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ON THE TAYLOR COEFFICIENTS OF BLASCHKE AND DJRBASHYAN PRODUCTS AND THEIR CONNECTING FUNCTIONS

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Abstract

Using the Riemann-Liouville integration-differentiation operator, Djrbashyan generalized the class of Nevanlinna's meromorphic functions in the unit circle including the product $B_\alpha(-1 < \alpha < +\infty)$, which in the special case of $\alpha = 0$ coincide with the Blaschke product. Furthermore, when $-1 < \alpha < 0$, Djrbashyan and Zakaryan showed a connection between the products B_α and B of Blaschke.

In this work, we show the existence of Blaschke and Djrbashyan products with the same null sets, Taylor-Maclaurin coefficients satisfying certain new constraints. To achieve that result, we use a theorem of Shapiro and Shields and a remark from Zakaryan. We further estimate the Taylor coefficients of these functions.

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1. Introduction

Let $\mathbb{D} = \{z; |z|; <\}$ be the unit circle on the complex plane. [1] studied classes S_α of analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ in $\mathbb{D}$, which Taylor-Maclaurin coefficients satisfy the following condition:

$$\sum_{n=1}^{\infty} n^\alpha |a_n|^2 < +\infty, \quad 0 < \alpha < 1.$$

Let $0 < p < +\infty, -1 < \alpha < +\infty$ and define D_α^p as the set of analytic functions satisfying the following condition:

$$\left\|D_\alpha^p\right\|^p = \int_0^r \int_0^{2\pi} (1-r)^\alpha \left|f'\left(re^{i\theta}\right)\right|^p r dr d\theta < +\infty.$$

When $\alpha + 1 < p$, classes D_α^p are called Dirichlet-type classes.

The set $\{a_n\} \subset D$ is called the uniqueness set for class D_α^2, if from $f(a_n) = 0$ and $f \in D_\alpha^2$ it follows that $f \equiv 0$. [9] solves the problem of when the sequence $\{a_n\}$ is a set of uniqueness for the class D_α^2, $0 \le \alpha < 1$ and a comprehensive answer is given. For more general classes $D^2(\omega)$ using the same approach a similar result is given in [10].

In this work, we study the question of certain analytic in the unit circle functions belonging to the Dirichlet-type class D_α^2, $0 < \alpha < 1$. The connection between classes S_α and D_α^2, $0 < \alpha < 1$, was first given (to the best of the authors knowledge) in [11], where it was shown that $f \in D_\alpha^2$, $0 < \alpha < 1$ if and only if $f \in S_{1-\alpha}$. Using this remark and the fact that functions of our interest belong to the class D_α^2, $0 < \alpha < 1$, we obtain several new results on the Taylor-Maclaurin coefficients of these functions.

Let the sequence of the complex numbers $\{z_n\} \subset \mathbb{D}$ be such that

$$\sum_{n=1}^{\infty}(1-|z_n|) < +\infty. \tag{1}$$

Blaschke product is the following function:

$$B(z;\{z_n\}) = \prod_{n=1}^{\infty}\frac{z_n - z}{1-\bar{z}_n z}\cdot\frac{|z_n|}{z_n}, \quad z \in \mathbb{D}.$$

The properties of Blaschke product can be found, for example, in [7] and [8].

In ([3], Chapter IX), [6] has introduced the classes $N_\alpha(-1 < \alpha < +\infty)$ of meromorphic in the unit circle functions and derived their parametric representation. The class $N_\alpha(-1 < \alpha < +\infty)$ is defined by the α-characteristic

$$T_\alpha(r, F) = m_\alpha(r, F) + N_\alpha(r, F),$$

as the set of those meromorphic in the circle $\mathbb{D}\ |z| < 1$ functions F, for which

$$\sup_{0<r<1}\{T_\alpha(r, F)\} < +\infty.$$

Note that the functions $m_\alpha(r, F)$, $N_\alpha(r, F)$ and $T_\alpha(r, F)$ are original analogues of known Nevanlinna functions $m(r, F)$, $N(r, F)$, and $T(r, F)$, coinciding with them for $\alpha = 0$, therefore the class N_0 coincides with Nevanlinna class N.

Furthermore, an important property of classes N_α is the fact that for any values $-1 < \alpha_1 < \alpha_2 < +\infty$, the following strict inclusion $N_{\alpha_1} \subset N_{\alpha_2}$ holds, and in particular,

$$N_\alpha \subset N_0 = N, \quad (-1 < \alpha < 0).$$

The Riemann-Liouville integration-differentiation operator $D^{-\alpha}$ (for $-1 < \alpha < +\infty$) with the zero origin initial null point is defined as

$$D^{-\alpha}\{\varphi(r)\} = \frac{1}{\Gamma(\alpha)}\int_0^r (r-t)^{\alpha-1}\varphi(t)dt, \quad (0 < \alpha < +\infty),$$

$$D^0\{\varphi(r)\} = \varphi(r),$$

$$D^{-\alpha}\{\varphi(r)\} = \frac{d}{dr}D^{-(1+\alpha)}\{\varphi(r)\}, \quad (-1 < \alpha < 0).$$

Define function $T_\alpha(r,\ F)$ for analytic functions from class N_α as follows:

$$T_\alpha(r,\ F) = \int_{-\pi}^{\pi} D_{(+)}^{-\alpha}\ \log\left|f\left(re^{i\theta}\right)\right| d\theta,$$

where

$$D_{(+)}^{-\alpha}\{\varphi(r)\} \overset{\text{def}}{=} \max\left\{D^{-\alpha}\{\varphi(r)\};\ 0\right\}.$$

It is well-known (see [3], [6]) that analytic functions from class N_α have the following form:

$$f(z) = e^{i\gamma+\lambda k_\alpha} z^\lambda B_\alpha(z;\ \{a_n\}) \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} s_\alpha\left(e^{-i\theta}z\right)d\psi(\theta)\right\},$$

where γ is a real number, λ is a positive integer, $K_\alpha = \alpha\sum_{n=1}^{\infty}\frac{1}{n(n+\alpha)}$, and $\psi(\theta)$ is a realvalued function with a finite total variation on $[-\pi;\ \pi]$,

$$B_\alpha(z;\ \{a_n\}) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{-W_\alpha(z;a_n)},$$

$$W_\alpha(z,\,\xi) = \int_{|\xi|}^{1} \frac{(1-x)^\alpha}{x}\,dx$$

$$-\sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha+k)}{\Gamma(1+\alpha)\Gamma(1+k)} \left\{ \xi^{-k} \int_0^{|\xi|} (1-x)^\alpha x^{k-1} dx - \overline{\xi}^{k} \int_{|\xi|}^{1} (1-x)^\alpha x^{k-1} dx \right\} z^k,$$

$$|z| < 1, \quad |\xi| < 1,$$

$$S_\alpha(z) = \frac{1}{\Gamma(1+\alpha)} \left[\frac{2}{(1-z)^{1+\alpha}} \right], \quad |z| < 1.$$

The function $S_\alpha(z)$ is called the Djrbashyan kernel of Schwartz type. $\operatorname{Re} S_\alpha(z)$ is the Djrbashyan kernel of Poisson type. For $\alpha = 0$, those kernels coincide with the Schwartz and Poisson kernels, respectively.

$B_\alpha(z;\,\{a_n\})$ is called Djrbashyan product; and for the special case $\alpha = 0$ product B_α coincides with the Blaschke product ([3], [6]).

$$B_0(z;\,\{a_n\}) = B(z;\,\{a_n\}) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a}_n z} \frac{|a_n|}{a_n}.$$

In [5], the following was proven:

Theorem (On connection between B_α and B)**.** *Given that*

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{1+\alpha} < +\infty, \quad (-1 < \alpha < 0), \tag{2}$$

the following holds:

$$B_\alpha(z;\,\{a_n\}) = B(z;\,\{a_n\}) \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} S_\alpha\left(e^{-i\theta} z\right) d\omega(\theta) \right\},$$

where $\omega(\theta)$ *is a non-increasing function of bounded variation on* $[0,\,2\pi]$ *of the type*

$$\omega(\theta) = \lim_{n\to\infty} \int_0^{\theta} D^{-\alpha} \log \left| \frac{B_\alpha\left(r_n e^{i\theta}; \{a_n\}\right)}{B\left(r_n e^{i\theta}; \{a_n\}\right)} \right| d\theta, \tag{3}$$

$$\left(0 < r_1 < r_2 < \ldots < r_n < \ldots,\ r_n \uparrow 1\right).$$

We also remark the definitions of classes $T_\beta (0 < \beta < 1)$ included in class $N = N_0$. Let $w(z) \in T_\beta$, if

$$\int_0^1 \frac{A(r, w)}{(1-r)^\beta} dr < +\infty,$$

where

$$A(r, w) = \iint_{|z|<r} \frac{|w'(z)|^2}{\left(1+|w'(z)|^2\right)^2} dxdy.$$

It is known [4] that if $w(z) = \sum_{n=1}^{\infty} a_n z^n \in T_\beta (0 < \beta < 1)$ and it is bounded, then

$$\sum_{n=1}^{\infty} |a_n|^2 n^\beta < +\infty.$$

2. Main Results

Theorem 1. *Let* $-1 < \alpha < 0$ *and let* $\{z_n\} \subset \mathbb{D}$ *be a sequence of complex numbers such that condition* (2) *is satisfied. Then*

$$B_\alpha(z; \{z_n\}) \in D^2_{1+\alpha}, \quad B(z; \{z_n\}) \in D^2_{1+\alpha}.$$

Proof. Given (2) in [2] it was shown, that $B(z; \{z_n\}) \in T_{-\alpha}$, and in [4] it was shown, that $B_\alpha(z; \{z_n\}) \in T_{-\alpha}$. This means that the following statements hold for their Taylor-Maclaurin coefficients:

$$\sum_{n=1}^{\infty}\left|\hat{B}(n)\right|^2 n^{-\alpha} < +\infty,$$

$$\sum_{n=1}^{\infty}\left|\hat{B}_{\alpha}(n)\right|^2 n^{-\alpha} < +\infty, \quad -1 < \alpha < 0.$$

Therefore, the products $B(z; \{z_n\})$ and $B_{\alpha}(z; \{z_n\})$ belong to the class $S_{-\alpha}$. On the other hand, it is not difficult to verify the following statements (see, e.g., [11]):

$$f \in S_{-\alpha} \Leftrightarrow f \in D^2_{1+\alpha}.$$

Hence, the proof of the theorem follows.

Remark. This theorem is known for Blaschke product. But the authors believe that this proof presents particular interest since it is based on the early results of famous mathematicians.

Theorem 2. *Let* $-1 < \alpha < 0$ *and let* $\{z_n\} \subset \mathbb{D}$ *be a sequence of complex numbers such that the condition* (2) *is satisfied. Then*

$$g_{\alpha}(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} S_{\alpha}\left(e^{-i\theta}z\right)d\omega(\theta)\right\}, \quad |z| < 1,$$

belongs to the class $D^2_{1+\alpha}$, *where* ω *is a non-increasing function of bounded variation on* $[0, 2\pi]$ *having the form* (3).

Proof. From the theorem about the Blaschke and Djrbashyan products, we have

$$B_{\alpha}(z; \{z_n\}) = g_{\alpha}(z) \cdot B(z; \{z_n\}).$$

Hence,

$$|B'_{\alpha}(z; \{z_n\})|^2 = |g'_{\alpha}(z) \cdot B(z; \{z_n\}) + g_{\alpha}(z) \cdot B'(z; \{z_n\})|^2$$

$$\geq (|g'_{\alpha}(z) \cdot B(z; \{z_n\})| - |g_{\alpha}(z) \cdot B'(z; \{z_n\})|)^2$$

$$= |g'_\alpha(z)|^2 \cdot |B(z;\ \{z_n\})|^2 - 2|g'_\alpha(z)| \cdot |B(z;\ \{z_n\})|\,|g_\alpha(z)|$$

$$\times |B'(z;\ \{z_n\})| + |g_\alpha(z)|^2 \cdot |B'(z;\ \{z_n\})|^2.$$

Applying Hölder's inequality, we have

$$\int_0^{2\pi} \left|B'_\alpha\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi \geq \int_0^{2\pi} \left|g'_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi$$

$$- 2\left\{\int_0^{2\pi} \left|g'_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi\right\}^{\frac{1}{2}}$$

$$\times \left\{\int_0^{2\pi} \left|g_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi\right\}^{\frac{1}{2}}$$

$$+ \int_0^{2\pi} \left|g_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi.$$

Once again applying Hölder's inequality, we obtain

$$\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|B'_\alpha\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr$$

$$\geq \int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g'_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr$$

$$- 2\left\{\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g'_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr\right\}^{\frac{1}{2}}$$

$$\times \left\{\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr\right\}^{\frac{1}{2}}$$

$$+ \int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr.$$

Therefore,

$$\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|B'_\alpha\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr$$

$$\geq \left\{ \left[\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g'_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr \right]^{\frac{1}{2}} \right.$$

$$\left. - \left[\int_0^1 (1-r)^{1+\alpha} \int_0^{2\pi} \left|g_\alpha\left(re^{i\varphi}\right)\right|^2 \cdot \left|B'\left(re^{i\varphi};\ \{z_n\}\right)\right|^2 d\varphi dr \right]^{\frac{1}{2}} \right\}^2.$$

It is known from [1] that, if (2) holds, then

$$\lim_{r\to 1-} \left|B\left(re^{i\varphi};\ \{z_n\}\right)\right| = 1,$$

except may be of a set $E \subset \partial D$ with zero $(1+\alpha)$-capacity.

Then, since $0 < |g_\alpha(z)| \leq 1$, it is not difficult to see that there exists a positive constant c, such that

$$\|B_\alpha\|_{D^2_{1+\alpha}} \geq c\|g_\alpha\|_{D^2_{1+\alpha}} - \|B\|_{D^2_{1+\alpha}}.$$

From Theorem 1, we have $B_\alpha \in D^2_{1+\alpha}$ and $B \in D^2_{1+\alpha}$. From these facts and taking into account the last inequality, it follows that $g_\alpha \in D^2_{1+\alpha}$.

Remark. If ω is a non-increasing function of bounded variation not of form (3), then the statements of the theorem may not be true.

From Theorem 2 and from the fact that for $-1 < \alpha < 0$,

$$f \in S_{-\alpha} \Leftrightarrow f \in D^2_{1+\alpha},$$

the following holds:

Theorem 3. *Let* $-1 < \alpha < 0$ *and let* $\{z_n\} \subset \mathbb{D}$ *be a sequence of complex numbers such that the condition* (2) *is satisfied. Then for the Taylor-Maclaurin coefficients of the function* $g_\alpha(z)$, *the following holds*:

$$\sum_{n=1}^{\infty} |\hat{g}_\alpha(z)|^2 n^{-\alpha} < +\infty.$$

It is not difficult to prove the following theorem:

Theorem 4. *Let* $-1 < \alpha < 0$ *and let* $\{z_n\} \subset \mathbb{D}$ *be a sequence of complex numbers such that the condition* (2) *is satisfied. Then*

$$\hat{B}_\alpha(n) = \sum_{k=0}^{n} \hat{g}_\alpha(k) \cdot \hat{B}(n-k); \tag{1}$$

$$\hat{B}(n) = \sum_{k=0}^{n} \hat{h}_\alpha(k) \cdot \hat{B}_\alpha(n-k), \tag{2}$$

where $h_\alpha(z) = [g_\alpha(z)]^{-1}$.

Similarly, one can verify that the following theorem holds:

Theorem 5. *Let* $-1 < \alpha < 0$ *and let* $\{z_n\} \subset \mathbb{D}$ *be a sequence of complex numbers such that condition* (2) *is satisfied. Then we have the following connection between the Taylor-Maclaurin coefficients of the functions* $g_\alpha(z)$ *and* $h_\alpha(z) = [g_\alpha(z)]^{-1}$:

$$\hat{g}_\alpha(0) \cdot \hat{h}_\alpha(0) = 1,$$

$$\sum_{k=0}^{n} \hat{g}_\alpha(k) \cdot \hat{h}_\alpha(n-k) = 0, \quad n = 1, 2, 3, \cdots.$$

Theorem 6. *Let* $0 < \gamma < 1$, *then for each* δ, $0 < \delta < 1 - \gamma$, *there exist Blaschke and Djrbashyan products, having coinciding null sets, such that their Taylor-Maclaurin coefficients of corresponding those products satisfy the following conditions*:

(1) $$\sum_{n=0}^{\infty} n^{\gamma} \left|\hat{B}(n)\right|^2 < +\infty,$$

(2) $$\sum_{n=0}^{\infty} n^{\gamma} \left|\hat{B}_{\alpha}(n)\right|^2 < +\infty,$$

(3) $$\sum_{n=0}^{\infty} n^{\gamma+\delta} \left|\hat{B}(n)\right|^2 = +\infty,$$

(4) $$\sum_{n=0}^{\infty} n^{\gamma+\delta} \left|\hat{B}_{\alpha}(n)\right|^2 = +\infty.$$

Proof. Let $\varphi(t) = t^{\delta}$, $t \in [0, +\infty)$, then from the aforementioned theorem of Shapiro and Shields follows the existence of the null set $\{z_n\} \subset \mathbb{D}$ for the class $D^2_{\gamma+\delta}$ satisfying the following condition:

$$\sum_{n=0}^{\infty} (1 - |z_n|)^{1-\gamma-\delta} \varphi(1 - |z_n|) < +\infty,$$

i.e.,

$$\sum_{n=0}^{\infty} (1 - |z_n|)^{1-\gamma} < +\infty.$$

Then from Theorem 1, the products $B_{-\gamma}(z; \{z_n\})$ and $B(z; \{z_n\})$ belong to the class $D^2_{1-\gamma}$. This means that these products also belong to the class S_{γ}, therefore the conditions (1) and (2) are satisfied.

As $\{z_n\}$ is a uniqueness set for the class $D^2_{\gamma+\delta}$, then from conditions $f(z) \in D^2_{\gamma+\delta}$ and $f(z_n) = 0$, $n = 1, 2, \ldots$, it follows that $f(z) \equiv 0$. Therefore, the products $B_{-\gamma}(z; \{z_n\})$ and $B(z; \{z_n\})$ do not belong to the class $D^2_{1-\gamma-\delta}$. This in turn means that those products do not belong to the class $S_{\gamma+\delta}$. Thus we derive conditions (3) and (4).

Theorem 7. *Let* $-1 < \alpha < 0$ *and assume that the sequence of complex numbers* $\{z_n\} \subset \mathbb{D}$ *satisfies the condition* (2) *and has a finite number of boundary points on* $\partial\mathbb{D}$, *such that for each point* $e^{i\varphi}$ *of unit circle, the Frostman type condition takes place*

$$\sum_{n=1}^{\infty}\left(\frac{1-|z_n|}{\left|e^{i\varphi}-z_n\right|}\right)^{1+\alpha} < +\infty. \tag{4}$$

Then the function

$$f(z) = -\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} S_\alpha\left(e^{-i\theta}z\right)d\omega(\theta), \quad |z| < 1,$$

where $\omega(\theta)$ *is defined by equality* (3), *is a bounded function.*

Proof. From the theorem about the connection of Blaschke and Djrbashyan products, we have

$$-\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} S_\alpha\left(e^{-i\theta}z\right)d\omega(\theta) = \log\left|\frac{B_0(z; \{z_n\})}{B_\alpha(z; \{z_n\})}\right| = \sum_{n=1}^{\infty}\operatorname{Re}[W_\alpha(z; z_n) - W_0(z; z_n)]$$

$$\le \sum_{n=1}^{\infty}|W_\alpha(z; z_n) - W_0(z; z_n)|.$$

Therefore, from the well-known inequality (see [12]), we get

$$|W_\alpha(z; \xi) - W_0(z; \xi)| \le C \cdot \left(\frac{1-|\xi|}{\left|e^{i\varphi}-\xi\right|}\right)^{1+\alpha}, \quad |\xi| < 1, \quad z = r \cdot e^{i\varphi},$$

which gives

$$-\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} S_\alpha\left(e^{-i\theta}z\right)d\omega(\theta) \le C\cdot\sum_{n=1}^{\infty}\left(\frac{1-|z_n|}{\left|e^{i\varphi}-z_n\right|}\right)^{1+\alpha}.$$

From the last inequality follows the statement of the theorem.

Theorem 8. *Let* $-1 < \alpha < 0$ *and assume the sequence* $\{z_n\}$ *satisfies the conditions of the Theorem* 7. *Then for the Taylor coefficients of the function* $h_\alpha(z) = [g_\alpha(z)]^{-1}$ *takes place the following statement*:

$$\sum_{n=1}^{\infty}\left|\hat{h}_\alpha(n)\right|^2 n^\beta < +\infty,$$

where $\alpha < -\beta < 0$.

Proof. Let $|z| < 1$. Then

$$A(r,\, h) = \iint\limits_{|z|<r}\frac{|h'_\alpha(z)|^2}{\left(1+|h(z)|^2\right)^2}\,dxdy$$

$$= \iint\limits_{|z|<r}\frac{\left|\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}S_\alpha\left(e^{-i\theta}z\right)d\omega(\theta)\right\}\right|^2\cdot\left|\frac{1}{2\pi}\int_0^{2\pi}S'_\alpha\left(e^{-i\theta}z\right)d\omega(\theta)\right|^2}{\left(1+\left|\exp\left\{-\frac{1}{2\pi}\int_0^{2\pi}S_\alpha\left(e^{-i\theta}z\right)d\omega(\theta)\right\}\right|^2\right)^2}\,dxdy = A(r,\, g).$$

Therefore, if the sequence $\{z_n\}$ satisfies the Frostman type condition, then $h_\alpha(z) \in T_\beta$, $(0 < \beta < 1)$ if and only if $g_\alpha(z) \in T_\beta$. But in [4], it is shown that for β, $\alpha < -\beta < 0$, $g_\alpha(z) \in T_\beta$, that means that $h_\alpha(z) \in T_\beta$. This completes the proof.

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