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HOM-LIE ALGEBRAS ON STRICTLY UPPER TRIANGULAR MATRICES LIE ALGEBRA OVER A COMMUTATIVE RING: DERIVATION CASE

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Abstract

Let $\mathbf{n}$ be the nilpotent Lie algebra consisting of all strictly upper triangular $n \times n$ matrices over a commutative ring. In this paper, we characterize the decomposition of the derivations of $\mathbf{n}$ when $(\mathbf{n}, \phi)$ is a hom-Lie algebra. Meanwhile, the isomorphic classes are also determined.

1. Introduction and Preliminaries

The concept of hom-Lie algebra was first introduced by Hartwig et al. in their paper [7], which satisfies the skew symmetry and σ-twisted Jacobi identity, where σ is an algebra homomorphism. The motivations to study hom-Lie structures are related to physics and to deformations of Lie algebras, in particular Lie algebras of vector fields. The paradigmatic

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examples are q-deformations of Witt and Virasoro algebras constructed in pioneering works (see [1], [2]-[4], [6]). Jin and Li [8] discussed the hom-Lie algebra structures on finite dimensional semisimple Lie algebras. They proved that the hom-Lie algebra structures are almost trivial and determined the isomorphic classes of non-trivial hom-Lie algebras. Firstly, we give some definitions which follows those of [10].

Definition 1. (1) A hom-Lie algebra is a triple $(L, [., .], \alpha)$ consisting of a vector space L, a skew-symmetric bilinear map (bracket) $[., .] : L \times L \to L$ and a linear map $\alpha : L \to L$ satisfying the following hom-Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0. \tag{1.1}$$

(2) A hom-Lie algebra is called a multiplicative hom-Lie algebra, if α is a Lie algebra endomorphism, i.e., for any $x, y \in L$, we have

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

(3) A hom-Lie algebra is called a regular hom-Lie algebra, if α is a Lie algebra automorphism.

This paper mainly focuses on the case that the linear map α is a derivation, i.e., $\alpha([x, y]) = [\alpha(x), y] + [x, \alpha(y)]$, for any $x, y \in L$. In the other words, for an arbitrary derivation α, when $(\mathbf{n}, \alpha)$ is a hom-Lie algebra, we get the decomposition of α and determine all the isomorphic classes of hom-Lie algebra $(\mathbf{n}, \alpha)$.

We state some notation and list five types of automorphisms of $\mathbf{n}$ as in [5] and three types of derivations as in [11], which will be used later.

Let R be a commutative ring with identity, R^* be the set of invertible elements of R, and $M_n(R)$ be the set of $n \times n$ matrices over R, where n is a positive integer. Obviously, $M_n(R)$ is a Lie algebra under the Lie product defined as follows:

$$[A, B] = AB - BA, \quad \forall A, B \in M_n(R).$$

Denote by $\mathbf{n}$ the $n \times n$ strictly upper triangular matrices over R, which is a nilpotent Lie subalgebra of $M_n(R)$. Define a lower central series of $\mathbf{n}$

$$\mathbf{n} = \mathbf{n}_1,\ \mathbf{n}_2 = [\mathbf{n}, \mathbf{n}_1],\ \mathbf{n}_3 = [\mathbf{n}, \mathbf{n}_2], \cdots. \tag{1.2}$$

Let $Aut(\mathbf{n})$ (resp., $Der(\mathbf{n})$) be denoted the set of all the automorphisms (resp., derivations) of $\mathbf{n}$. Let E be the identity matrix of $M_n(R)$ and E_{ij} be the matrix whose sole nonzero entry is 1 in the (i, j) position. One can check that $\{E_{ij} | 1 \leq i < j \leq n\}$ is a basis of $\mathbf{n}$ as a vector space and $\{E_{i,i+1} | 1 \leq i \leq n-1\}$ is the generators of $\mathbf{n}$ as a Lie algebra.

● Five types of automorphisms

(1) Inner automorphisms

For any $z \in \mathbf{n}$, $x = E + z$ is invertible and the map $\sigma_x : y \mapsto xyx^{-1}$ is an automorphism of $\mathbf{n}$, which is called the inner automorphism.

(2) Diagonal automorphisms

Let $d_i \in R^*$ and $d = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}$. Define a map $\eta_d : x \mapsto dxd^{-1}$ and it is easy to testify that this is an automorphism of $\mathbf{n}$, which is called the diagonal automorphism.

(3) Central automorphisms

When $n \geq 4$, we define a linear map μ_c as follows:

$$\mu_c(E_{i,i+1}) = E_{i,i+1} + c_i E_{in}, \quad 2 \leq i \leq n-2; \quad \mu_c(E_{ij}) = E_{ij}, \text{ otherwise.}$$

This definition is called proper central automorphism in [5].

(4) Graph automorphisms

For $x \in M_n(R)$, let x^t denote the transpose of x. Set

$$r = E_{1n} + E_{2,n-1} + \cdots + E_{n-1,2} + E_{n1}.$$

It is clear that $r^2 = E$ and $r = r^t$. The map $\omega_0 : x \mapsto -rx^t r$ is an automorphism of $\mathbf{n}$, which is called the graph automorphism.

(5) Extremal automorphisms

Assume that $n \geq 3$, $b = (b_1, b_2) \in R^2$, define a linear map

$\xi_b : E_{12} \mapsto E_{12} + b_1 E_{2n}, E_{n-1,n} \mapsto E_{n-1,n} + b_2 E_{1,n-1}, E_{ij} \mapsto E_{ij}$, otherwise;

which is called extremal automorphism.

By [5], any automorphism φ of $\mathbf{n}$ can be uniquely expressed as

$$\varphi = \omega \cdot \eta \cdot \xi \cdot \mu \cdot \sigma. \tag{1.3}$$

● Three types of derivations

(1) Inner derivations

For any $x \in \mathbf{n}$, define $adx : y \mapsto [x, y]$, which is called the inner derivation of $\mathbf{n}$.

(2) Central derivations

For $t = (t_2, t_3, \cdots, t_{n-2}) \in R^{n-3}$, define linear map

$$\lambda_t : E_{i,i+1} \mapsto t_i E_{1n}, \quad 2 \leq i \leq n-2; \quad E_{ij} \mapsto 0, \text{ otherwise;}$$

which is called the central derivation.

(3) Extremal derivations

Define linear map ρ as follows:

$$\begin{cases} \rho(E_{12}) = a_1 E_{2n}; \\ \rho(E_{n-1,n}) = a_2 E_{1,n-1}; \\ \rho(E_{ij}) = 0, \text{ otherwise.} \end{cases} \tag{1.4}$$

As it is checked in [11], ρ is a derivation of $\mathbf{n}$ and called the extremal derivation.

Remark 1. The notions of diagonal derivation and central derivation are different with those in [11] in order not to make confusion with diagonal automorphism and central automorphism.

2. Hom-Lie Algebra Structures of n and Isomorphic Classes

In this section, we firstly show that any derivation ϕ can be written as the sum of three inner derivations, the central derivation, and the extremal derivation when $(\mathbf{n}, \phi)$ is a hom-Lie algebra. In addition, all the isomorphic classes of $(\mathbf{n}, \phi)$ are found.

Lemma 1. *Let* $(\mathbf{n}, \phi)$ *be a hom-Lie algebra over a commutative ring* R *with* $2 \in R^*$, *then* ϕ *can be written as*

$$\phi = adw + \lambda_t + \rho,$$

where $w \in \mathbf{n}_{n-3}$ *and* λ_t, ρ *are central, extremal derivation, respectively.*

Remark 2. By the result in [11], any derivation ϕ can be expressed as $\phi = adw + \chi_d + \lambda_c + \rho$, where $adw, \chi_d, \lambda_c, \rho$ are called inner, diagonal, central, extremal derivation, respectively. When 2 is in R^* the extremal derivation ρ is slight different with that in [11].

Proof. (1) If $\phi = adw$, $w = \sum_{i<j} a_{ij} E_{ij}$. In (1.1), by taking

$$x = E_{i,i+1}, \quad y = E_{i+1,i+2}, \quad z = E_{i+2,i+3}, \quad 1 \le i \le n-3,$$

we can figure out that all $a_{ij} = 0$ but $a_{1,n-2}, a_{1,n-1}, a_{1n}, a_{2,n-1}, a_{2n}$, $a_{3,n-1}$ and a_{3n}. That

$$a_{3,n-1} = 0,$$

follows from

$$[adw(E_{12}), [E_{23}, E_{n-1,n}]] + [adw(E_{23}), [E_{n-1,n}, E_{12}]] + [adw(E_{n-1,n}), [E_{12}, E_{23}]] = 0.$$

So if $(\mathbf{n}, adw)$ is a hom-Lie algebra, $w \in \mathbf{n}_{n-3}$.

(2) If $\phi = \chi_d$, where $d = (d_1, d_2, \cdots, d_n)$. In (1.1), by taking

$$x = E_{ij}, \quad y = E_{jk}, \quad z = E_{kp}, \quad 1 \leq i < j < k < p \leq n,$$

it is not difficult to check that $d_1 = d_2 = \cdots = d_n$. In view of the construction of diagonal derivation in [11], we can get $d_i = 0, 1 \leq i \leq n$.

(3) If $\phi = \lambda_c$ or ρ, it satisfies (1.1).

So $\phi = adw + \lambda_c + \rho$, which completes the lemma. □

Definition 2. Let (L_1, σ) and (L_2, τ) be both hom-Lie algebras, $\varphi : L_1 \to L_2$ an algebra isomorphism, we call (L_1, σ) is isomorphic to (L_2, τ), if $\varphi \circ \sigma = \tau \circ \varphi$. In the other words, the following diagram commutes:

$$\begin{tikzcd} L_1 \arrow[r, "\varphi"] \arrow[d, "\sigma"] & L_1 \arrow[d, "\tau"] \\ L_2 \arrow[r, "\varphi"] & L_2 \end{tikzcd}$$

Suppose $\sigma, \tau \in Der(\mathbf{n})$, if $(\mathbf{n}, \sigma)$ and $(\mathbf{n}, \tau)$ are isomorphic as hom-Lie algebra, then there exists a Lie algebra automorphism φ such that $\varphi \circ \sigma = \tau \circ \varphi$. In the other words, $\varphi \circ \sigma \circ \varphi^{-1} = \tau$.

Theorem 1. *Let* $\mathbf{n}$ *be defined an above, then two hom-Lie algebra structures* $(\mathbf{n}, \sigma)$ *and* $(\mathbf{n}, \tau)$ *are isomorphic if and only if the two Lie derivations* σ *and* τ *are conjugate.*

Lemma 2. $ad(RE_{1n})$ *is an isomorphic class.*

Proof. Notice that E_{1n} is in the center of $\mathbf{n}$, $[E_{1n}, E_{ij}] = 0$, we therefore have, for any $\varphi \in Aut(\mathbf{n})$,

$$\varphi adE_{1n}\varphi^{-1} = adE_{1n}.$$

□

Lemma 3. *$ad(RE_{1,n-1} + RE_{2n})$ is an isomorphic class.*

Proof. We divide it into five steps to prove this lemma.

Case 1. Suppose φ is an inner automorphism σ_x.

$$\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n})\sigma_x^{-1}(E_{12}) = -a_2E_{1n} = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{12});$$

$$\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n})\sigma_x^{-1}(E_{n-1,n}) = a_1E_{1n} = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{n-1,n});$$

$$\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n})\sigma_x^{-1}(E_{i,i+1}) = 0 = ad(a_1E_{1,n-1} + a_2E_{2n})(E_{i,i+1});$$

$$(2 \le i \le n-2).$$

So $\sigma_x ad(a_1E_{1,n-1} + a_2E_{2n})\sigma_x^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

Case 2. Suppose φ is a diagonal automorphism η_d.

$$\eta_d ad(a_1E_{1,n-1} + a_2E_{2n})\eta_d^{-1}(E_{12}) = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_n^{-1}E_{2n})(E_{12});$$

$$\eta_d ad(a_1E_{1,n-1} + a_2E_{2n})\eta_d^{-1}(E_{n-1,n}) = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_n^{-1}E_{2n})(E_{n-1,n});$$

$$\eta_d ad(a_1E_{1,n-1} + a_2E_{2n})\eta_d^{-1}(E_{i,i+1}) = 0 = ad(a_1d_1d_{n-1}^{-1}E_{1,n-1} - a_2d_2d_n^{-1}E_{2n})$$

$$\times (E_{i,i+1});\ (2 \le i \le n-2).$$

So $\eta_d ad(a_1E_{1,n-1} + a_2E_{2n})\eta_d^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

Case 3. Suppose φ is a central automorphism μ_c.

$$\mu_c ad(a_1E_{1,n-1} + a_2E_{2n})\mu_c^{-1}(E_{12}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{12});$$

$$\mu_c ad(a_1E_{1,n-1} + a_2E_{2n})\mu_c^{-1}(E_{n-1,n}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{n-1,n});$$

$$\mu_c ad(a_1E_{1,n-1} + a_2E_{2n})\mu_c^{-1}(E_{i,i+1}) = 0 = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{i,i+1});$$

$$(2 \le i \le n-2).$$

So $\mu_c ad(a_1E_{1,n-1} + a_2E_{2n})\mu_c^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

Case 4. Suppose φ is a graph automorphism ω_0.

$$\omega_0 ad(a_1E_{1,n-1} + a_2E_{2n})\omega_0^{-1}(E_{12}) = ad(a_2E_{1,n-1} - a_1E_{2n})(E_{12});$$

$$\omega_0 ad(a_1E_{1,n-1} + a_2E_{2n})\omega_0^{-1}(E_{n-1,n}) = ad(a_2E_{1,n-1} - a_1E_{2n})(E_{n-1,n});$$

$$\omega_0 ad(a_1E_{1,n-1} + a_2E_{2n})\omega_0^{-1}(E_{i,i+1}) = 0 = ad(a_2E_{1,n-1} - a_1E_{2n})(E_{i,i+1});$$

$$(2 \le i \le n-2).$$

So $\omega_0 ad(a_1E_{1,n-1} + a_2E_{2n})\omega_0^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

Case 5. Suppose φ is a extremal automorphism ξ_b.

$$\xi_b ad(a_1E_{1,n-1} + a_2E_{2n})\xi_b^{-1}(E_{12}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{12});$$

$$\xi_b ad(a_1E_{1,n-1} + a_2E_{2n})\xi_b^{-1}(E_{n-1,n}) = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{n-1,n});$$

$$\xi_b ad(a_1E_{1,n-1} + a_2E_{2n})\xi_b^{-1}(E_{i,i+1}) = 0 = ad(a_1E_{1,n-1} - a_2E_{2n})(E_{i,i+1});$$

$$(2 \le i \le n-2).$$

So $\xi_b ad(a_1E_{1,n-1} + a_2E_{2n})\xi_b^{-1} \in ad(RE_{1,n-1} + RE_{2n})$.

We therefore can conclude that for any $\varphi \in Aut(\mathbf{n})$,

$$\varphi ad(RE_{1,n-1} + RE_{2n})\varphi^{-1} \in ad(RE_{1,n-1} + RE_{2n}),$$

which completes the lemma. □

Lemma 4. *$ad(RE_{1,n-2} + RE_{2,n-1} + RE_{3n})$ is an isomorphic class.*

Proof. This process is similar to that of Lemma 3. □

Lemma 5. *λ_t is an isomorphic class.*

Proof. On one hand, for any $\varphi \in Aut(\mathbf{n})$ and $2 \leq i \leq n-2$,

$$\varphi\lambda_t\varphi^{-1}(E_{i,i+1}) = \varphi\lambda_t(E_{i,i+1}) \bmod \mathbf{n}_2 = \varphi(t_i E_{1n}) = \lambda_t(E_{i,i+1}).$$

On the other hand, λ_t vanishes on other generators of $\mathbf{n}$ and then $\varphi\lambda_t\varphi^{-1} = \lambda_t$.

□

Lemma 6. ρ *is in the isomorphic class of* $\rho + ad(RE_{1,n-1} + RE_{2n})$.

Proof. (i) For any inner automorphism σ_x, $x = E + \sum_{i<j} a_{ij}E_{ij}$, assume $x^{-1} = E + \sum_{i<j} b_{ij}E_{ij}$, one can figure out that

$$\sigma_x\rho\sigma_x^{-1}(E_{12}) = (\rho + ad(a_2 b_{n-1,n}E_{1,n-1} - a_1 a_{12}E_{2n}))(E_{12});$$

$$\sigma_x\rho\sigma_x^{-1}(E_{n-1,n}) = (\rho + ad(a_2 b_{n-1,n}E_{1,n-1} - a_1 a_{12}E_{2n}))(E_{n-1,n}).$$

Meanwhile, both of them vanish on the other elements of $\mathbf{n}$, so

$$\sigma_x\rho\sigma_x^{-1} = \rho + ad(a_2 b_{n-1,n}E_{1,n-1} - a_1 a_{12}E_{2n}).$$

(ii) For any diagonal automorphism η_d, it acts on E_{ij} as a constant, so $\eta_d\rho\eta_d^{-1} = \rho'$ for some extremal derivation ρ'.

(iii) For any central automorphism μ_c, it acts on E_{12} and $E_{n-1,n}$ as identity and ρ vanishes at other elements, which follows that $\mu_c\rho\mu_c^{-1} = \rho$.

(iv) For any graph automorphism ω_0, define an extremal derivation

$$\hat{\rho} : E_{12} \mapsto a_2 E_{2n};\ E_{1,n-1} \mapsto a_1 E_{1,n-1};\ E_{ij} \mapsto 0, \text{ otherwise.}$$

It is easy to show that $\omega_0^{-1}\rho\omega_0 = \hat{\rho}$.

(v) For any extremal automorphism ξ_b, it is not difficult to verify that $\xi_b\rho\xi_b^{-1} = \rho$. Hence, for any $\varphi \in Aut(\mathbf{n})$, $\varphi\rho\varphi^{-1} \in \rho + ad(RE_{1,n-1} + RE_{2n})$.

□

Theorem 2. *Suppose that R is a commutative ring such that $2 \in R^*$, ϕ is a derivation of $\boldsymbol{n}$ and $(\mathbf{n}, \phi)$ a hom-Lie algebra. Set*

$$x = s_1E_{1n},\ y = s_2E_{1,n-1} + s_3E_{2n},\ z = s_4E_{1,n-2} + s_5E_{2,n-1} + s_6E_{3n},\ s_i \in R.$$

If $n \geq 4$, each hom-Lie algebra is isomorphic to one of the following 24 types:

$$\{(\mathbf{n}, adx + ady + adz + \lambda_t + \rho)\}; \tag{2.1}$$

$$\{(\mathbf{n}, adx + ady + adz + \lambda_t), (\mathbf{n}, adx + ady + adz + \rho),$$
$$(\mathbf{n}, adx + ady + \lambda_t + \rho), (\mathbf{n}, ady + adz + \lambda_t + \rho)\}; \tag{2.2}$$

$$\{(\mathbf{n}, adx + ady + adz), (\mathbf{n}, adx + ady + \lambda_t), (\mathbf{n}, adx + ady + \rho),$$
$$(\mathbf{n}, adx + adz + \lambda_t), (\mathbf{n}, ady + adz + \lambda_t), (\mathbf{n}, ady + adz + \rho),$$
$$(\mathbf{n}, ady + \lambda_t + \rho)\}; \tag{2.3}$$

$$\{(\mathbf{n}, adx + ady), (\mathbf{n}, adx + adz), (\mathbf{n}, adx + \lambda_t), (\mathbf{n}, ady + adz),$$
$$(\mathbf{n}, ady + \lambda_t), (\mathbf{n}, ady + \rho), (\mathbf{n}, adz + \lambda_t)\}; \tag{2.4}$$

$$\{(\mathbf{n}, adx), (\mathbf{n}, ady), (\mathbf{n}, adz), (\mathbf{n}, \lambda_t)\}; \tag{2.5}$$

$$\{(\mathbf{n}, 0)\}. \tag{2.6}$$

Proof: Claim 1. Suppose D_1 is isomorphic to D_2, D_3 is isomorphic to itself, then $D_1 + D_3$ is isomorphic to $D_2 + D_3$.

If there is an automorphism φ of $\mathbf{n}$ such that $\varphi D_1 \varphi^{-1} = D_2$ and $\varphi D_3 \varphi^{-1} = D_3$, then

$$\varphi(D_1 + D_3)\varphi^{-1} = D_2 + D_3.$$

Claim 2. Suppose D_i is isomorphic to D_i, $i = 1, 2$ and D_1 is not isomorphic to D_2, then $D_1 + D_2$ is isomorphic to itself.

It follows from

$$\varphi(D_1 + D_2)\varphi^{-1} = \varphi D_1 \varphi^{-1} + \varphi D_2 \varphi^{-1} = D_1 + D_2.$$

In view of Claim 1 and Claim 2 and Lemma 6, we can conclude that there are 24 isomorphic classes of hom-Lie algebra $(\mathbf{n}, \phi)$ stated as above.

□

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