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LIE TRIPLE DERIVATIONS AND BOSONIC AND FERMIONIC REPRESENTATIONS FOR FILIFORM LIE ALGEBRA L_N

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Abstract

The aim of this paper is to present the decomposition of Lie triple derivations for filiform Lie algebra L_N. Meanwhile its bosonic and fermionic representations are also constructed.

1. Introduction and Preliminaries

Let L be a Lie algebra over a commutative ring R. Recall that a linear mapping ϕ from L to L is called a Lie derivation of L, if it satisfies

$$\phi([x, y]) = [\phi(x), y] + [x, \phi(y)], \text{ for all } x, y \in L.$$

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It is called a Lie triple derivation (in the paper [1], it is called prederivation), if the following equation holds:

$$\phi([[x, y], z]) = [[\phi(x), y], z] + [[x, \phi(y)], z] + [[x, y], \phi(z)], \text{ for all } x, y, z \in L.$$

Lie derivations are important research objects in the structure theory for Lie algebras. It is well-known that any Lie derivation for finite dimensional semisimple Lie algebras is inner. So people focused on those for solvable and nilpotent ones [10, 12]. Lie triple derivation, as a natural generalization of Lie derivations, also attracted many authors' attentions [8, 13, 14]. Meanwhile, Lie triple system on infinite-dimensional case is also studied by Martín and González in [9].

Define a decreasing central series for a finite dimensional Lie algebra L as follows:

$$L^0 = L, \quad L^1 = [L, L], \cdots, L^{i+1} = [L, L^i], \cdots. \tag{1.1}$$

If there exists a positive integer s such that $L^s = 0$, then L is called a nilpotent Lie algebra. Filiform Lie algebras are a class of important nilpotent Lie algebras, which were first introduced by Vergne in [11]. They are defined as follows:

Definition 1. A Lie algebra L with $\dim L = n$ is called filiform, or a filiform Lie algebra, if

$$\dim L^k = n - k - 1, \quad k \geq 1.$$

Here we give four examples of filiform Lie algebras. Actually there are some other types [7]. By [5], any filiform Lie algebra is a deformation of L_N. So L_N has a role importance in filiform Lie algebras. Here the non displayed commutators are either zero or obtained by skew symmetry.

(1) Let L_N be the n-dimensional Lie algebra with basis $\{e_1, e_2, \cdots, e_N\}$ satisfying the following brackets:

$$[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq N - 1.$$

The undefined brackets are zeros. One can check that it is indeed a filiform Lie algebra. It is, in a certain manner, the simplest filiform Lie algebra.

(2) Let $Q_n (n = 2k+1)$ be the $n+1$-dimensional Lie algebra with basis $\{X_0, X_1, \cdots, X_n\}$. The nonzero Lie brackets are

$$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-1;$$

$$[X_i, X_{n-i}] = (-1)^i X_n, \quad 1 \le i \le k.$$

This Lie algebra is also filiform.

(3) Let R_n be defined in the basis $\{X_0, X_1, \cdots, X_n\}$ by

$$[X_0, X_i] = X_{i+1}, \quad 2 \le i \le n-1;$$

$$[X_1, X_j] = X_{j+2}, \quad 2 \le j \le n-2.$$

It is evidently filiform.

(4) Let W_n be the Lie algebra whose brackets defined in the basis $\{X_0, X_1, \cdots, X_n\}$ are

$$[X_0, X_i] = X_{i+1}, \quad 2 \le i \le n-1;$$

$$[X_i, X_j] = \frac{6(i-1)!\,(j-1)!\,(j-i)}{(i+j)!} X_{i+j+1}, \quad 1 \le i < j \le n-1,\ i+j+1 \le n.$$

This Lie algebra is filiform. It is isomorphic to the Lie algebra defined by

$$[Y_i, Y_j] = (j-i)Y_{i+j},\ i+j \le n+1,\ 1 \le i < j \le n-1,$$

which is a finite quotient of the nilpotent part of Witt algebra.

Significant researches of bosonic and fermionic representations have been done on Kac-Moody algebras, affine Lie algebras, and other infinitely dimensional Lie algebras. For instance, fermionic representations for the affine Kac-Moody algebras were first given by Frenkel [3]. The representations for classical affine Lie algebras were constructed by Feingold and Frenkel [4] by using Clifford or Weyl

algebras with infinitely many generators. Chen and Gao [2] developed the bosonic and fermionic representations for a class of BC_N-graded Lie algebras. However, bosonic and fermionic representations have not been given for filiform Lie algebras, so the other aim of this paper is to construct the bosonic and fermionic representations of L_N.

Throughout this paper, $\mathrm{span}\{v_1, v_2, \cdots, v_n\}$ stands for the space spanned by $\{v_1, v_2, \cdots, v_n\}$. The symbols $\mathbb{C}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$ denote the set of complex, positive integers, and negative integers, respectively.

2. Lie Triple Derivations of L_N

In this section, we shall fix our discussion on the complex number field $\mathbb{C}$ and give the decomposition of any Lie triple derivation of L_N. Before this we define four types of standard Lie triple derivations.

(1) For any $x \in L_N$, $adx(y) = [x, y]$ defines a Lie derivation (also a Lie triple derivation), which is called the Lie triple derivation induced by x.

(2) Let α be a linear mapping on L_N and defined as

$$\alpha(e_1) = k_1 e_1 + k_1' e_2.$$

For even i, $\alpha(e_i) = (i-2)k_1 e_1 + k_2 e_2$.

For odd i and $i \geq 3$, $\alpha(e_i) = (i-3)k_1 e_1 + k_3 e_3$. Here $k_1, k_1', k_2, k_3 \in \mathbb{C}$. One can check that α is a Lie triple derivation of L_N.

(3) For any integer $j \geq 1$ and $l \geq -1$, $l \neq 0$, define linear mappings s_j and t_l,

$$s_j, t_l : L_N \to L_N,$$

$$s_j(e_i) = \varepsilon_j e_{i+j}, \quad t_l(e_i) = \epsilon_l e_{i+l}.$$

If i is even and $i + j \leq N$, ε_j is a nonzero constant; otherwise 0. If i is odd, $i \geq 3$ and $i + l \leq N$, ϵ_l is a nonzero constant; otherwise 0.

Obviously s_j's and t_l's are all Lie triple derivations on L_N.

(4) For $a_1, a_2, b_1, b_2 \in \mathbb{C}$, let β be a linear mapping of L_N defined as follows:

$$\begin{cases} \beta(e_2) = a_1 e_{N-1} + a_2 e_N; \\ \beta(e_3) = b_1 e_{N-1} + b_2 e_N; \\ \beta(e_i) = 0, \text{ otherwise.} \end{cases}$$

It is not difficult to verify that β is a Lie triple derivation of L_N.

In order to get the decomposition of ϕ, we assume that

$$\phi(e_i) = \sum_{j=1}^{N} a_j^{(i)} e_j, \tag{2.1}$$

where $a_j^{(i)} \in \mathbb{C}$, for all $i, j = 1, 2, \cdots, N$.

Lemma 1. *For any Lie triple derivation ϕ of L_N, we have*

$$\phi(e_i) \in \begin{cases} span\{e_1, e_2, \cdots, e_N\}, & i = 1; \\ span\{e_i, e_{i+1}, \cdots, e_N\}, & i \text{ even}; \\ span\{e_{i-1}, e_i, \cdots, e_N\}, & i \text{ odd and } i \geq 3. \end{cases}$$

Proof. Suppose ϕ is an arbitrary Lie triple derivation. By applying ϕ on the both sides of $[e_1, [e_2, e_3]] = 0$, we have

$$\begin{aligned} 0 &= \phi([e_1, [e_2, e_3]]) \\ &= [e_1, [\phi(e_2), e_3]] + [e_1, [e_2, \phi(e_3)] \\ &= a_1^{(2)} e_5 - a_1^{(3)} e_4. \end{aligned}$$

So $a_1^{(2)} = a_1^{(3)} = 0$. For any $2 \le i \le N-2$, by considering the operation of ϕ on $[e_1, [e_1, e_i]] = e_{i+2}$, we have the desired result. □

We now give the theorem of decomposition for any Lie triple derivation of L_N.

Theorem 1. *If $N \ge 6$, any Lie triple derivation ϕ of L_N can be uniquely expressed as*

$$\phi = adx + \alpha + \beta + \sum_{j=3}^{N-2} s_j + \sum_{\substack{l=2 \\ l\neq 3}}^{N-2} t_l. \tag{2.2}$$

Proof. By (2.1), we can choose $x = -\sum_{i=2}^{N-1} a_{i+1}^{(1)} e_i$, then $(\phi - adx)(e_1) = a_1^{(1)} e_1 + a_2^{(1)} e_2$ and $e_i (2 \le i \le N)$ are stable under adx. Denote $\phi_1 = \phi - adx$.

For any even i, by applying ϕ_1 on $[e_1, [e_1, e_i]] = e_{i+2}$, we can get that

$$2a_1^{(1)} + a_2^{(2)} = a_4^{(4)},$$

$$2a_1^{(1)} + a_4^{(4)} = a_6^{(6)},$$

.....,

that is to say $(i-2)a_1^{(1)} + a_2^{(2)} = a_i^{(i)}$. Similarly for odd i and $i \ge 3$, we have $(i-3)a_1^{(1)} + a_3^{(3)} = a_i^{(i)}$.

Let $k_1 = a_1^{(1)}$, $k_1' = a_2^{(1)}$, $k_2 = a_2^{(2)}$ and denote $\phi_2 = \phi_1 - \alpha$. By applying ϕ_2 on $[e_1, [e_1, e_i]] = e_{i+2}$, we can show that the following two equalities hold:

$$\begin{cases} a_j^{(2)} = a_{j+2}^{(4)} = a_{j+4}^{(6)} = \cdots, & 3 \le j \le N-2; \\ a_l^{(3)} = a_{l+2}^{(5)} = a_{l+4}^{(7)} = \cdots, & 2 \le l \le N-2,\ l \neq 3. \end{cases} \tag{2.3}$$

Let $a_j^{(2)} = \varepsilon_j$, $a_l^{(3)} = \epsilon_l$ and $\phi_3 = \phi_2 - \sum_{j=3}^{N-2} s_j - \sum_{\substack{l=2 \\ l\neq 3}}^{N-2} t_l$, then

$$\begin{cases} \phi_3(e_i) \in \text{span}\{e_{N-1}, e_N\}, & i = 2, 3; \\ \phi_3(e_i) = 0, & \text{otherwise.} \end{cases}$$

By choosing $a_1 = a_{N-1}^{(2)}$, $a_2 = a_N^{(2)}$, $b_1 = a_{N-1}^{(3)}$, $b_2 = a_N^{(3)}$, we can prove that $(\phi_3 - \beta)(e_i) = 0$, for all i. So any Lie triple derivation ϕ has the decomposition of (2.2).

Suppose that $\phi = 0$, that is to say $\phi(e_i) = 0$ for all $i \neq 1, 2, 3$, we have $s_j = t_l = 0$, if $j + i \leq N$ and $l + i \leq N$. Meanwhile ε_j and ϵ_l are 0 when $j + i > N$ and $l + i > N$. Then all s_j and t_l are 0. It follows that $\beta = 0$ from $(adx + \alpha + \beta)(e_2) = (adx + \alpha + \beta)(e_3) = 0$, hence $adx + \alpha = 0$. That $(adx + \alpha)(e_1) = 0$ implies $adx = 0$ and then $\alpha = 0$, which completes the proof of uniqueness of the decomposition.

□

Remark 1. The length of the two equalities in (2.3) is variable. That is to say, if the subscript of $a_{j+2k}^{(i)}$ are greater than N, the equalities will end before this term.

Remark 2. The uniqueness of this decomposition depends on the definitions of standard Lie triple derivations. Once they are defined, the decomposition is unique. In another word, if the definition of α is changed, then those of s_j and t_l will be also changed, but the types of Lie triple derivations are the same.

3. The Representations of L_N

In this section, we construct the representations for filiform Lie algebra L_N given in Section 1. Let A be an associative algebra and $\rho = \pm 1$, the ρ-bracket is defined as follows:

$$\{a, b\}_\rho = ab + \rho ba, \quad a, b \in A.$$

It is easy to see that

$$\{a, b\}_\rho = \rho\{b, a\}_\rho \text{ and } [ab, c] = a\{b, c\}_\rho - \rho\{a, c\}_\rho b,$$

for $a, b, c \in A$, where $[a, b] = \{a, b\}_{-1}$ is the Lie bracket.

Define $\mathcal{V}$ to be a unital associative algebra with $2N$ generators $a_i, a_i^*, 1 \le i \le N$ with respect to the relations

$$\{a_i, a_j\}_\rho = \{a_i^*, a_j^*\}_\rho = 0 \text{ and } \{a_i, a_j^*\}_\rho = \rho\delta_{ij}.$$

Let $\mathcal{V}(N, \rho)$ be the unital associative algebra generated by

$$\left\{u(m) \mid u \in \left(\bigoplus_{i=1}^N \mathbb{C}a_i\right) \oplus \left(\bigoplus_{i=1}^N \mathbb{C}a_i^*\right), m \in \mathbb{Z}\right\},$$

with the relations

$$\{u(m), v(n)\}_\rho = \{u, v\}_\rho \delta_{m+n,0}.$$

We now define the normal ordering as in [4]

$$: u(m)v(n) := \begin{cases} u(m)v(n), & \text{if } n > m; \\ \dfrac{1}{2}(u(m)v(n) - \rho v(n)u(m)), & \text{if } m = n; \\ -\rho v(n)u(m), & \text{if } m > n. \end{cases}$$

One can check that the following equalities hold:

$$: a_i(m)a_j(n) := a_i(m)a_j(n) = -\rho a_j(n)a_i(m),$$

$$: a_i(m)a_j^*(n) := a_i(m)a_j^*(n) - \rho\delta_{ij}\delta_{m+n,0}\theta(m-n),$$

and

$$[a_i(m)a_j(n),\ a_k(p)] = 0,$$

$$[a_i(m)a_j(n),\ a_k^*(p)] = -\delta_{ik}\delta_{m+p,0}a_j(n) + \rho\delta_{jk}\delta_{n+p,0}a_i(m), \qquad (3.1)$$

where $m, n, p \in \mathbb{Z}, 1 \le i, j, k \le N$.

Define $\mathcal{V}(N, \rho)^+$ to be the subalgebra generated by $a_i(n), a_j^*(m), a_k^*(0)$, for $n, m \in \mathbb{Z}_+$, and $1 \le i, j, k \le N$. Let $\mathcal{V}(N, \rho)^-$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k(0)$, for $n, m \in \mathbb{Z}_-$, and $1 \le i, j, k \le N$. As in [2, 4], let $\mathcal{M}(N, \rho)$ be a simple $\mathcal{V}(N, \rho)$-module containing the vacuum vector v_0, and satisfying

$$\mathcal{V}(N, \rho)^+ v_0 = 0.$$

So all annihilation operators kill v_0 and

$$M(N, \rho) = \mathcal{V}(N, \rho)v_0 = \mathcal{V}(N, \rho)^- v_0.$$

For $s \in \mathbb{Z}, 1 \le i \le N-1$ and any fixed $m \in \mathbb{Z}$, we are now to construct the bosons (if $\rho = -1$) or fermions (if $\rho = 1$) as follows:

$$f(m) = \sum_{i=1}^{N-1} \sum_{s\in\mathbb{Z}} : a_i(m-s)a_{i+1}^*(s) :$$

$$g_j(m) = \sum_{s\in\mathbb{Z}} : a_1((j-1)m+s)a_{N+1-j}(m-s).$$

Theorem 2. *Suppose L_N is the filiform Lie algebra defined as above with basis $e_1, e_2, \cdots, e_N$. Let*

$$\tau_m(e_1) = f(m),$$

$$\tau_m(e_j) = g_j(m), \quad 2 \le j \le N.$$

Then τ_m *is a representation of* L_N *on* $V(N, \rho)$.

Proof. In view of (3.1), for any $2 \le j \le N-1$,

$$
\begin{aligned}
&[f(m),\ g_j(m)] \\
&= [\sum_{i=1}^{N-1} \sum_{s\in\mathbb{Z}} :\ a_i(m-s)a_{i+1}^*(s)\ :,\ \sum_{t\in\mathbb{Z}} :\ a_1((j-1)m+t)a_{N+1-j}(m-t)\ :\ ] \\
&= [\sum_{i=1}^{N-1} \sum_{s\in\mathbb{Z}} a_i(m-s)a_{i+1}^*(s),\ \sum_{t\in\mathbb{Z}} a_1((j-1)m+t)a_{N+1-j}(m-t)] \\
&= \sum_{i=1}^{N-1} \sum_{s,t\in\mathbb{Z}} a_i(m-s)[a_{i+1}^*(s),\ a_1((j-1)m+t)a_{N+1-j}(m-t)] \\
&\quad + \sum_{i=1}^{N-1} \sum_{s,t\in\mathbb{Z}} [a_i(m-s),\ a_1((j-1)m+t)a_{N+1-j}(m-t)]a_{i+1}^*(s) \\
&= -\sum_{s\in\mathbb{Z}} \rho a_{N-j}(m-s)a_1(jm-s) \\
&= \sum_{s\in\mathbb{Z}} :\ a_1(jm-s)a_{N-j}(m-s)\ : \\
&= g_{j+1}(m).
\end{aligned}
$$

Similarly, one can show that $[f(m),\ g_N(m)] = 0$.

According to (3.1), for any $i,\ j \ne 1$, we have

$$
\begin{aligned}
&[g_i(m),\ g_j(m)] \\
&= [\sum_{s\in\mathbb{Z}} :\ a_1((i-1)m+s)a_{N+1-i}(m-s)\ :,\ \sum_{t\in\mathbb{Z}} :\ a_1((j-1)m+t)a_{N+1-j}(m-t)\ :\ ] \\
&= [\sum_{s\in\mathbb{Z}} a_1((i-1)m+s)a_{N+1-i}(m-s),\ \sum_{t\in\mathbb{Z}} a_1((j-1)m+t)a_{N+1-j}(m-t)]
\end{aligned}
$$

$$= \sum_{s,t\in\mathbb{Z}} a_1((i-1)m+s)[a_{N+1-i}(m-s),\ a_1((j-1)m+t)a_{N+1-j}(m-t)]$$

$$+ \sum_{s,t\in\mathbb{Z}} [a_1((i-1)m+s),\ a_1((j-1)m+t)a_{N+1-j}(m-t)]a_{N+1-i}(m-s)$$

$$= 0.$$

□

For any $m \in \mathbb{Z}$, there is a bosonic or fermionic representation π_m. All of π_m's are equivalent mutually. Meanwhile, if we fix an integer m, let

$$\widetilde{f}(m) = \sum_{i=2}^{N}\sum_{s\in\mathbb{Z}} :\ a_{i-1}(m-s)a_i^*(s)\ :$$

$$\widetilde{g}_j(m) = \sum_{s\in\mathbb{Z}} :\ a_N^*((j-1)m+s)a_{j-1}^*(s)\ : (2 \le j \le N),$$

and define

$$\pi_m(e_1) = f(m),$$

$$\pi_m(e_j) = g_j(m), \quad 2 \le j \le N,$$

then π_m is also a representation of L_N.

It is known from [5] that any filiform Lie algebra arises as a linear deformation of the model algebra L_N. So by Theorem 2 and this linear deformation, we can get the bosonic and fermionic representations of any filiform Lie algebra.

For example, let P_N be a Lie algebra with basis $\{X_1, X_2, \cdots, X_N\}$ subject to the relations

$$[X_1, X_i] = X_{i+1}, \quad 2 \le i \le n-1;$$

$$[X_2, X_j] = X_{j+1}, \quad 3 \le j \le n-1.$$

It is not difficult to verify that this Lie algebra is a deformation of L_N. In another word, there exists a linear space isomorphism (not Lie isomorphism) σ from P_N to L_N, where σ is defined by

$$\sigma(X_2) = e_1 + e_2, \ \sigma(X_i) = e_i, \quad i = 1, 3, 4, \cdots, N,$$

then the bosonic and fermionic representations can be constructed as follows:

$$X_1 = \sum_{i=1}^{N-1} \sum_{s\in\mathbb{Z}} : a_i(m-s)a_{i+1}^*(s) :$$

$$X_2 = \sum_{i=1}^{N-1} \sum_{s\in\mathbb{Z}} : a_i(m-s)a_{i+1}^*(s) : + \sum_{s\in\mathbb{Z}} : a_1((j-1)m+s)a_{N-1}(m-s) :$$

$$X_j = \sum_{s\in\mathbb{Z}} : a_1((j-1)m+s)a_{N+1-j}(m-s) : (3 \le j \le N).$$

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