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VERTEX ALGEBRA GENERATION OF ALMOST HOLOMORPHIC MODULAR FORMS

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Abstract

Using computations related to vertex operator algebra characters, we derive the simplest example of almost holomorphic modular forms.

1. Introduction

It is well-known [9, 12, 14] that vertex operator algebras [2, 6, 9] serve as a natural algebraic source of automorphic forms generation. In this short note, we show how to generate the simplest example of almost holomorphic modular form [3] $\hat{E}_2(\tau) = E_2(\tau) + c / y$, where $E_2(\tau)$ is the second Eisenstein series [13], $y = \mathrm{Im}(\tau)$, τ is the torus modular parameter, and c is a constant. For this purpose, we use the machinery of character computations developed for vertex operator algebras/conformal field theory [12]. The holomorphic quasi-modular form $E_2(\tau)$ appeared at various points in computations related to vertex operator algebra module. At the same time, it was not know how to generate its non-holomorphic

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extension $\hat{E}_2(\tau)$ using similar reasoning. The author is also grateful to Bringmann for pointing the problem and useful discussions. In this note, we find answer for this question.

1.1. Almost holomorphic modular forms

Almost holmorphic modular forms [3], which originally defined in [8], transform like usual modular forms, but are polynomials in $1/v$ with holomorphic coefficients. Standard examples of almost holomorphic modular form include derivatives of holomorphic modular forms, as well as the non-holomorphic Eisenstein series $\hat{E}_2(\tau)$, defined by

$$\hat{E}_2(\tau) = E_2(\tau) + \frac{c}{\operatorname{Im} \tau},$$

where $c = -3/\pi$ Here its holomorphic part is given by

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $\sigma_1(n)$ is the sum of positive integer divisors of n, and $q = e^{2\pi i \tau}$. In general, the holomorphic part of an almost holomorphic modular form is called a *quasimodular form*. After their introduction [8], almost holomorphic modular forms have been shown to play numerous roles in mathematics and physics (see, for example, [1]).

1.2. Vertex operator algebras

Recall that a vertex operator algebra [2, 6, 9] is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded complex vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, a linear map $Y : V \to (\operatorname{End} V)[[z, z^{-1}]]$, and a pair of distinguished vectors, i.e., the vacuum vector $\mathbf{1} \in V_0$, and the conformal vector $\omega \in V_2$. A vertex operator is given by the map

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1},$$

with modes $v(n) \in \operatorname{End} V$, where $Y(v, z).\mathbf{1}|_{z=0} = v(-1).\mathbf{1} = v$, and z is a formal variable. For the vector ω, one has

$$Y(\omega,\ z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}.$$

The modes $L(n)$ form the Virasoro Lie algebra of central charge C:

$$[L(m),\ L(n)] = (m-n)L(m+n) + (m^3 - m)\delta_{m,-n} C / 12.$$

We define the homogeneous space of weight k to be $V_k = \{v \in V | L(0)v = kv\}$, where for v in V_k we write $wt(v) = k$. Then as an operator on V, we have

$$v(n) : V_m \to V_{m+k-n-1}. \tag{1.1}$$

In particular, the zero mode $o(v) = v(wt(v) - 1)$ is a linear operator on each homogeneous space of V.

1.3. The Heisenberg vertex operator algebra M

Consider $a \in V_1$ whose modes obey

$$[a(m),\ a(n)] = m\delta_{m,-n} I_V,$$

where I_V is the identity operator on V. The vertex operator $Y(a,\ z)$ generates the Heisenberg vertex operator algebra M with vector space V with Fock basis

$$a(-1)^{r_1} a(-2)^{r_2} \cdots a(-n)^{r_n}.\mathbf{1}; \quad r_i \geq 0,$$

with $a = a(-1).\mathbf{1}$, and $a(n).\mathbf{1} = 0$ for all $n \geq 0$. The Virasoro vector is $\omega = \frac{1}{2} a^2(-1).\mathbf{1}$ for central charge $C = 1$ with

$$L(0) = \frac{1}{2} a(0)^2 + \sum_{m \geq 0} a(-m)a(m).$$

In the context of modular-invariance for n-point functions at genus 1, Zhu introduced in [14] a second vertex operator algebra $(V,\ Y[,],\ \mathbf{1},\ \tilde{\omega})$ associated to a given vertex operator algebra $(V,\ Y(,),\ \mathbf{1},\ \omega)$. The underlying Fock space of the second vertex operator algebra is the same space V as the first, moreover, they share the same vacuum vector **1** and have the same central charge. The new vertex operators are defined by a change of co-ordinates, namely,

$$Y[v, z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1} = Y(q_z^{L(0)}v, q_z - 1),$$

where $q_z = e^z$. The new conformal vector $\tilde{\omega}$ is defined to be the state $\tilde{\omega} = \omega - \frac{C}{24}.\mathbf{1}$. We set

$$Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n]z^{-n-2},$$

and write $wt[v] = k$ if $L[0].v = kv$, $V_{[k]} = \{v \in V | wt[v] = k\}$. A Heisenberg vertex operator algebra module M_α is determined by

$$a(n)w = \delta_{n,0}\alpha \cdot w,$$

for all $w \in M_\alpha$, $n \geq 0$ for a complex $\alpha \in \mathbb{C}$.

1.4. Torus two-point function

Torus n-point function for a vertex operator algebra is defined by [14]

$$F_V^{(1)}(v_1, z_1; \ldots; v_n, z_n; q) = Tr_V\left(Y(v_1, z_1)\ldots Y(v_n, z_n)\, q^{L(0)-C/24}\right).$$

In particular, the partition (or zero-point) functions and one-point functions are given by

$$F_V^{(1)}(q) = Tr_V\left(q^{L(0)-C/24}\right),$$

$$F_V^{(1)}(v, z; q) = Tr_V\left(Y(v, z)\, q^{L(0)-C/24}\right) = Tr_V\left(o(v)\, q^{L(0)-C/24}\right).$$

The second equality in the last formula follows from (1.1). Note that one-point function does not depend on z.

Modular and elliptic functions arise in context of computation of n-point functions for vertex operator algebras [12, 14]. The Zhu recursion formula [14] relates n-point functions to $(n-1)$-point functions. Thus, two-point function

$$F_V^{(1)}(u,\ x;\ v,\ y;\ q) = Tr_V\left(o(u)\,o(v)\,q^{L(0)-C/24}\right)$$

$$+\sum_{m\geq 0}\frac{1}{m!}\frac{\partial^m}{\partial y^m}P_1(x-y)F_V^{(1)}(u[m]v,\ q), \tag{1.2}$$

for elliptic and modular functions

$$P_1(z) = -\sum_{n\in\mathbb{Z},\,n\neq 0}\frac{e^{nz}}{1-q^n} = \frac{1}{z}-\sum_{n\geq 1}E_{2n}(q)z^{2n-1},$$

and

$$E_{2n}(\tau) = -\frac{B_{2n}}{(2n)!}+\frac{2}{(2n-1)!}\sum_{r\geq 0}\frac{r^{2n-1}q^r}{1-q^r},$$

where $E_{2n}(\tau)$ are Eisenstein series, and B_{2n} being Bernoulli numbers [13]. Note that $E_2(\tau)$ is a quasi-modular form satisfying the transformation law:

$$(c\tau+d)^{-2}E_2(\gamma\tau) = E_2(\tau)-\frac{c}{2\pi i(c\tau+d)},\ \gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix},\ \gamma\tau = \frac{a\tau+b}{\gamma\tau+d},$$

for $\gamma\in SL(2,\ \mathbb{Z})$.

2. The Simplest Almost Holomorphic Modular Form

2.1. Identities for the Heisenberg vertex operator algebra

$E_2(\tau)$ has also important relations with derivatives of modular forms. One can show that

$$q\frac{d}{dq}\eta(\tau) = -\frac{1}{2}\eta(\tau)E_2(\tau).$$

Note, in particular, that the one-point Virasoro vector function for the Heisenberg vertex operator algebra is given by [7]

$$F^{(1)}_{M_0}(\tilde{\omega}) = q\,\frac{d}{dq}\,\eta(\tau)^{-1} = \frac{E_2(\tau)}{2\eta(\tau)}.$$

For the Virasoro vector $\tilde{\omega} = \omega - \frac{C}{24}$, we find [12]

$$F^{(1)}_V(\tilde{\omega}, q) = Tr_V\left((L(0) - C/24)q^{L(0)-C/24}\right) = q\partial_q F^{(1)}_V(q)$$

$$= \frac{1}{2\pi i}\,\partial_\tau F^{(1)}_V(q).$$

For n primary (i.e., satisfying $L(n)u = 0$, for all $n \geq 1$) vectors $u_1, \ldots, u_n$, the Zhu reduction gives the Ward identity [10, 12] for the $n+1$-point function:

$$F^{(1)}_V(\tilde{\omega}, x; u_1, z_1; \ldots; u_n, z_n; q) = q\partial_q F^{(1)}_V(u_1, z_1; \ldots; u_n, z_n; q)$$

$$+ \sum_{1\leq i\leq n} (P_1(x - z_i)\partial_{z_i} + wt[u_i]\partial_{z_i} P_1(x - z_i))F^{(1)}_V(u_1, z_1; \ldots; u_n, z_n; q).$$

$E_2(\tau)$ can be reproduced via the computation of a one-point function for the Heisenberg vertex operator algebra [11, 12, 14]. For the Heisenberg vertex operator algebra M, the Zhu reduction gives all n-pt functions, e.g., [10, 12]

$$F^{(1)}_M(a, z_1; a; z_2; q) = P_2(z_1 - z_2, \tau)F^{(1)}_M(q),$$

where $P_2(z, \tau) = -\frac{d}{dz}, P_1(z, \tau)$. Then,

$$F^{(1)}_M(\tilde{\omega}, q) = \frac{1}{2}\lim_{x\to y}\left(F^{(1)}_M(a, z_1; a, z_2; q) - (z_1 - z_2)^{-2} F^{(1)}_M(q)\right)$$

$$= \frac{1}{2}\,E_2(q)F^{(1)}_M(q). \tag{2.1}$$

Thus one has an ordinary differential equation

$$q\partial_q F_M^{(1)}(q) = \frac{1}{2} E_2(q) F_M^{(1)}(q),$$

therefore, the genus one partition function for the Heisenberg vertex operator algebra is given by $F_M^{(1)}(q) = \dfrac{1}{\eta(q)}$.

2.2. The simplest almost holomorphic modular form

Since, as we saw in (2.1) that $E_2(\tau)$ is given by the Heisenberg vertex operator algebra one point function for the Virasoro vector $\tilde{\omega}$, [12], we need to extend this vertex operator algebra in order to reconstruct the non-holomorphic part of $\hat{E}_2(\tau)$. For that purpose, we consider Virasoro state $\tilde{\omega}$ one-point function for a Heisenberg module. We formulate the following:

Proposition 1. *For* $\alpha = \pm(\dfrac{c}{\operatorname{Im}\tau})^{1/2}$, *the torus Virasoro state character generates the simplest almost holomorphic modular form, i.e.,*

$$F_{M_\alpha}^{(1)}(\tilde{\omega}, q) = \frac{1}{2} \hat{E}_2(\tau) F_{M_\alpha}^{(1)}(q).$$

Proof. The Zhu reduction formula (1.2) also applies to n-pt correlation functions for a vertex operator algebra module [12]. For a Heisenberg module M_α (for some $\alpha \in \mathbb{C}$), one finds that

$$F_{M_\alpha}^{(1)}(q) = Tr_{M_\alpha}(q^L(0) - 1/24) = \frac{q^{\alpha^2}}{\eta(q)}.$$

The one-point function for the Virasoro vector $\tilde{\omega}$ is [7]

$$F_{M_\alpha}^{(1)}(\tilde{\omega}, q) = q\partial_q F_{M_\alpha}^{(1)}(q).$$

On the other hand, the Zhu reduction formula (1.2) gives

$$F_{M_\alpha}^{(1)}(a;\, z_1;\, a;\, z_2;\, q) = \left(\frac{\alpha^2}{2} + P_2(z_1 - z_2)\right) F_{M_\alpha}^{(1)}(q),$$

so that

$$F^{(1)}_{M_\alpha}(\tilde{\omega}, q) = \frac{1}{2}(\alpha^2 + E_2(q))\, F^{(1)}_{M_\alpha}(q).$$

Now let us choose $\alpha = \pm\left(\frac{c}{\mathrm{Im}\,\tau}\right)^{1/2}$. Then we obtain

$$F^{(1)}_{M_\alpha}(\tilde{\omega}, q) = \frac{1}{2}\left(E_2(\tau) + \frac{c}{\mathrm{Im}\,\tau}\right) F^{(1)}_{M_\alpha}(q) = \frac{1}{2}\,\hat{E}_2(\tau) F^{(1)}_{M_\alpha}(q).$$

□

Thus we see that the Zhu recursion formula (1.2) which is the main tool in computation of the torus as well as genus two (see [10]) n-point functions for vertex operator algebras results in generation of non-holomorphic modular forms as coefficients, and provides us with the simplest example of non-holomorphic quasi-modular forms. Similar reasoning can be used to derive [15] more complicated examples of mock forms [3, 16].

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