}_{\theta_0}((x_k)_{k\in N}) = \overline{\lim}\widetilde{T_n}((x_k)_{k\in N})$ *if* $\underline{\lim}\widetilde{T_n}((x_k)_{k\in N}) \in \Theta \setminus \{\theta_0\}$ *and* $\mathbf{T}^{(2)}_{\theta_0}((x_k)_{k\in N}) = \theta_0$, *otherwise, where* $\underline{\lim}\widetilde{T_n} = \sup_n \inf_{m\ge n} \widetilde{T_m}$ *and*

$$\widetilde{T_n}((x_k)_{k\in N}) = n^{-1}\#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*]), \tag{3.8}$$

for $(x_k)_{k\in N} \in \mathbf{R}^N$. *Then* $\mathbf{T}^{(2)}_{\theta_0}$ *is an objective infinite sample consistent estimator of a parameter* θ *for the family* $(\mu_\theta^N)_{\theta\in\Theta}$.

Proof. Following [23] (see p. 189), the function $\underline{\lim}\widetilde{T_n}$ is Borel measurable which implies that the function $\overline{\lim}\widetilde{T_n}$ is $(\mathcal{B}(\mathbf{R}^n), L(\Theta))$-measurable. Following Corollary 2.16, we have $\mu_\theta^N(A_\theta) = 1$ for $\theta \in \Theta$, where A_θ is defined by (3.2). Hence, we get

$$\mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \mathbf{T}_{\theta_0}^{(2)}(x_k)_{k\in N} = \theta\})$$

$$\geq \mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \underline{\lim}\widetilde{T_n}(x_k)_{k\in N} = \theta\})$$

$$\geq \mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n}(x_k)_{k\in N}$$

$$= \underline{\lim}\widetilde{T_n}(x_k)_{k\in N} = F_\theta(x_*)\}) \geq \mu_\theta^N(A_\theta) = 1, \tag{3.9}$$

for $\theta \in \Theta$.

Thus, we have proved that the estimator $\mathbf{T}_{\theta_0}^{(2)}$ is an infinite sample consistent estimators of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$.

Now, let us show that $\mathbf{T}_{\theta_0}^{(2)}$ is an objective infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$.

Let us show that $B(\theta) = (\mathbf{T}_{\theta_0}^{(2)})^{-1}(\theta)$ is Haar ambivalent set for each $\theta \in \Theta$.

Let $(x_k)_{k\in N}$ be μ_θ-uniformly distributed sequence. Then we get

$$\lim_{n\to\infty} n^{-1}\#(\{x_1, \cdots, x_n\} \cap (-\infty; x_*]) = \theta. \tag{3.10}$$

Let consider a set

$$C(\theta) = \{(y_k)_{k\in N} : (y_k)_{k\in N} \in \mathbf{R}^N \ \&\ y_k \leq x_k \text{ if } x_k \leq x_*$$
$$\&\ y_k > x_k \text{ if } x_k > x_*\}. \tag{3.11}$$

Setting $J = \{k : x_k \leq x_*\}$, we deduce that $C(\theta) - (x_k)_{k \in N} = A_J$, where A_J comes from Lemma 2.7. Since any translate of Haar ambivalent set is again Haar ambivalent set, we claim that $C(\theta)$ is Haar ambivalent set. A set $B(\theta)$ which contains the Haar ambivalent set $C(\theta)$ is non-shy. Since $\theta \in \Theta$ was taken arbitrary we deduce that each B_θ is Haar ambivalent set. The latter relation means that the estimator $\mathbf{T}_{\theta_0}^{(2)}$ is an objective infinite sample consistent estimator of a parameter θ for the family $(\mu_\theta^N)_{\theta \in \Theta}$. □

Remark 3.6. It can be shown that Theorems 3.4 and 3.5 extend the recent result obtained in [20] (see Theorem 3.1). Indeed, let consider the linear one-dimensional stochastic system

$$(\xi_k)_{k \in N} = (\theta_k)_{k \in N} + (\Delta_k)_{k \in N}, \tag{3.12}$$

where $(\theta_k)_{k \in N} \in \mathbf{R}^N$ is a sequence of useful signals, $(\Delta_k)_{k \in N}$ is sequence of independent identically distributed random variables (the so-called generalized "white noise") defined on some probability space $(\Omega, \mathcal{F}, P)$ and $(\xi_k)_{k \in N}$ is a sequence of transformed signals. Let μ be a Borel probability measure on $\mathbf{R}$ defined by a random variable Δ_1. Then the N-power of the measure μ denoted by μ^N coincides with the Borel probability measure on $\mathbf{R}^N$ defined by the generalized "white noise", i.e.,

$$(\forall X)(X \in \mathcal{B}(\mathbf{R}^N) \rightarrow \mu^N(X) = P(\{\omega : \omega \in \Omega \;\&\; (\Delta_k(\omega))_{k \in N} \in X\})), \tag{3.13}$$

where $\mathcal{B}(\mathbf{R}^N)$ is the Borel σ-algebra of subsets of $\mathbf{R}^N$.

Following [26], a general decision in the information transmission theory is that the Borel probability measure λ, defined by the sequence of transformed signals $(\xi_k)_{k \in N}$ coincides with $(\mu^N)_{\theta_0}$ for some $\theta_0 \in \Theta$ provided that

$$(\exists\theta_0)(\theta_0 \in \Theta \to (\forall X)(X \in \mathcal{B}(\mathbf{R}^N) \to \lambda(X) = (\mu^N)_{\theta_0}(X))), \tag{3.14}$$

where $(\mu^N)_{\theta_0}(X) = \mu^N(X - \theta_0)$ for $X \in \mathcal{B}(\mathbf{R}^N)$.

In [27] has been considered a particular case of the above model (3.12) for which

$$(\theta_k)_{k \in N} \in \{(\theta, \theta, \cdots) : \theta \in \mathbf{R}\}. \tag{3.15}$$

For $\theta \in \mathbf{R}$, a measure μ_θ^N defined by

$$\mu_\theta^N = \mu_\theta \times \mu_\theta \times \cdots, \tag{3.16}$$

where μ_θ is a θ-shift of μ(i.e., $\mu_\theta(X) = \mu(X - \theta)$ for $(X \in \mathcal{B}(\mathbf{R}))$, is called the N-power of the θ-shift of μ on $\mathbf{R}$.

Let denote by F_θ a distribution function defined by μ_θ for $\theta \in \Theta$. Notice that the family $\mathcal{F} = \{F_\theta : \theta \in \Theta\}$ satisfies all conditions participated in Theorem 3.1. Indeed, under x_* we can take the zero of the real axis. Then following Theorems 3.4 and 3.5, estimators $T_{\theta_0}^{(1)}$ and $T_{\theta_0}^{(2)}$ are objective infinite sample consistent estimators of a useful signal θ in the linear one-dimensional stochastic system (3.12). Notice that these estimators exactly coincide with estimators constructed in [20] (see Theorem 3.1).

Theorem 3.7. *Let $\mathcal{F}$ be a family of all strictly increasing and continuous distribution functions in* $\mathbf{R}$ *and p_F be a Borel probability measure on* $\mathbf{R}$ *defined by F for each $F \in \mathcal{F}$. Then the family of Borel probability measures $\{p_F^N : F \in \mathcal{F}\}$ is strong separable.*

Proof. We denote by D_F the set of all p_F-equidistributed sequences on $\boldsymbol{R}$ for each $F \in \mathcal{F}$. By Lemma 2.17, we know that $D_{F_1} \cap D_{F_2} = \emptyset$ for each different $F_1, F_2 \in \mathcal{F}$. By Corollary 2.16, we know that

$p_F^N(D_F) = 1$ for each $F \in \mathcal{F}$. Let fix $F_0 \in \mathcal{F}$ and define a family $(C_F)_{F\in\mathcal{F}}$ of subsets of $\mathbf{R}^N$ as follows: $C_F = D_F$ for $F \in \mathcal{F} \setminus \{F_0\}$ and $C_{F_0} = \mathbf{R}^N \setminus \cup_{F\in\mathcal{F}\setminus\{F_0\}} D_F$. Notice that since D_F is a Borel subset of $\mathbf{R}^N$ for each $F \in \mathcal{F}$, we claim that $C_F \in S(\mathbf{R}^N)$ for each $F \in \mathcal{F}\setminus\{F_0\}$, where $S(\mathbf{R}^N)$ comes from Definition 2.19. Since $\overline{p_F^N}(\mathbf{R}^N \setminus \cup_{F\in\mathcal{F}} D_F) = 0$ for each $F \in \mathcal{F}$, we deduce that $\mathbf{R}^N \setminus \cup_{F\in\mathcal{F}} D_F \in \cap_{F\in\mathcal{F}} dom(\overline{p_F^N}) = S(\mathbf{R}^N)$. Since $S(\mathbf{R}^N)$ is the σ-algebra we claim that $C_{F_0} \in S(\mathbf{R}^N)$, because $\overline{p_F^N}(\mathbf{R}^N \setminus \cup_{F\in\mathcal{F}} D_F) = 0$ for each $F \in \mathcal{F}$ (equivalently, $\mathbf{R}^N \setminus \cup_{F\in\mathcal{F}} D_F \in S(\mathbf{R}^N)$), and

$$C_{F_0} = \mathbf{R}^N \setminus \cup_{F\in\mathcal{F}\setminus\{F_0\}} D_F = (\mathbf{R}^N \setminus \cup_{F\in\mathcal{F}} D_F) \cup D_{F_0}. \tag{3.17}$$

This ends the proof of the theorem. □

Remark 3.8. By virtue the results of Lemma 2.21 and Theorem 3.7, we get that there exists a consistent estimate of an unknown distribution function $F(F \in \mathcal{F})$ for the family of Borel probability measures $\{p_F^N : F \in \mathcal{F}\}$, where $\mathcal{F}$ comes from Theorem 3.7. This estimate $T : \mathbf{R}^N \to \mathcal{F}$ is defined by : $T((x_k)_{k\in N}) = F$ if $(x_k)_{k\in N} \in C_F$, where the family $(C_F)_{F\in\mathcal{F}}$ of subsets of $\mathbf{R}^N$ also comes from Theorem 3.7. Notice that this result extends the main result established in [27] (see Lemma 2.6, p. 476).

At end of this section, we state the following:

Problem 3.1. Let $\mathcal{F}$ be a family of all strictly increasing and continuous distribution functions on $\mathbf{R}$ and p_F be a Borel probability measure in $\mathbf{R}$ defined by F for each $F \in \mathcal{F}$. Does there exist an objective infinite sample consistent estimate of an unknown distribution function F for the family of Borel probability measures $\{p_F^N : F \in \mathcal{F}\}$?

4. An Effective Construction of the Strong Objective infinite Sample Consistent Estimate of a Useful Signal in the Linear one-Dimensional Stochastic Model

In [18], the examples of objective and strong objective infinite sample consistent estimates ([18], $T^{\star}$ (p. 63), T° (p. 67)) of a useful signal in the linear one-dimensional stochastic model were constructed by using the axiom of choice and a certain partition of the non-locally compact abelian Polish group R^N constructed in [17].

In this section, in the same model, we present an effective example of the strong objective infinite sample consistent estimate of a useful signal constructed in [19].

For each real number $a \in R$, we denote by $\{a\}$ its fractal part in the decimal system.

Theorem 4.1. *Let consider the linear one-dimensional stochastic model* (3.12), *for which "white noise" has a infinite absolute moment of the first order and its moment of the first order is equal to zero. Suppose that the Borel probability measure* λ, *defined by the sequence of transformed signals* $(\xi_k)_{k \in N}$ *coincides with* $\left(\mu_{\theta_0}^N\right)$ *for some* $\theta_0 \in [0, 1]$.

Let $T : R^N \to [0, 1]$ *be defined by:* $T((x_k)_{k \in N}) = \{ \lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{n} \}$ *if* $\lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{n} \neq 1$; $T((x_k)_{k \in N}) = 1$ *if* $\lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{n} = 1$; *and* $T((x_k)_{k \in N}) = \sum_{k \in N} \frac{\chi_{(0, +\infty)}(x_k)}{2^k}$, *otherwise, where* $\chi_{(0, +\infty)}(\cdot)$ *denotes an indicator function of the set* $(0, +\infty)$ *defined on the real axis R. Then T is a strong objective infinite sample consistent estimate of the parameter* θ *for the statistical structure* $(R^N, \mathcal{B}(R^N), \mu_{\theta}^N)_{\theta \in \Theta}$ *describing the linear one-dimensional stochastic system* (3.12).

Proof. Step 1. We have to show that T is an infinite sample consistent estimate of the parameter θ for the statistical structure $(R^N, \mathcal{B}(R^N), \mu_\theta^N)_{\theta\in\Theta}$ and $T^{-1}(\theta)$ is a Haar ambivalent set for each $\theta = \sum_{k=1}^{\infty} \frac{\theta_k}{2^k} \in \Theta$, where $\sum_{k=1}^{\infty} \frac{\theta_k}{2^k}$ is representation of the number θ in the binary system.

Indeed, we have

$$(\forall\theta)(\theta \in (0, 1) \to T^{-1}(\theta) = (B_{H(\theta)} \setminus S) \cup \cup_{z\in Z} S_{\theta+z}), \tag{4.1}$$

where $H(\theta) = \{k : k \in N \ \& \ \theta_k = 1\}$, $B_{H(\theta)} = \theta - A_{H(\theta)}$, $A_{H(\theta)}$ comes from Lemma 2.7,

$$S = \{(x_k)_{k\in N} \in \mathbf{R}^N : \text{exists a finite limit } \lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k}{n} \}, \tag{4.2}$$

and

$$S_{\theta+z} = \{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k}{n} = \theta + z\}, \tag{4.3}$$

for each $\theta \in \Theta$ and $z \in Z$. Notice that the set S like $\cup_{z\in Z} S_{\theta+z}$ is Borel shy set (see [18], Lemma 4.14, p. 60). Taking into account this fact, the results of Lemmas 2.4 and 2.7 invariance of Haar ambivalent sets under translations and symmetric transformation and the simple statement that difference of non-shy and shy sets is non-shy, we deduce that $T^{-1}(\theta)$ is a Borel measurable Haar ambivalent sets for each $\theta \in \Theta$.

Notice that

$$T^{-1}(1) = (B_{H(1)} \setminus S) \cup S_1 = (B_N \setminus S) \cup S_1, \tag{4.4}$$

and

$$T^{-1}(0) = (B_{H(0)} \setminus S) \cup \cup_{z\in Z\setminus\{1\}} S_{0+z} = (B_\emptyset \setminus S) \cup \cup_{z\in Z\setminus\{1\}} S_{0+z}, \tag{4.5}$$

which also are Borel measurable Haar ambivalent sets.

Now, it is not hard to show that T is $(\mathcal{B}(\mathbf{R}^N), L(\Theta))$-measurable because the class $\mathcal{B}(\mathbf{R}^N)$ is closed under countable family of set-theoretical operations and each element of $L(\Theta)$ is countable or co-countable in the interval $\theta = [0, 1]$. Since $S_\theta \subseteq T^{-1}(\theta)$ for $\theta \in \Theta$, we deduce that $\mu_\theta(T^{-1}(\theta)) = 1$. The later relation means that T is an infinite sample consistent estimate of a parameter θ.

Step 2. Let us show that for each different $\theta_1, \theta_2 \in [0, 1]$ there exists an isometric (with respect to Tychonoff metric) transformation $A_{(\theta_1, \theta_2)}$ such that

$$A_{(\theta_1, \theta_2)}(T^{-1}(\theta_1))\Delta T^{-1}(\theta_2) \tag{4.6}$$

is shy.

We define $A_{(\theta_1, \theta_2)}$ as follows: for $(x_k)_{k \in N} \in R^N$ we put $A_{(\theta_1, \theta_2)}((x_k)_{k \in N}) = (y_k)_{k \in N}$, where $y_k = -x_k$ if $k \in H(\theta_1)\Delta H(\theta_2)$ $(:= (H(\theta_1) \setminus H(\theta_2)) \cup (H(\theta_2) \setminus H(\theta_1))$ and $y_k = x_k$, otherwise. It is obvious that $A_{(\theta_1, \theta_2)}$ is isometric (with respect to Tychonoff metric) transformation of the R^N.

Notice that

$$A_{(\theta_1, \theta_2)}(T^{-1}(\theta_1))\Delta T^{-1}(\theta_2) \subseteq \cup_{k \in N} \{0\}_k \times R^{N \setminus \{k\}} \cup S. \tag{4.7}$$

Since both sets $\cup_{k \in N} \{0\}_k \times R^{N \setminus \{k\}}$ and S are shy, by Lemmas 2.4 and Definition 2.2, we claim that the set

$$A_{(\theta_1, \theta_2)}(T^{-1}(\theta_1))\Delta T^{-1}(\theta_2) \tag{4.8}$$

is also shy.

This ends the proof of the theorem. □

5. On Infinite Sample Consistent Estimates of an Unknown Probability Density Function

Let $X_1, X_2, \cdots$ be independent identically distributed real-valued random variables having a common probability density function *f*. After a so-called kernel class of estimates f_n of *f* based on $X_1, X_2, \cdots, X_n$ was introduced by Rosenblatt [21], various convergence properties of these estimates have been studied. The stronger result in this direction was due to Nadaraya [13] who proved that if *f* is uniformly continuous, then for a large class of kernels the estimates f_n converges uniformly on the real line to *f* with probability one. In [22] has been shown that the above assumptions on *f* are necessary for this type of convergence. That is, if f_n converges uniformly to a function *g* with probability one, then *g* must be uniformly continuous and the distribution *F* from which we are sampling must be absolutely continuous with $F'(x) = g(x)$ everywhere. When in addition to the mentioned above, it is assumed that *f* and its first $r+1$ derivatives are bounded, it is possible to show that how to construct estimates f_n such that $f_n^{(s)}$ converges uniformly to $f^{(s)}$ as a given rate with probability one for $s = 0, \cdots, r$. Let $f_n(x)$ be a kernel estimate based on $X_1, X_2, \cdots, X_n$ from *F* as given in [21], that is,

$$f_n(x) = (na_n)^{-1}\sum_{i=1}^{n} k\left(\frac{X_i - x}{a_n}\right), \tag{5.1}$$

where $(a_n)_{n \in N}$ is a sequence of positive numbers converging to zero and k is a probability density function such that $\int_{-\infty}^{+\infty} |x| k(x) dx$ is finite and $k^{(s)}$ is continuous function of bounded variation for $s = 0, \cdots, r$. The density function of the standard normal, for example, satisfies all these conditions.

In the sequel, we need the following wonderful statement:

Lemma 5.1 ([22], Theorem 3.11, p. 1194)**.** *A necessary and sufficient condition for*

$$\lim_{n \to \infty} \sup_{x \in \mathbf{R}} |f_n(x) - g(x)| = 0, \tag{5.2}$$

with probability one for a function g is that g be the uniformly continuous derivative of F.

Let $X_1, X_2, \cdots$ be independent and identically distributed real-valued random variables with an unknown probability density function *f*. Assume that we know that *f* belongs to the class of probability density function $\mathcal{SC}$ each element of which is uniformly continuous.

Let denote by $\ell^{\infty}(R)$ an infinite-dimensional non-separable Banach space of all bounded real-valued functions on **R** equipped with norm $\|\cdot\|_{\infty}$ defined by

$$\|h\|_{\infty} = \sup_{x \in \mathbf{R}} |h(x)|, \tag{5.3}$$

for all $h \in \ell^{\infty}(R)$. We say that $(\ell^{\infty}(R)) \lim_{n \to \infty} h_n = h_0$ if $\lim_{n \to \infty} \|h_n - h_0\|_{\infty} = 0$.

Theorem 5.2. *Let* ϕ *denotes a normal density function. We set* $\Theta = \mathcal{SC}$. *Let* μ_{θ} *be a Borel probability measure on* **R** *with probability density function* $\theta \in \Theta$. *Let fix* $\theta_0 \in \Theta$. *For each* $(x_i)_{i \in N}$, *we set* $T_{\mathcal{SC}}((x_i)_{i \in N}) = (\ell^{\infty}(R)) \lim_{n \to \infty} f_n$ *if this limit exists and is in* $\Theta \setminus \{\theta_0\}$, *and* $T_{\mathcal{SC}}((x_i)_{i \in N}) = \theta_0$, *otherwise. Then* $T_{\mathcal{SC}}$ *is a consistent infinite-sample estimate of an unknown parameter* θ *for the family* $(\mu_{\theta}^N)_{\theta \in \Theta}$.

Proof. By Lemma 5.1, for each $\theta \in \Theta$, we have

$$\begin{aligned} &\mu_{\theta}^N(\{(x_i)_{i \in N} \in \mathbf{R}^N : T_{\mathcal{SC}}((x_i)_{i \in N}) = \theta\}) \\ &\geq \mu_{\theta}^N(\{(x_i)_{i \in N} \in \mathbf{R}^N : (\ell^{\infty}(R)) \lim_{n \to \infty} f_n = \theta\}) \\ &= \mu_{\theta}^N(\{(x_i)_{i \in N} \in \mathbf{R}^N : \lim_{n \to \infty} \|f_n - \theta\|_{\infty} = 0\}) = 1. \end{aligned}$$

This ends the proof of theorem. □

Concerning with Theorem 5.2, we state the following problems:

Problem 5.1. Let $T_{\mathcal{SC}}$ comes from the Theorem 5.2. Is $T_{\mathcal{SC}}$ an objective infinite sample consistent estimate of the parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$?

Problem 5.2. Let the statistical structure $\{(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta^N): \theta \in \Theta\}$ comes from the Theorem 5.2. Does there exist an objective (or strong objective) infinite sample consistent estimate of the parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$?

Let $X_1, X_2, \cdots$ be independent and identically distributed real-valued random variables with positive continuous probability density function *f*. Assume that we know that *f* belongs to the separated class $\mathcal{A}$ of positive continuous probability densities provided that there is a point x_* such that $g_1(x_*) \neq g_2(x_*)$ for each $g_1, g_2 \in \mathcal{A}$. Suppose that we have an infinite sample $(x_k)_{k\in N}$ and we want to estimate an unknown probability density function. Setting $\Theta = \{\theta = g(x_*) : g \in \mathcal{A}\}$, we can parameterise the family $\mathcal{A}$ as follows: $\mathcal{A} = \{f_\theta : \theta \in \Theta\}$, where f_θ is such a unique element *f* from the family $\mathcal{A}$ for which $f(x_*) = \theta$. Let μ_θ be a Borel probability measure defined by the probability density function f_θ for each $\theta \in \Theta$. It is obvious that $\{(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta^N) : \theta \in \Theta\}$ will be the statistical structure described our experiment.

Theorem 5.3. *Let $(h_m)_{m\in N}$ be a sequence of a strictly decreasing sequence of positive numbers tending to zero. Let $\theta_0 \in \Theta$. For each $(x_k)_{k\in N} \in \mathbf{R}^N$, we put*

$$T((x_k)_{k\in N}) = \lim_{m\to\infty} \lim_{n\to\infty} \frac{\#(\{x_1, \cdots, x_n\} \cap [x_* - h_m, x_* + h_m])}{2nh_m}, \quad (5.4)$$

if this repeated limit exists and belongs to the set $\Theta \setminus \{\theta\}$, *and* $T((x_k)_{k\in N}) = \theta_0$, *otherwise. Then* T *is an infinite sample consistent estimate of the parameter* θ *for the family* $(\mu_\theta^N)_{\theta\in\Theta}$.

Proof. For each $\theta \in \Theta$, we put

$$A_\theta = \{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \;\&\; (x_k)_{k\in N} \text{ is } \mu_\theta\text{-equidistributed}\}. \tag{5.5}$$

By Corollary 2.16, we know that $\mu_\theta^N(A_\theta) = 1$ for each $\theta \in \Theta$.

For each $\theta \in \Theta$, we have

$$\begin{aligned}
\mu_\theta^N(T^{-1}(\theta)) &= \mu_\theta^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : T((x_k)_{k\in N}) = \theta\}) \\
&\geq \mu_\theta^N(\{(x_k)_{k\in N} \in A_\theta : T((x_k)_{k\in N}) = \theta\}) \\
&= \mu_\theta^N\left(\left\{(x_k)_{k\in N} \in A_\theta : \lim_{m\to\infty} \frac{F_\theta(x_* + h_m) - F_\theta(x_* - h_m)}{2h_m} = \theta\right\}\right) \\
&= \mu_\theta^N\left(\left\{(x_k)_{k\in N} \in A_\theta : \lim_{m\to\infty} \frac{\int_{x_*-h_m}^{x_*+h_m} f_\theta(x)dx}{2h_m} = \theta\right\}\right) \\
&= \mu_\theta^N(\{(x_k)_{k\in N} \in A_\theta : f_\theta(x^*) = \theta\}) = \mu_\theta^N(A_\theta) = 1.
\end{aligned} \tag{5.6}$$

□

Concerning with Theorem 5.3, we state the following:

Problem 5.3. Let T comes from the Theorem 5.3. Is T an objective infinite sample consistent estimate of the parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$?

Problem 5.4. Let the statistical structure $\{(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta^N) : \theta \in \Theta\}$ comes from the Theorem 5.3. Does there exist an objective (or strong objective) infinite sample consistent estimate of the parameter θ for the family $(\mu_\theta^N)_{\theta\in\Theta}$?

Example 5.4. Let $X_1, X_2, \cdots$ be independent normally distributed real-valued random variables with parameters (a, σ) where a is a mean and σ is a standard deviation. Suppose that we know the mean a and want to estimate an unknown standard deviation σ by an infinite sample $(x_k)_{k\in N}$. For each $\sigma > 0$, let denote by μ_σ the Gaussian probability measure on $\mathbf{R}$ with parameters (a, σ) (here $a \in \mathbf{R}$ is fixed). Let $(h_m)_{m\in N}$ be a sequence of a strictly decreasing sequence of positive numbers tending to zero.

By virtue of Theorem 5.4, we know that for each $\sigma > 0$ the following condition:

$$\mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N \ \&$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{\#(\{x_1, \cdots, x_n\}\cap[a - h_m, a + h_m])}{2nh_m} = \frac{1}{\sqrt{2\pi}\sigma}\}) = 1 \tag{5.7}$$

holds true.

Let $\sigma_0 > 0$. For $(x_k)_{k\in N} \in \mathbf{R}^N$, we put

$$T_1((x_k)_{k\in N}) = \lim_{m\to\infty}\lim_{n\to\infty}\frac{2nh_m}{\sqrt{2\pi}\#(\{x_1, \cdots, x_n\}\cap[a - h_m, a + h_m])}, \tag{5.8}$$

if this limit exists and belongs to the set $(0, +\infty)\setminus\{\sigma_0\}$, and $T_1((x_k)_{k\in N}) = \sigma_0$, otherwise. Then for each $\sigma > 0$, we get

$$\mu_\sigma^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \mathbf{R}^N \ \& \ T_1((x_k)_{k\in N}) = \sigma\}) = 1, \tag{5.9}$$

which means that T_1 is an infinite sample consistent estimate of the standard deviation σ for the family $(\mu_\sigma^N)_{\sigma>0}$.

Theorem 5.5. *Let $X_1, X_2, \cdots$ be independent normally distributed real-valued random variables with parameters (a, σ), where a is a mean and σ is a standard deviation. Suppose that we know the mean a. Let $(a_n)_{n\in N}$ be a sequence of positive numbers converging to zero and ϕ be a standard Gaussian density function in* $\mathbf{R}$. *We denote by μ_σ a Borel*

Gaussian probability measure in $\mathbf{R}$ *with parameters* (a, σ) *for each* $\sigma \in \Sigma = (0, \infty)$. *Let fix* $\sigma_0 \in \Sigma$. *Let define an estimate* $T^{(1)}_{\sigma_0} : \mathbf{R}^N \to \Sigma$ *as follows*: $T^{(1)}_{\sigma_0}((x_k)_{k \in N}) = \overline{\lim}\widetilde{T^{(1)}_n}((x_k)_{k \in N})$ *if* $\overline{\lim}\widetilde{T^{(1)}_n}((x_k)_{k \in N}) \in \Sigma \setminus \{\sigma_0\}$ *and* $T^{(1)}_{\sigma_0}((x_k)_{k \in N}) = \sigma_0$, *otherwise, where* $\overline{\lim}\widetilde{T^{(1)}_n} := \inf_n \sup_{m \geq n} \widetilde{T^{(1)}_m}$ *and*

$$\widetilde{T^{(1)}_n}((x_k)_{k \in N}) = T^{(1)}_n(x_1, \cdots, x_n) = \frac{1}{\sqrt{2\pi}(na_n)^{-1}\sum_{i=1}^{n}\phi\left(\frac{x_i - a}{a_n}\right)}, \tag{5.10}$$

for $(x_k)_{k \in N} \in \mathbf{R}^N$. *Then* $T^{(1)}_{\sigma_0}$ *is an infinite sample consistent estimator of a parameter* σ *for the family* $(\mu^N_\sigma)_{\sigma \in \Sigma}$.

Proof. Following [23] (see p. 189), the function $\overline{\lim}\widetilde{T^{(1)}_n}$ is Borel measurable which implies that the function $\overline{\lim}\widetilde{T^{(1)}_n}$ is $(\mathcal{B}(\mathbf{R}^N), L(\Sigma))$-measurable.

For each $\sigma \in \Sigma$, we put

$$A_\sigma = \{(x_k)_{k \in N} \in \mathbf{R}^N : \lim_{n \to \infty}(na_n)^{-1}\sum_{i=1}^{n}\phi\left(\frac{x_i - a}{a_n}\right) = f_\sigma(a)\}. \tag{5.11}$$

Since uniformly convergence implies pointwise convergence, by Lemma 5.1, we deduce that $\mu^N_\sigma(A_\sigma) = 1$ for $\sigma \in \Sigma$ which implies

$$\mu^N_\sigma(\{(x_k)_{k \in N} \in \mathbf{R}^N : T^{(1)}_{\theta_0}(x_k)_{k \in N} = \sigma\})$$

$$\geq \mu^N_\sigma(\{(x_k)_{k \in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T^{(1)}_n}(x_k)_{k \in N} = \sigma\})$$

$$\geq \mu^N_\sigma(\{(x_k)_{k \in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T^{(1)}_n}(x_k)_{k \in N} = \underline{\lim}\widetilde{T^{(1)}_n}(x_k)_{k \in N} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k \in N} \in \mathbf{R}^N : \lim_{n \to \infty} \widetilde{T_n^{(1)}}((x_k)_{k \in N}) = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k \in N} \in \mathbf{R}^N : \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}(na_n)^{-1}\sum_{i=1}^n \phi(\frac{x_i - a}{a_n})} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k \in N} \in \mathbf{R}^N : \lim_{n \to \infty}(na_n)^{-1}\sum_{i=1}^n \phi(\frac{x_i - a}{a_n}) = \frac{1}{\sqrt{2\pi}\sigma}\})$$

$$= \mu_\sigma^N(\{(x_k)_{k \in N} \in \mathbf{R}^N : \lim_{n \to \infty}(na_n)^{-1}\sum_{i=1}^n \phi(\frac{x_i - a}{a_n}) = f_\sigma(a)\})$$

$$= \mu_\sigma^N(A_\sigma) = 1. \tag{5.12}$$

□

The following theorem gives a construction of the objective infinite sample consistent estimate of an unknown parameter σ in the same model.

Theorem 5.6. *Let* $X_1, X_2, \cdots$ *be independent normally distributed real-valued random variables with parameters* (a, σ), *where a is a mean and* σ *is a standard deviation. Suppose that we know the mean a is nonzero. Let* Φ *be a standard Gaussian distribution function in* $\mathbf{R}$. *We denote by* μ_σ *a Borel Gaussian probability measure in* $\mathbf{R}$ *with parameters* (a, σ) *for each* $\sigma \in \Sigma = (0, \infty)$. *Let fix* $\sigma_0 \in \Sigma$. *Let define an estimate* $T_{\sigma_0}^{(2)} : \mathbf{R}^N \to \Sigma$ *as follows*: $T_{\sigma_0}^{(2)}((x_k)_{k \in N}) = \overline{\lim}\widetilde{T_n^{(2)}}((x_k)_{k \in N})$ *if* $\overline{\lim}\widetilde{T_n^{(2)}}((x_k)_{k \in N}) \in \Sigma \setminus \{\sigma_0\}$ *and* $T_{\sigma_0}^{(2)}((x_k)_{k \in N}) = \sigma_0$, *otherwise, where* $\overline{\lim}\widetilde{T_n^{(2)}} := \inf_n \sup_{m \geq n} \widetilde{T_m^{(2)}}$ *and*

$$\widetilde{T_n^{(2)}}((x_k)_{k \in N}) = T_n^{(2)}(x_1, \cdots, x_n) = -\frac{a}{\Phi^{-1}\left(\frac{\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0])}{n}\right)}, \tag{5.13}$$

for $(x_k)_{k\in N} \in \mathbf{R}^N$. *Then* $T^{(2)}_{\sigma_0}$ *is an objective infinite sample consistent estimator of a parameter* σ *for the family* $(\mu_\sigma^N)_{\sigma\in\Sigma}$.

Proof. Following [23] (see p. 189), the function $\overline{\lim}\widetilde{T_n^{(2)}}$ is Borel measurable which implies that the function $\overline{\lim}\widetilde{T_n^{(2)}}$ is $(\mathcal{B}(\mathbf{R}^N), L(\Sigma))$ -measurable.

For each $\sigma \in \Sigma$, we put

$$A_\sigma = \{(x_k)_{k\in N} \in \mathbf{R}^N : (x_k)_{k\in N} \text{ is } \mu_\alpha\text{-equidistributed in } \mathbf{R}\}. \tag{5.14}$$

By Corollary 2.16, we know that $\mu_\sigma^N(A_\sigma) = 1$ for $\sigma \in \Sigma$ which implies

$$\mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n^{(2)}}(x_k)_{k\in N} = \sigma\})$$

$$\geq \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n^{(2)}}(x_k)_{k\in N} = \underline{\lim}\widetilde{T_n^{(2)}}(x_k)_{k\in N} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty}\widetilde{T_n^{(2)}}((x_k)_{k\in N}) = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} - \frac{a}{\Phi^{-1}\left(\frac{(\#(\{x_1, \cdots x_n\} \cap (-\infty, 0]))}{n}\right)} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \Phi^{-1}\left(\frac{(\#(\{x_1, \cdots x_n\} \cap (-\infty, 0])}{n}\right) = -\frac{a}{\sigma}\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \frac{(\#(\{x_1, \cdots x_n\} \cap (-\infty, 0])}{n} = \Phi(-\frac{a}{\sigma})\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \frac{(\#(\{x_1, \cdots x_n\} \cap (-\infty, 0])}{n} = \Phi_{(a,\sigma)}(0)\})$$

$$\geq \mu_\sigma^N(A_\sigma) = 1. \tag{5.15}$$

The latter relation means that $\overline{\lim}\widetilde{T_n^{(2)}}$ is a infinite-sample consistent estimate of a parameter σ for the family of measures $(\mu_\sigma^N)_{\sigma>0}$.

Let show that $\widetilde{\overline{\lim} T_n^{(2)}}$ is objective.

We have to show that for each $\sigma < 0$ the set $(\widetilde{\overline{\lim} T_n^{(2)}})^{-1}(\sigma)$ is a Haar ambivalent set.

Let $(x_k)_{k \in N}$ be μ_σ-equidistributed sequence. Then we get

$$\lim_{n \to \infty} \frac{(\#(\{x_1, \cdots x_n\} \cap (-\infty, 0])}{n} = \Phi_{(a, \sigma)}(0), \tag{4.16}$$

which means

$$T_{\sigma_0}^{(2)}((x_k)_{k \in N}) = \widetilde{\overline{\lim} T_n^{(2)}}((x_k)_{k \in N}) = \sigma. \tag{5.17}$$

Setting $J_\sigma = \{i : x_i \leq 0\}$, it is not hard to show that a set

$$B_{J_\sigma} = \{(y_i)_{i \in N} : y_i \leq x_i \text{ for } i \in J_\sigma \ \& \ y_i > x_i \text{ for } i \in N \setminus J_\sigma\}, \tag{5.18}$$

is a Haar ambivalent set.

It is clear also that for each $(y_i)_{i \in N} \in B_{J_\sigma}$, we have

$$T_{\sigma_0}^{(2)}((y_k)_{k \in N}) = \widetilde{\overline{\lim} T_n^{(2)}}((y_k)_{k \in N}) = \sigma,$$

which implies that $B_{J_\sigma} \subseteq (\widetilde{\overline{\lim} T_n^{(2)}})^{-1}(\sigma)$.

Since $\{(\widetilde{\overline{\lim} T_n^{(2)}})^{-1}(\sigma) : \sigma > 0\}$ is a partition of the $\mathbf{R}^N$ and each of them contains a Haar ambivalent set B_{J_σ}, we deduce that $(\widetilde{\overline{\lim} T_n^{(2)}})^{-1}(\sigma)$ is a Haar ambivalent set for each $\sigma > 0$.

This ends the proof of the theorem. □

Theorem 5.7. *Let* $X_1, X_2, \cdots$ *be independent normally distributed real-valued random variables with parameters* (a, σ), *where a is a mean and* σ *is a standard deviation. Suppose that both parameters are* *unknown. Let* Φ *be a standard Gaussian distribution function in* $\mathbf{R}$. *We denote by* μ_σ *a Borel Gaussian probability measure in* $\mathbf{R}$ *with parameters* (a, σ) *for each* $\sigma \in \sum = (0, \infty)$ *and* $a \in \mathbf{R}$. *Let* $\sigma_0 \in \sum$. *Let define an estimate* $T^{(3)}_{\sigma_0} : \mathbf{R}^N \to \sum$ *as follows*: $T^{(3)}_{\sigma_0}((x_k)_{k\in N}) = \overline{\lim}\widetilde{T^{(3)}_n}((x_k)_{k\in N})$ *if* $\overline{\lim}\widetilde{T^{(3)}_n}((x_k)_{k\in N}) \in \sum \setminus \{\sigma_0\}$ *and* $T^{(3)}_{\sigma_0}((x_k)_{k\in N}) = \sigma_0$, *otherwise, where* $\overline{\lim}\widetilde{T^{(3)}_n} := \inf_n \sup_{m \ge n} \widetilde{T^{(3)}_m}$, *and*

$$\widetilde{T^{(3)}_n}((x_k)_{k\in N}) = T^{(3)}_n(x_1, \cdots, x_n) = -\frac{\sum_{i=1}^{n} x_k}{n\Phi^{-1}\left(\frac{\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0])}{n}\right)}, \tag{5.19}$$

for $(x_k)_{k\in N} \in \mathbf{R}^N$. *Then* $T^{(3)}_{\sigma_0}$ *is an infinite sample consistent estimator of a parameter* σ *for the family* $(\mu^N_\sigma)_{\sigma\in\sum}$.

Proof. Following [23] (see p. 189), the function $\overline{\lim}\widetilde{T^{(3)}_n}$ is Borel measurable which implies that the function $\overline{\lim}\widetilde{T^{(3)}_n}$ is $(\mathcal{B}(\mathbf{R}^N), L(\sum))$ -measurable.

For each $\sigma \in \sum$, we put

$$A_\sigma = \{(x_k)_{k\in N} \in \mathbf{R}^N : (x_k)_{k\in N} \text{ is } \mu_\alpha\text{-equidistributed in } \mathbf{R}\}, \tag{5.20}$$

and

$$B_\sigma = \{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \frac{\sum_{k=1}^{n} x_k}{n} = a\}. \tag{5.21}$$

On the one hand, by By Corollary 2.16, we know that $\mu^N_\sigma(A_\sigma) = 1$ for $\sigma \in \sum$. On the other hand, by strong law of large numbers, we know that $\mu^N_\sigma(B_\sigma) = 1$ for $\sigma \in \sum$. These relations imply that

$$\mu_\sigma^N(A_\sigma \cap B_\sigma) = 1, \tag{5.22}$$

for $\sigma \in \sum$.

Take into account (5.22), we get

$$\mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n^{(3)}}(x_k)_{k\in N} = \sigma\})$$

$$\geq \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \overline{\lim}\widetilde{T_n^{(3)}}(x_k)_{k\in N} = \underline{\lim}\widetilde{T_n^{(3)}}(x_k)_{k\in N} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty}\widetilde{T_n^{(3)}}((x_k)_{k\in N} = \sigma\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} - \frac{\frac{\sum_{k=1}^n x_k}{n}}{\Phi^{-1}(\frac{(\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0]))}{n}} = \sigma\})$$

$$\geq \mu_\sigma^N(\{(x_k)_{k\in N} \in A_\sigma \cap B_\sigma : \lim_{n\to\infty} \Phi^{-1}(\frac{(\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0])}{n})$$

$$= -\lim_{n\to\infty} \frac{\frac{\sum_{k=1}^n x_k}{n}}{\sigma}\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in A_\sigma \cap B_\sigma : \lim_{n\to\infty} \Phi^{-1}(\frac{(\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0])}{n}) = -\frac{a}{\sigma}\})$$

$$= \mu_\sigma^N(\{(x_k)_{k\in N} \in \mathbf{R}^N : \lim_{n\to\infty} \frac{(\#(\{x_1, \cdots, x_n\} \cap (-\infty, 0])}{n} = \Phi_{(a,\sigma)}(0)\})$$

$$= \mu_\sigma^N(A_\sigma \cap B_\sigma) = 1. \tag{5.23}$$

The latter relation means that $T_{\sigma_0}^{(3)}$ is an infinite-sample consistent estimate of a parameter σ for the family of measures $(\mu_\sigma^N)_{\sigma>0}$.

This ends the proof of the theorem.

□

Example 5.8. Since a sequence of real numbers $(\pi \times n - [\pi \times n])_{n \in N}$, where $[\cdot]$ denotes an integer part of a real number, is uniformly distributed on $(0, 1)$ (see [10], Example 2.1, p. 17), we claim that a simulation of a $\mu_{(3,5)}$-equidistributed sequence $(x_n)_{n \leq M}$ on R (M is a "sufficiently large" natural number and depends on a representation quality of the irrational number π), where $\mu_{(3,5)}$ denotes a a linear Gaussian measure with parameters (3, 5), can be obtained by the formula

$$x_n = \Phi^{-1}_{(3,5)}(\pi \times n - [\pi \times n]), \tag{5.24}$$

for $n \leq M$, where $\Phi_{(3,5)}$ denotes a Gaussian distribution function with parameters (3, 5).

Suppose that we know a mean $a = 3$ and want to estimate an "unknown" standard deviation σ.

We set: n − the number of trials; S_n – a square root from the sample variance; S'_n – a square root from the corrected sample variance; $T_n^{(2)}$ - an estimate defined by the formula (5.13); $T_n^{(3)}$ - an estimate defined by the formula (5.19); and σ - an unknown standard deviation.

The numerical data placed in Table 3 were obtained by using Microsoft Excel. Notice that results of computations presented in Table 3 show us that both statistics $T_n^{(2)}$ and $T_n^{(3)}$ work correctly. Unfortunately, we can not present the results of computations for the statistics $T_n^{(1)}$ defined by the formula (5.10) because it needs a quite exact calculations.

Table 3. Estimates of an unknown standard deviation $\sigma = 5$

n	S_n	S'_n	$T_n^{(2)}$	$T_n^{(3)}$
200	4.992413159	5.004941192	5.205401325	4.895457577
400	4.992413159	5.004941192	5.141812921	4.835655399
600	5.10523925	5.109498942	5.211046737	4.855457413
800	5.106390271	5.109584761	5.19369988	4.92581015
1000	5.066642282	5.069177505	5.028142523	4.944169095
1200	5.072294934	5.074409712	5.235885276	4.935995814
1400	5.081110418	5.082926073	5.249446371	4.96528786
1600	5.079219075	5.080807075	5.205452797	4.9564705
1800	5.060850283	5.06225666	5.207913228	4.963326232
2000	5.063112113	5.064378366	5.239119585	4.981223889

At end of this section, we state the following:

Problem 5.5. Let $\mathcal{D}$ be a class of positive continuous probability densities and p_f be a Borel probability measure on $\mathbf{R}$ with probability density function f for each $f \in \mathcal{D}$. Does there exist an objective (or a subjective) infinite sample consistent estimate of an unknown probability density function f for the family of Borel probability measures $\{p_f^N : f \in \mathcal{D}\}$?

6. On Orthogonal Statistical Structures in a Non-Locally-Compact Polish Group Admitting an Invariant Metric

Let G be a Polish group, by which we mean a separable group with a complete invariant metric ρ (i.e., $\rho(fh_1g, fh_2g) = \rho(h_1, h_2)$ for each $f, g, h_1, h_2 \in G$) for which the transformation (from $G \times G$ onto G), which sends (x, y) into $x^{-1}y$ is continuous. Let $\mathcal{B}(G)$ denotes the σ-algebra of Borel subsets of G.

Definition 6.1 ([12]). A Borel set $X \subseteq G$ is called shy, if there exists a Borel probability measure μ over G such that $\mu(fXg) = 0$ for all $f, g \in G$. A measure μ is called a testing measure for a set X. A subset of a Borel shy set is called also shy. The complement of a shy set is called a prevalent set.

Definition 6.2 ([1]). A Borel set is called a Haar ambivalent set if it is neither shy nor prevalent.

Remark 6.3. Notice that if $X \subseteq G$ is shy then there exists such a testing measure μ for a set X which has a compact carrier $K \subseteq G$(i.e., $\mu(G \setminus K) = 0$). The collection of shy sets constitutes an σ-ideal and in the case where G is locally compact a set is shy iff it has Haar measure zero.

Definition 6.4. If G is a Polish group and $\{\mu_\theta : \theta \in \Theta\}$ is a family of Borel probability measures on G, then the family of triplets $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$, where Θ is a non-empty set equipped with an σ algebra $L(\Theta)$ generated by all singletons of Θ, is called a statistical structure. A set Θ is called a set of parameters.

Definition 6.5. $(\mathcal{O})$ The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called orthogonal if the measures μ_{θ_1} and μ_{θ_2} are orthogonal for each different parameters θ_1 and θ_2.

Definition 6.6. $(\mathcal{WS})$ The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called weakly separated if there exists a family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ such that $\mu_{\theta_1}(X_{\theta_2}) = \delta(\theta_1, \theta_2)$, where δ denotes Kroneckers function defined on the Cartesian square $\Theta \times \Theta$ of the set Θ.

Definition 6.7. $(\mathcal{SS})$ The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called strong separated (or strictly separated) if there exists a partition of the group G into family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ such that $\mu_\theta(X_\theta) = 1$ for each $\theta \in \Theta$.

Definition 6.8. $(\mathcal{CE})$ A $(\mathcal{B}(G), L(\Theta))$-measurable mapping $T: G \to \Theta$ is called a consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ if the condition $\mu_\theta(T^{-1}(\theta)) = 1$ holds true for each $\theta \in \Theta$.

Definition 6.9. $(\mathcal{OCE})$ A $(\mathcal{B}(G), L(\Theta))$-measurable mapping $T : G \to \Theta$ is called an objective consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ if the following two conditions hold:

(i) $\mu_\theta(T^{-1}(\theta)) = 1$ for each $\theta \in \Theta$;

(ii) $T^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$.

If the condition (i) holds but the condition (ii) fails, then T is called a subjective consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$.

Definition 6.10. $(\mathcal{SOCE})$ An objective consistent estimate $T: G \to \Theta$ of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called strong if for each $\theta_1, \theta_2 \in \Theta$ there exists an isometric Borel measurable bijection $A_{(\theta_1, \theta_2)} : G \to G$ such that the set $A_{(\theta_1, \theta_2)}(T^{-1}(\theta)) \Delta T^{-1}(\theta_2)$ is shy in G.

Remark 6.11. Let G be a Polish non-locally-compact group admitting an invariant metric. The relations between statistical structures introduced in Definitions 6.5-6.10 for such a group can be presented by the following diagram:

$$\mathcal{SOCE} \to \mathcal{OCE} \to \mathcal{CE} \leftrightarrow \mathcal{SS} \to \mathcal{WS} \to \mathcal{O}. \tag{6.1}$$

To show that the converse implications sometimes fail we consider the following examples:

Example 6.12. $\rceil(\mathcal{WS} \leftarrow \mathcal{O})$ Let $F \subset G$ be a closed subset of the cardinality $2^{\aleph_0}$. Let $\phi : [0, 1] \to F$ be a Borel isomorphism of $[0, 1]$ onto F. We set $\mu(X) = \lambda(\phi^{-1}(X \cap F))$ for $X \in \mathcal{B}(G)$, where λ denotes a linear Lebesgue measure on $[0, 1]$. We put $\Theta = F$. Let $\theta_0 \in \Theta$ and put: $\mu_\theta = \mu$ if $\theta = \theta_0$, and $\mu_\theta = \delta_\theta|_{\mathcal{B}(G)}$, otherwise, where δ_θ denotes a Dirac measure on G concentrated at the point θ and $\delta_\theta|_{\mathcal{B}(G)}$ denotes the restriction of the δ_θ to the class $\mathcal{B}(G)$. Then the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ stands $\mathcal{O}$ which is not $\mathcal{WS}$.

Example 6.13. (SM) $\rceil(\mathcal{SS} \leftarrow \mathcal{WS})$ Following [15] (see Theorem 1, p. 335), in the system of axioms (*ZFC*) the following three conditions are equivalent:

(1) The continuum hypothesis $(c = 2^{\aleph_0} = \aleph_1)$.

(2) For an arbitrary probability space $(E; S; \mu)$, the μ-measure of the union of any family $(E_i)_{i \in I}$ of μ-measure zero subsets, such that $card(I) < c$, is equal to zero.

(3) an arbitrary weakly separated family of probability measures, of cardinality continuum, is strictly separated.

The latter relation means that under continuum hypothesis in ZFC we have $\mathcal{SS} \leftarrow \mathcal{WS}$. This is just Skorohod well known result (see [26]). Moreover, following [15] (see Theorem 2, p.339), if (F, ρ) is a Radon metric space and $(\mu_i)_{i \in I}$ is a weakly separated family of Borel probability measures with $card(I) \leq c$, then in the system of axioms (*ZFC*) & (*MA*), the family $(\mu_i)_{i \in I}$ is strictly separated.

Let consider a counter example to the implication $\mathcal{SS} \leftarrow \mathcal{WS}$ in the Solovay model (*SM*) [25], which is the following system of axioms: (*ZF*) + *DC* + "every subset of the real axis **R** is Lebesgue measurable", where (*ZF*) denotes the Zermelo-Fraenkel set theory and (*DC*) denotes the axiom of dependent choices.

For $\theta \in (0; 1)$, let b_θ be a linear classical Borel measure defined on the set $\{\theta\} \times (0; 1)$. For $\theta \in (1; 2)$, let b_θ be a linear classical Borel measure defined on the set $(0; 1) \times \{\theta - 1\}$. By λ_θ we denote a Borel probability measure on $(0; 1) \times (0; 1)$ produced by b_θ, i.e.,

$$(\forall X)(\forall \theta_1)(\forall \theta_2)(X \in \mathcal{B}((0; 1) \times (0; 1)) \;\&\; \theta_1 \in (0; 1) \;\&\; \theta_2 \in (1; 2) \rightarrow$$

$$\lambda_{\theta_1}(X) = b_{\theta_1}((\{\theta_1\} \times (0; 1)) \cap X) \;\&\; \lambda_{\theta_2}(X) = b_{\theta_2}(((0; 1) \times \{\theta_1 - 1\}) \cap X)). \tag{6.2}$$

If we put $\theta = (0; 1) \cup (1; 2)$, then we get a statistical structure

$$((0; 1) \times (0; 1), \mathcal{B}((0; 1) \times (0; 1)), \lambda_\theta)_{\theta \in \Theta}. \tag{6.3}$$

Setting $X_\theta = \{\theta\} \times (0; 1)$ for $\theta \in (0; 1)$, and $X_\theta = (0; 1) \times \{\theta - 1\}$ for $\theta \in (1.2)$, we observe that for the family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ we have $\lambda_{\theta_1}(X_{\theta_2}) = \delta(\theta_1, \theta_2)$, where δ denotes Kroneckers function defined on the Cartesian square $\Theta \times \Theta$ of the set Θ. In the other words, $(\lambda_\theta)_{\theta \in \Theta}$ is weakly separated. Now, let assume that this family is strong separated. Then, there will be a partition $\{Y_\theta : \theta \in \Theta\}$ of the set $(0; 1) \times (0; 1)$ into Borel subsets $(Y_\theta)_{\theta \in \Theta}$ such that $\lambda_\theta(Y_\theta) = 1$ for each $\theta \in \Theta$. If we consider $A = \cup_{\theta \in (0;1)} Y_\theta$ and $B = \cup_{\theta \in (1;2)} Y_\theta$, then we observe by Fubini theorem that $\ell_2(A) = 1$ and $\ell_2(B) = 1$, where ℓ_2 denotes the 2-dimensional Lebesgue measure defined in $(0; 1) \times (0; 1)$. This is the contradiction and we proved that $(\lambda_\theta)_{\theta \in \Theta}$ is not strictly separated. An existence of a Borel isomorphism g between $(0; 1) \times (0; 1)$ and G allows us to construct a family $(\mu_\theta)_{\theta \in \Theta}$ in G as follows: $\mu_\theta(X) = \lambda_\theta(g^{-1}(X))$ for each $X \in \mathcal{B}(G)$ and $\theta \in \Theta$, which is $\mathcal{WS}$ but no $\mathcal{SS}$ (equivalently, $\mathcal{CE}$). By virtue the celebrated result of Mycielski and Swierczkowski (see [11]) asserted that under the axiom of determinacy (*AD*) every subset of the real axis **R** is Lebesgue measurable, the same example can be used as a counter example to the implication $\mathcal{SS} \leftarrow \mathcal{WS}$ in the theory (ZF) + (DC) + (AD). Since the answer to the question asking "whether $(\mu_\theta)_{\theta\in\Theta}$ has a consistent estimate?" is **yes** in the theory (*ZFC*) & (*CH*), and **no** in the theory (*ZF*) + (*DC*) + (*AD*), we deduce that this question is not solvable within the theory (*ZF*) + (*DC*).

Example 6.14. $\rceil(\mathcal{OCE} \leftarrow \mathcal{CE})$ Setting $\Theta = G$ and $\mu_\theta = \delta_\theta|\mathcal{B}(G)$ for $\theta \in \Theta$, where δ_θ denotes a Dirac measure in G concentrated at the point θ and $\delta_\theta|\mathcal{B}(G)$ denotes its restriction to $\mathcal{B}(G)$, we get a statistical structure $(G, \mathcal{B}(G), \mu_\theta)_{\theta\in\Theta}$. Let $L(\Theta)$ denotes a minimal σ-algebra of subsets of Θ generated by all singletons of Θ. Setting $T(g) = g$ for $g \in G$, we get a consistent estimate of an unknown parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$. Notice that there does not exist an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$. Indeed, if we assume the contrary and T_1 be such an estimate, we get that $T_1^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$. Since T_1 is a consistent estimate of an unknown parameter θ for each $\theta \in \Theta$, we get that the condition $\mu_\theta(T_1^{-1}(\theta)) = 1$ holds true which implies that $\theta \in T_1^{-1}(\theta)$ for each $\theta \in \Theta$. Let fix any parameter $\theta_0 \in \Theta$. Since $T_1^{-1}(\theta_0)$ is a Haar ambivalent set there is $\theta_1 \in T_1^{-1}(\theta_0)$ which differs from θ_0. Then $T_1^{-1}(\theta_0)$ and $T_1^{-1}(\theta_1)$ are not disjoint because $\theta_1 T_1^{-1}(\theta_0) \cap T_1^{-1}(\theta_1)$ and we get the contradiction.

Remark 6.15. Notice that if (Θ, ρ) is a metric space and if in the Definition 2.8 the requirement of a $(\mathcal{B}(G), L(\Theta))_i$-measurability will be replaced with a $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurability, then the implication $\mathcal{SS} \rightarrow \mathcal{CE}$ may be false. Indeed, let G be a Polish group and $f : G \leftarrow \Theta(:= G)$ be a non-measurable (in the Borel sense) bijection. For each $\theta \in \Theta$ denote by μ_θ the restriction of the Dirac measure $\delta_{f(\theta)}$ to the σ-algebra of Borel subsets of the group G. It is clear that the statistical structure

$\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is strictly separated. Let show that there does not exist a consistent estimate for that statistical structure. Indeed, let $T : G \to \Theta$ be $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable mapping such that $\mu_\theta(\{x : T(x) = \theta\}) = 1$ for each $\theta \in \Theta$. Since the measure μ_θ is concentrated at the point $f(\theta)$, we have that $f(\theta) \in \{x : T(x) = \theta\}$ for each $\theta \in \Theta$ which implies that $T(f(\theta)) = \theta$ for each $\theta \in \Theta$. The latter relation means that $T = f^{-1}$. Since f is not $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable we claim that $f^{-1} = T$ is not also $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable and we get the contradiction.

There naturally arises a question asking whether there exists such a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in a Polish non-locally-compact group admitting an invariant metric, which has an objective consistent estimate of a parameter θ. To answer positively to this question, we need the following two lemmas:

Lemma 6.16 ([24], Theorem, p. 206)**.** *Assume G is a Polish non-locally-compact group admitting an invariant metric. Then, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$, there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.*

Lemma 6.17 ([5], Proposition 12, p.87)**.** *Let G be a non-locally-compact Polish group with an invariant metric. Then any compact subset (and hence any K_σ subset) of G is shy.*

Remark 6.18. In [19] (see proof of Theorem 4.1, Step 2) has been constructed a partition $\Phi = \{A_\theta : \theta \in [0, 1]\}$ of the $\mathbf{R}^N$ into Haar ambivalent sets such that for each $\theta_1, \theta_2 \in [0, 1]$, there exists an isometric (with respect to Tychonoff metric which is invariant under translates) Borel measurable bijection $A_{(\theta_1, \theta_2)}$ of $\mathbf{R}^N$ such that $A_{(\theta_1, \theta_2)}(A_{\theta_1}) \Delta A_{\theta_2}$ is shy. In this context and concerning with Lemma

6.16, it is natural to ask whether an arbitrary Polish non-locally-compact group with an invariant metric admits a similar partition into Haar ambivalent sets. Notice that we have no any information in this direction.

Theorem 6.19. *Let G be a Polish non-locally-compact group admitting an invariant metric. Then, there exists a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G which has an objective consistent estimate of a parameter θ such that*:

(i) $\Theta \subseteq G$ *and* $card(\Theta) = 2^{\aleph_0}$;

(ii) μ_θ *is the restriction of the Dirac measure concentrated at the point θ to the Borel σ-algebra $\mathcal{B}(G)$ for each $\theta \in \Theta$.*

Proof. By virtue of Lemma 6.16, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$, there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$. For $x \in 2^N \setminus \{(0, 0, \dots)\}$, we put

$$X_x = \phi^{-1}(x). \tag{6.4}$$

We set $X_{(0,0,\dots)} = \phi^{-1}((0, 0, \cdots)) \cup (G \setminus F)$. Thus, we have a partition $\{X_x : x \in 2^N\}$ of G into Borel subsets such that each element of the partition is Borel measurable and Haar ambivalent set. Let $\{\theta_x : x \in 2^N\}$ be any selector. We put $\Theta = \{\theta : \theta = \theta_x$ for some $x \in 2^N\}$ and denote by μ_θ the restriction of the Dirac measure concentrated at the point θ to the σ-algebra $\mathcal{B}(G)$. Thus, we have constructed a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G. We put $T(g) = \theta$ for each $g \in X_\theta$. Now, it is obvious that T is the objective consistent estimate of a parameter θ for the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G such that the conditions (i)-(ii) are fulfilled. □

Theorem 6.20. *Let G be a Polish non-locally-compact group admitting an invariant metric. Let μ be a Borel probability measure whose carrier is a compact set K_0(i.e., $\mu(G \setminus K_0) = 0$). Then, there exists a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G which has an objective consistent estimate of a parameter θ such that*

(i) $\Theta \subseteq G$ *and* $card(\Theta) = 2^{\aleph_0}$;

(ii) μ_θ *is a* θ*-shift of the measure* μ(*i.e.*, $\mu_\theta(X) = \mu(\theta^{-1}X)$ *for* $X \in \mathcal{B}(G)$ *and* $\theta \in \Theta$).

Proof. By virtue of Lemma 6.16, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$, there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$. For $x \in 2^N \setminus\{(0, 0, \cdots)\}$, we put $X_x = \phi^{-1}(x)$. We set $X_{(0,0,\ldots)} = \phi^{-1}((0, 0, \cdots)) \cup (G\setminus F)$. Thus, we have a partition $\{X_x : x \in 2^N\}$ of G into Borel subsets such that each element of the partition is Borel measurable, Haar ambivalent set and for any $x \in 2^N$ and any compact set $K \subseteq G$, there is $g \in G$ with $gK \subseteq X_x$. If we take under K a set K_0, then for any $x \in 2^N$, there is $g(K_0, x) \in G$ with $g(K_0, x)K_0 \subseteq X_x$. We put $\Theta = \{\theta : \theta = g(K_0, x)\ \&\ x \in 2^N\}$. For each $\theta \in \Theta$ and $X \in \mathcal{B}(G)$, we put $\mu_\theta(X) = \mu(\theta^{-1}X)$. For $g \in X_x$, we put $T(g) = g(K_0, x)$. Let us show that $T : G \to \Theta$ is an objective consistent estimate of a parameter θ. Indeed, on the one hand, for each $\theta \in \Theta$, we have

$$\mu_\theta(T^{-1}(\theta)) = \mu_{g(K_0,x)}(T^{-1}(g(K_0, x))) = \mu_{g(K_0,x)}(X_x) = \mu(g(K_0, x)^{-1}X_x)$$

$$\geq \mu(g(K_0, x)^{-1}g(K_0, x)K_0) = \mu(K_0) = 1, \tag{6.5}$$

which means that $T : G \to \Theta$ is a consistent estimate of a parameter θ. On the other hand, for each $\theta = g(K_0, x) \in \Theta$, we have that a set $T^{-1}(\theta)$

$= T^{-1}(g(K_0, x)) = X_x$ is Borel measurable and a Haar ambivalent set which together with the latter relation implies that $T : G \to \Theta$ is an objective consistent estimate of a parameter θ. Now, it is obvious to check that for the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ the conditions (i)-(ii) are fulfilled. □

The next theorem shows whether can be constructed an objective consistent estimates by virtue of some consistent estimates in a Polish non-locally-compact group admitting an invariant metric.

Theorem 6.21. *Assume G is a Polish non-locally-compact group admitting an invariant metric. Let $card(\Theta) = 2^{\aleph_0}$ and $T : G \to \Theta$ be a consistent estimate of a parameter θ for the family of Borel probability measures $(\mu_\theta)_{\theta \in \Theta}$ such that there exists $\theta_0 \in \Theta$ for which $T^{-1}(\theta_0)$ is a prevalent set. Then, there exists an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta \in \Theta}$.*

Proof. For $\theta \in \Theta$ we put $S_\theta = T^{-1}(\theta)$. Since S_{θ_0} is a prevalent set, we deduce that

$$\cup_{\theta \in \Theta \setminus \{\theta_0\}} S_\theta = \mathbf{R}^N \setminus S_{\theta_0}, \tag{6.6}$$

is shy in G.

By Lemma 2.6, we know that the measure μ_{θ_0} is concentrated on a union of a countable family of compact subsets $\{F_k^{(\theta_0)} : k \in N\}$. By Lemma 6.7, we know that $\cup_{k \in N} F_k^{(\theta_0)}$ is shy in G.

We put $\widetilde{S}_\theta = S_\theta$ for $\theta \in \Theta \setminus \{\theta_0\}$ and $\widetilde{S}_{\theta_0} = \cup_{k \in N} F_k^{(\theta_0)}$. Clearly, $S = \cup_{\theta \in \Theta} \widetilde{S}_\theta$ is shy in G.

By virtue of Lemma 6.16, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$, there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$. Let $f : 2^N \to \Theta$ be any bijection. For $\theta \in \Theta$, we put

$$B_\theta = (\phi^{-1}(f^{-1}(\theta)) \setminus S) \cup S_\theta. \tag{6.7}$$

Notice that $(B_\theta)_{\theta \in \Theta}$ is a partition of G into Haar ambivalent sets. We put $T_1(g) = \theta$ for $g \in B_\theta(\theta \in \Theta)$. Since

$$\mu_\theta(T_1^{-1}(\theta)) = \mu_\theta(B_\theta) \geq \mu_\theta(S_\theta) = 1, \tag{6.8}$$

for $\theta \in \Theta$, we claim that T_1 is a consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta \in \Theta}$. Since $T_1^{-1}(\theta) = B_\theta$ is a Borel and Haar ambivalent set for each $\theta \in \Theta$ we end the proof of the theorem. □

Example 6.22. Let F be a distribution function on $\mathbf{R}$ such that the integral $\int_{\mathbf{R}} x dF(x)$ exists and is equal to zero. Suppose that p is a Borel probability measure on $\mathbf{R}$ defined by F. For $\theta \in \Theta(:= \mathbf{R})$, let p_θ be θ-shift of the measure p(i.e., $p_\theta(X) = p(X - \theta)$ for $X \in \mathcal{B}(\mathbf{R})$). Setting, $G = \mathbf{R}^N$, for $\theta \in \Theta$, we put $\mu_\theta = p_\theta^N$, where p_θ^N denotes the infinite power of the measure p_θ. We set $T((x_k)_{k \in N}) = \lim_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n}$, if $\lim_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n}$ exists, is finite and differs from the zero, and $T((x_k)_{k \in N}) = 0$, otherwise. Notice that $T : \mathbf{R}^N \to \Theta$ is a consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta \in \Theta}$ such that $T^{-1}(0)$ is a prevalent set. Indeed, by virtue the strong law of large numbers, we know that

$$\mu_\theta(\{(x_k)_{k \in N} : \lim_{n \to \infty} \frac{\sum_{k=1}^n x_k}{n} = \theta\}) = 1, \tag{6.9}$$

for $\theta \in \Theta$.

Following [18] (Lemma 4.14, p. 60), a set S defined by

$$S = \{(x_k)_{k \in N} : \lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{n} \text{ exists and is finite}\}, \tag{6.10}$$

is Borel shy set, which implies that $\mathbf{R}^N \setminus S$ is a prevalent set. Since $\mathbf{R}^N \setminus S \subseteq T^{-1}(0)$, we deduce that $T^{-1}(0)$ is a prevalent set. Since for the statistical structure $\{(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta) : \theta \in \Theta\}$ all conditions of the Theorem 6.21 are fulfilled, we claim that there exists an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta \in \Theta}$.

Notice that in Theorem 4.1 (see also [19], Theorem 3.1, p. 117) has been considered an example of a strong objective infinite sample consistent estimate of an unknown parameter for a certain statistical structure in the Polish nonlocally compact abelian group $\mathbf{R}^N$. In context with this example, we state the following:

Problem 6.1. Let G be a Polish non-locally-compact group admitting an invariant metric. Does there exist a statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ with $\mathrm{card}(\Theta) = 2^{\aleph_0}$ for which there exists a strong objective consistent estimate of a parameter θ?

7. On Objective and Strong Objective Consistent Estimates of an Unknown Parameter in a Compact Polish Group $\{0, 1\}^N$

Let $x_1, x_2, \cdots, x_k, \cdots$ be an infinite sample obtained by coin tosses. Then the statistical structure described this experiment has the form:

$$\{(\{0, 1\}^N, B(\{0, 1\}^N), \mu_\theta^N) : \theta \in (0, 1)\}, \tag{7.1}$$

where $\mu_\theta(\{1\}) = \theta$ and $\mu_\theta(\{0\}) = 1 - \theta$. By virtue of the strong law of large numbers, we have

$$\mu_\theta^N(\{(x_k)_{k\in N} : (x_k)_{k\in N} \in \{0, 1\}^N \ \& \ \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = \theta\}) = 1, \qquad (7.2)$$

for each $\theta \in (0, 1)$.

Notice that for each $k \in N$, $G_k = \{0, 1\}$ can be considered as a compact group with an addition group operation (mod 2). Hence, the space of all infinite samples $G := \{0, 1\}^N$ can be presented as an infinite product of compact groups $\{G_k : k \in N\}$, i.e., $G = \prod_{k\in N} G_k$. Also, that the group G admits an invariant metric ρ, which is defined by $\rho((x_k)_{k\in N}, (y_k)_{k\in N}) = \sum_{k\in N} \frac{|x_k - y_k(\text{mod } 2)|}{2^{k+1}(1 + |x_k - y_k(\text{mod } 2)|)}$ for $(x_k)_{k\in N}$, $(y_k)_{k\in N} \in G$. It is obvious that the measure λ_k on G_k defined by $\lambda_k(\{0\}) = \lambda_k(\{1\}) = 1/2$ is a probability Haar measure in G_k for each $k \in N$ and for the probability Haar measure λ in G the following equality $\lambda = \prod_{k\in N} \lambda_k$ holds true, equivalently, $\lambda = \mu_{0,5}^N$.

By virtue (7.2), we deduce that the set

$$A(0, 5) = \{(x_k)_{k\in N} : (x_k)_{k\in N} \in \{0, 1\}^N \ \& \ \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = 0, 5\}, \qquad (7.3)$$

is prevalence. Since $A(\theta) \subset G \setminus A(0, 5)$ for each $\theta \in (0; 1) \setminus \{1/2\}$, where

$$A(\theta) = \{(x_k)_{k\in N} : (x_k)_{k\in N} \in \{0, 1\}^N \ \& \ \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = \theta\}, \qquad (7.4)$$

we deduce that they all are shy (equivalently, of Haar measure zero) sets. In terms of [7], this phenomena can be expressed in the following form:

Theorem 7.1. *For "almost every" sequence* $(x_k)_{k \in N} \in \{0, 1\}^N$, *its Cezaro means* $(\frac{\sum_{k=1}^{n} x_k}{n})_{n \in N}$ *converges to* 0, 5 *whenever n tends to* ∞.

By virtue the strong law of large numbers, we get

Theorem 7.2. *Let fix* $\theta_0 \in (0, 1)$. *For each* $(x_k)_{k \in N} \in G$, *we set* $T((x_k)_{k \in N}) = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} x_k}{n}$ *if this limit exists and differs from* θ_0, *and* $T((x_k)_{k \in N}) = \theta_0$, *otherwise. Then T is a consistent estimate of an unknown parameter* θ *for the statistical structure* $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$.

Remark 7.3. Following Definition 6.9, the estimate T is subjective because $T^{-1}(1/2)$ is a prevalent set. Unlike Theorem 6.21, there does not exist an objective consistent estimate of an unknown parameter θ for any statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ for which $\mathrm{card}(\Theta) > \aleph_0$, where $\aleph_0$ denotes the cardinality of the set of all natural numbers. Indeed, assume the contrary and let T_1 be such an estimate. Then, we get the partition $\{T_1^{-1}(\theta) : \theta \in \Theta\}$ of the compact group G into Haar ambivalent sets. Since each Haar ambivalent set is of positive λ-measure, we get that the probability Haar measure λ does not satisfy Suslin property provided that the cardinality of an arbitrary family of pairwise disjoint Borel measurable sets of positive λ-measure in G is not more than countable.

Remark 7.4. Let consider a mapping $F : G \to [0, 1]$ defined by $F((x_k)_{k \in N}) = \sum_{k \in N} \frac{x_k}{2^k}$ for $(x_k)_{k \in N} \in G$. This is a Borel isomorphism between G and $[0, 1]$ such that the following equality $\lambda(X) = \ell_1(F(X))$ holds true for each $X \in \mathcal{B}(G)$. By virtue the latter relation, for each natural number m, the exists a partition $\{X_k : 1 \leq k \leq m\}$ of the group G

into Haar ambivalent sets such that for each $1 \leq i \leq j \leq m$, there is an isometric Borel measurable bijection $f_{(i,j)} : G \to G$ such that the set $f_{(i,j)}(X_i)\Delta X_j$ is shy, equivalently, of the λ-measure zero.

By the scheme presented in the proof of the Theorem 6.21, one can get the validity of the following assertions:

Theorem 7.5. *Let* Θ_1 *be a subset of the* Θ *with* $card(\Theta) \geq 2$. *Then, there exists an objective consistent estimate of an unknown parameter* θ *for the statistical structure* $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta_1\}$ *if and only if* $card(\Theta_1) \leq \aleph_0$ *and* $1/2 \notin \Theta_1$.

Theorem 7.6. *Let* Θ_2 *be a subset of the* Θ *with* $card(\Theta) \geq 2$. *Then, there exists a strong objective consistent estimate of an unknown parameter* θ *for the statistical structure* $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta_2\}$ *if and only if* $card(\Theta_2) < \aleph_0$ *and* $1/2 \notin \Theta_2$.

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