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EXACT TRAVELLING WAVE SOLUTIONS OF THE SCHAMEL-KORTEWEG-DE VRIES (SCHAMEL-KdV) EQUATION

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Abstract

The Schamel-Korteweg-de Vries (Schamel-KdV) equation containing a square root nonlinearity is a very attractive model for the study of ion-acoustic waves in plasma and dusty plasma. In this work, we obtain the soliton-like solutions, the kink solutions, and the plural solutions of the Schamel-KdV equation by using the sine-cosine method and the extended tanh method. These solutions may be of important significance for the explanation of some practical physical problems. It is shown that these two methods provide a powerful mathematical tool for solving a great many nonlinear partial differential equations in mathematical physics.

1. Introduction

Many phenomena in physics and other fields are often described by nonlinear partial differential equations. In order to better understand the nonlinear phenomena as well as further applications in real life, it is

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important to seek their more exact travelling wave solutions. Recently, it has become more and more interesting to obtain exact analytical solutions to nonlinear partial differential equations by using appropriate techniques and symbolical computer programs, such as Maple or Mathematica. The capability and power of these softwares have increased dramatically over the past decade. To generate exact solutions and to understand their properties, several important techniques have been developed, such as the inverse scattering method [1], the tanh-function method [2, 3], the extended tanh-function method [4, 5], the homogeneous balance method [6], the exp-function method [7], the sine-cosine method [8, 9], the mapping method [10-12], the F-expansion method [13], the extended F-expansion method [14], the generalized Jacobi elliptic function method [15], and other methods [16-20].

Some of the nonlinear models in plasma and dust plasma are described by canonical models, such as the Korteweg-de Vries (KdV), the modified KdV, and so on [21]. The ion-acoustic solitary wave is one of the fundamental nonlinear wave phenomena appearing in plasma physics. In [22], Hans Schamel studies a modified Korteweg-de Vries equation for ion-acoustic waves. The Schamel-Korteweg-de Vries (Schamel-KdV) equation containing a square root nonlinearity is a very attractive model for the study of ion-acoustic waves in plasmas and dusty plasmas. Hence, seeking new exact solutions of the Schamel-KdV equation is important. It is of interest that the solution of the Schamel-KdV equation due to Tagare and Chakrabarti [23]

$$u_t + (a + bu^{\frac{1}{2}})u^{\frac{1}{2}}u_x + \alpha u_{xxx=0}, \tag{1.1}$$

where u refers to the perturbed ion density in a plasma with non-isothermal electrons, and a, b, α are real constants.

By direct integration, Tagare and Chakrabarti obtain:

$$u(x, t) = \left[\frac{4a}{15c} + \frac{\sqrt{75bc + 16a^2}}{15c}\cosh\left(\frac{1}{2}\sqrt{\frac{c}{\alpha}}(x - ct) + \frac{\Delta}{2}\right)\right]^{-2}, \tag{1.2}$$

where Δ is the extension or compression of the spring between two neighbouring masses. This solution is valid for all c but

$$c = \frac{16a^2}{75b}, \tag{1.3}$$

for which (1.2) would reduce to a constant. Some may wonder if there exists a solitary wave solution for the critical velocity (1.3). By using the direct method and MACSYMA [24], Willy Hereman and Masanori Takaoka get

$$u(x, t) = \frac{4a^2}{25b^2}\left[1 - \tanh\left(\frac{a}{15}\sqrt{\frac{-3}{b\alpha}}\left(x + \frac{16a^2}{75b}t\right) + \frac{\Delta}{2}\right)\right]^2. \tag{1.4}$$

This particular solution was clearly overlooked in Tagare and Chakrabarti.

To generate exact solutions and to understand their properties, several important techniques have been developed to study Equation (1.1), such as the exp-function method [25], the F-expansion method [26], the extended mapping method [21], and so on.

In this paper, we will study Equation (1.1) by using the sine-cosine method and the extended tanh method.

2. Analysis of Two Methods

It is known that the sine-cosine method and the extended tanh method have been applied for a wide variety of nonlinear problems. Here main features of these two methods will be reviewed briefly.

For both methods, we first use the wave variable $\xi = x_i - ct$ to carry a PDE in two independent variables

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \dots) = 0, \tag{2.1}$$

into an ODE

$$Q(u, u', u'', u''', \dots) = 0. \tag{2.2}$$

The Equation (2.2) is then integrated as long as all terms contain derivatives, where integration constants are considered as zeros.

As the sine-cosine method and the extended tanh method have been explained in [27, 28], we need not to repeat them here.

3. Using the Sine-Cosine Method

We use the wave variable $\xi = x - ct$, the Equation (1.1) is carried to ODE,

$$cu' = (a + bu^{\frac{1}{2}})u^{\frac{1}{2}}u' + \alpha u'''. \tag{3.1}$$

Integrating (3.1) once, and considering the constants of integration as zero, we can find

$$-cu + \frac{2}{3}au^{3/2} + \frac{1}{2}bu^2 + \alpha u'' = 0. \tag{3.2}$$

The sine-cosine method admits the use of the solution in the form

$$u(\xi) = \begin{cases} \lambda \cos^{\beta}(\mu\xi), & |\xi| \le \dfrac{\pi}{2\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

or in the form

$$u(\xi) = \begin{cases} \lambda \sin^{\beta}(\mu\xi), & |\xi| \le \dfrac{\pi}{\mu}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

where λ, μ, and β are parameters that will be determined.

As a consequence, the derivatives of (3.3) become

$$\begin{aligned} u &= \lambda \cos^{\beta}(\mu\xi), \\ u^n &= \lambda^n \cos^{n\beta}(\mu\xi), \\ (u^n)' &= -n\beta\mu\lambda^n \cos^{n\beta-1}(\mu\xi)\sin(\mu\xi), \\ (u^n)'' &= -n^2\mu^2\beta^2\lambda^n \cos^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \end{aligned} \tag{3.5}$$

and the derivatives of (3.4) become

$$u = \lambda \sin^{\beta}(\mu\xi),$$

$$u^n = \lambda^n \sin^{n\beta}(\mu\xi),$$

$$(u^n)' = n\beta\mu\lambda^n \sin^{n\beta-1}(\mu\xi)\cos(\mu\xi),$$

$$(u^n)'' = -n^2\mu^2\beta^2\lambda^n \sin^{n\beta}(\mu\xi) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi), \quad (3.6)$$

and so on for other derivatives.

Substituting (3.5) into (3.2) yields

$$-\left(c\lambda + \alpha\mu^2\beta^2\lambda\right)\cos^{\beta}(\mu\xi) + \frac{2}{3}a\lambda^{\frac{3}{2}}\cos^{\frac{3}{2}\beta}(\mu\xi) + \frac{1}{2}b\lambda^2\cos^{2\beta}(\mu\xi) + \alpha\mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu\xi) = 0. \quad (3.7)$$

Equating the exponents and the coefficients of each pair of the cosine functions, we find the following system of algebraic equations:

Case 1.

$$\beta - 1 \neq 0,$$

$$2\beta = \beta - 2,$$

$$c\lambda + \alpha\mu^2\beta^2\lambda = 0,$$

$$\frac{2}{3}a\lambda^{\frac{3}{2}} = 0,$$

$$\frac{1}{2}b\lambda^2 + \alpha\mu^2\lambda\beta(\beta-1) = 0. \quad (3.8)$$

Solving the system (3.8) yields

$$\beta = -2,$$

$$\mu = \sqrt{\frac{-c}{4\alpha}},$$

$$a = 0,$$

$$\lambda = \frac{3c}{b}. \tag{3.9}$$

We point out that the results (3.9) are valid if we also use the sine method (3.6). Consequently, the following solutions:

$$u(x, t) = \frac{3c}{b} \sec^2\left(\sqrt{\frac{-c}{4\alpha}}(x - ct)\right), \quad c < 0, \tag{3.10}$$

and

$$u(x, t) = \frac{3c}{b} \csc^2\left(\sqrt{\frac{-c}{4\alpha}}(x - ct)\right), \quad c < 0, \tag{3.11}$$

are readily obtained. It is worth noting that the results (3.10) and (3.11) are valid only if $c < 0$.

However, for $c > 0$, the following solutions:

$$u(x, t) = \frac{3c}{b} \operatorname{sech}^2\left(\sqrt{\frac{c}{4\alpha}}(x - ct)\right), \quad c > 0, \tag{3.12}$$

and

$$u(x, t) = \frac{3c}{b} \operatorname{csch}^2\left(\sqrt{\frac{c}{4\alpha}}(x - ct)\right), \quad c > 0, \tag{3.13}$$

are obtained.

Case 2.

$$\beta - 1 \neq 0,$$

$$\frac{3\beta}{2} = \beta - 2,$$

$$c\lambda + \alpha\mu^2\beta^2\lambda = 0,$$

$$\frac{b\lambda^2}{2} = 0,$$

$$\frac{2}{3}a\lambda^{\frac{3}{2}} + \alpha\mu^2\lambda\beta(\beta - 1) = 0. \tag{3.14}$$

Solving the system (3.14) yields

$$\beta = -4,$$

$$\mu = \sqrt{\frac{-c}{16\alpha}},$$

$$b = 0,$$

$$\lambda = \frac{225c^2}{64a^2}. \tag{3.15}$$

We point out that the results (3.15) are valid if we also use the sine method (3.6). Consequently, we can readily get the following solutions:

$$u(x, t) = \frac{225c^2}{64a^2}\sec^2\left(\sqrt{\frac{-c}{16\alpha}}(x - ct)\right), \quad c < 0, \tag{3.16}$$

$$u(x, t) = \frac{225c^2}{64a^2}\csc^2\left(\sqrt{\frac{-c}{16\alpha}}(x - ct)\right), \quad c < 0, \tag{3.17}$$

$$u(x, t) = \frac{225c^2}{64a^2}\operatorname{sech}^2\left(\sqrt{\frac{c}{16\alpha}}(x - ct)\right), \quad c > 0, \tag{3.18}$$

$$u(x, t) = \frac{225c^2}{64a^2}\operatorname{csch}^2\left(\sqrt{\frac{c}{16\alpha}}(x - ct)\right), \quad c > 0. \tag{3.19}$$

Case 3.

It is interesting to point out that for where $\beta - 1 = 0$, this is satisfied only when $a = 0$ and $b = 0$.

Consequently, we obtain

$$\lambda = \text{any real number}, \quad \mu = \sqrt{\frac{-c}{a}}. \tag{3.20}$$

This in turn gives the solutions

$$u(x, t) = A\cos\left(\sqrt{\frac{-c}{\alpha}}(x - ct)\right) + B\sin\left(\sqrt{\frac{-c}{\alpha}}(x - ct)\right), \quad c < 0,$$

$$u(x, t) = A\cosh\left(\sqrt{\frac{c}{\alpha}}(x - ct)\right) + B\sinh\left(\sqrt{\frac{c}{\alpha}}(x - ct)\right), \quad c > 0. \quad (3.21)$$

where A and B are arbitary constants.

4. Using the Extended Tanh Method

When we consider the transformations

$$u = v^2,$$

$$u' = 2vv',$$

$$u'' = 2((v')^2 + vv''), \quad (4.1)$$

and we can rewrite Equation (3.2) as the form

$$-cv^2 + \frac{2}{3}av^3 + \frac{1}{2}bv^4 + 2\alpha\left((v')^2 + vv''\right) = 0. \quad (4.2)$$

Balancing vv'' and v^4 gives

$$4M = M + M + 2, \quad (4.3)$$

so that

$$M = 1. \quad (4.4)$$

The extended tanh method allows us to use the substitution

$$u(x, t) = v^2(x, t) = S^2(Y) = \left(\sum_{i=0}^{M} A_i y^i + \sum_{i=1}^{M} B_i y^{-i}\right)^2 = \left(A_0 + A_1 y + B_1 y^{-1}\right)^2. \quad (4.5)$$

Substituting (4.5) into (4.2), collecting the coefficients of each power of $Y(Y = \tanh(\mu\xi))$, and using Maple to solve the resulting system of algebraic equations to find the sets of solutions:

Case 1. $A_0 = 0,\ A_1 = 0,\ B_1 = 0,\ \mu^2 = \mu^2,\ c = c.$

Case 2. $A_0 = A_0,\ A_1 = 0,\ B_1 = 0,\ \mu^2 = \mu^2,\ c = \frac{2}{3}aA_0 + \frac{1}{2}bA_0^2.$

Case 3. $A_0 = -\frac{2}{5}\frac{a}{b}, A_1 = 0,\ B_1 = \frac{2}{5}\frac{a}{b}, \mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha}, c = -\frac{16}{75}\frac{a^2}{b}.$

Case 4. $A_0 = -\frac{2}{5}\frac{a}{b},\ A_1 = 0,\ B_1 = -\frac{2}{5}\frac{a}{b},\ \mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha},\ c = -\frac{16}{75}\frac{a^2}{b}.$

Case 5. $A_0 = -\frac{2}{5}\frac{a}{b}, A_1 = \frac{2}{5}\frac{a}{b}, B_1 = 0, \mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha}, c = -\frac{16}{75}\frac{a^2}{b}.$

Case 6. $A_0 = -\frac{2}{5}\frac{a}{b}, A_1 = -\frac{2}{5}\frac{a}{b}, B_1 = 0, \mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha}, c = -\frac{16}{75}\frac{a^2}{b}.$

Case 7. $A_0 = -\frac{2}{5}\frac{a}{b}, A_1 = -\frac{1}{5}\frac{a}{b}, B_1 = -\frac{1}{5}\frac{a}{b}, \mu^2 = -\frac{1}{300}\frac{a^2}{b\alpha}, c = -\frac{16}{75}\frac{a^2}{b}.$

Case 8. $A_0 = -\frac{2}{5}\frac{a}{b},\ A_1 = \frac{1}{5}\frac{a}{b},\ B_1 = \frac{1}{5}\frac{a}{b},\ \mu^2 = -\frac{1}{300}\frac{a^2}{b\alpha},\ c = -\frac{16}{75}\frac{a^2}{b}.$

When $\mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha} > 0,$ we obtain the soliton-like solutions and the kink solutions:

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \frac{2}{5}\frac{a}{b}\coth\left[\sqrt{\frac{-1}{75}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right\}^2. \tag{4.6}$$

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \frac{2}{5}\frac{a}{b}\tanh\left[\sqrt{\frac{-1}{75}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right\}^2. \tag{4.7}$$

When $\mu^2 = -\frac{1}{300}\frac{a^2}{b\alpha} > 0$, we obtain the soliton-like solutions:

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \left(\frac{1}{5}\frac{a}{b}\tanh\left[\sqrt{\frac{-1}{300}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right.\right.$$

$$\left.\left. + \frac{1}{5}\frac{a}{b}\coth\left[\sqrt{\frac{-1}{300}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right)\right\}^2. \tag{4.8}$$

When $\mu^2 = -\frac{1}{75}\frac{a^2}{b\alpha} < 0$, we obtain the plural solutions:

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \frac{2}{5}\frac{a}{b}i\cot\left[-\sqrt{\frac{1}{75}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right\}^2. \tag{4.9}$$

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \frac{2}{5}\frac{a}{b}i\tan\left[-\sqrt{\frac{1}{75}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right\}^2. \tag{4.10}$$

When $\mu^2 = -\frac{1}{300}\frac{a^2}{b\alpha} < 0$, we obtain the plural solutions:

$$u(x, t) = \left\{-\frac{2}{5}\frac{a}{b} \pm \left(\frac{1}{5}\frac{a}{b}i\tan\left[-\sqrt{\frac{1}{300}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right.\right.$$

$$\left.\left. - \frac{1}{5}\frac{a}{b}i\cot\left[-\sqrt{\frac{1}{300}\frac{a^2}{b\alpha}}\left(x + \frac{16}{75}\frac{a^2}{b}t\right)\right]\right)\right\}^2. \tag{4.11}$$

5. Discussion

In this paper, we use the sine-cosine method and the extended tanh method to study the Schamel-Korteweg-de Vries (Schamel-KdV) equation. These two methods provide the soliton-like solutions, the kink solutions, and plural solutions. The presented exact solutions can describe various new features of waves and then may be useful in the

theoretical and numerical studies of the considered equation. The study emphasizes the fact that these two methods are reliable in handling nonlinear problems. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

References

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1991.

[2] W. Maliet, Solitary wave solutions of nonlinear wave equations, Am. J. Phys. 60 (1992), 650-654.

[3] W. Maliet, The tanh method: A tool for solving certain classes of nonlinear evolution and wave equations, J. Comput. Appl. Math. 164 (156) (2004), 529-541.

[4] E. G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000), 212-218.

[5] E. G. Fan and Y. C. Hong, Generalized tanh method to special types of nonlinear equations, Z. Naturforsch. A 57 (2002), 692-700.

[6] M. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A 213 (1996), 279-287.

[7] J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, Chaos, Solitons Fractals 30 (2006), 700-708.

[8] A. M. Wazwaz, Variants of the generalized KdV equation with compact and non-compact structures, Computers Mathematics with Applications 47 (2004), 583-591.

[9] E. Yusufoglu, A. Bekir and M. Alp, Periodic and solitary wave solutions of Kawahara and modified Kawahara equations by using sine-cosine method, Chaos, Solitons Fractals 37 (2008), 1193-1197.

[10] Y. Peng, Exact periodic wave solutions to a new Hamiltonian amplitude equation, J. Phys. Soc. Japan 72 (2003), 1356-1359.

[11] Y. Peng, New exact solutions to a new Hamiltonian amplitude equation II, J. Phys. Soc. Japan 73 (2004), 1156-1158.

[12] Y. Peng, Exact periodic wave solutions to the Melnikov equation, Z. Naturforsch. A 60 (2005), 321-327.

[13] Y. B. Zhao, M. L. Wang and Y. M. Wang, Periodic wave solutions to a coupled KdV equations with variable coefficients, Phys. Lett. A 308 (2003), 31-36.

[14] J. Liu and K. Yang, The extended *F*-expansion method and exact solutions of non-linear PDEs, Chaos, Solitons and Fractals 22 (2004), 111-121.

[15] H. T. Chen and H. Q. Zhang, New double periodic and multiple soliton solutions of the generalized $(2+1)$-dimensional Boussinesq equation, Chaos, Solitons and Fractals 20 (2004), 765-769.

[16] S. A. Elwakil, S. K. El-labany, M. A. Zahran and R. Sabry, Modified extended tanh-function method for solving nonlinear partial differential equations, Phys. Lett. A 299 (2002), 179-188.

[17] M. A. Abdou and S. Zhang, New periodic wave solutions via extended mapping method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 2-11.

[18] D. Baldwin, U. Goktas and W. Hereman et al., Symbolic computions of exact solutions expressible in hyperbolic and elliptic functions for nonlinear PDEs, J. Symb. Comput. 37 (2004), 669-705.

[19] Taogetusang and Sirendaoerji, The Jacobi elliptic function-like exact solutions to two kinds of KdV equations with variable coefficients and KdV equation with forcible term, Chinese Phys. 15 (2006), 2809-2818.

[20] A. H. Khater, M. M. Hassan, E. V. Krishnan and Y. Z. Peng, Applications of elliptic functions to ion-acoustic plasma waves, Eur. Phys. J. D 50 (2008), 177-184.

[21] M. M. Hassan, New exact solutions of two nonlinear physical models, Commun. Theor. Phys. China 53 (2010), 596-604.

[22] Hans Schamel, A modified Korteweg-de Vries equation for ion acoustic wavess due to resonant electrons, Journal of Plasma Physics 9 (1973), 377-387.

[23] S. G. Tagare and A. Chakrabarti, Solution of a generalized Korteweg-de Vries equation, Phys. Fluids A 17 (1974), 1331-1332.

[24] Willy A. Hereman and Takaoka Masanori, Solitary wave solutions of nonlinear evolution and wave equations using a direct method and MACSYMA, J. Phys. A: Math. Gen. 23 (1990), 4805-4822.

[25] Jonu Lee and Rathinasamy Sakthivel, Exact travelling wave solutions of the Schamel-Korteweg-de Vries equation, Reports on Mathematical Physics 68 (2011), 153-161.

[26] A. H. Khater and M. M. Hassan, Exact solutions expressible in hyperbolic and Jacobi elliptic functions of some important equations of ion-acoustic waves, Acoustic Waves-From Microdevices to Helioseismology (2011), 67-78.

[27] Jian Yang, Ruizhi Shen and Shengqiang Tang, Exact Travelling wave solutions for the $(2+1)$-dimensional generalized Camassa-Holm equation, Journal of Pure and Applied Mathematics: Advances and Applications 3 (2010), 249-263.

[28] Jian Yang and Shengqiang Tang, Analytic solutions of the $(2+1)$-dimensional nonlinear evolution equations using the extended tanh method, Journal of Mathematical Sciences: Advances and Applications 4 (2010), 359-369.

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