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NON-POLYNOMIAL SPLINE METHOD FOR A SYSTEM OF THIRD-ORDER OBSTACLE BOUNDARY VALUE PROBLEMS

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Abstract

We use a new class of non-polynomial spline functions constructed with exponential function e^{kx}, trigonometric function and polynomial functions as basis functions to develop a numerical method for computing approximations to the solution of a system of third-order boundary value problems associated with obstacle, unilateral, and contact problems. Such problems arise in physical oceanography and can be studied in the framework of variational inequality theory. The convergence analysis of the new method is studied and an upper bound for the error is derived. A numerical example is given to illustrate the efficiency of the new method.

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1. Introduction

We use non-polynomial spline functions constructed with exponential function e^{kx}, trigonometric function and polynomial functions as basis functions to develop a numerical method for solving a system of third-order boundary value problem of the type

$$y''' = \begin{cases} f(x), & a \le x \le c, \\ g(x)y(x) + f(x) + r(x), & c \le x \le d, \\ f(x), & d \le x \le b, \end{cases} \tag{1.1}$$

with the boundary conditions:

$$y(a) = \alpha, \quad y'(a) = \beta_1, \quad y'(b) = \beta_2, \tag{1.2}$$

and the continuity conditions of y, y', and y'' at c and d. Here, $f(x)$, $g(x)$, and $r(x)$ are continuous functions on $[a, b]$ and $[c, d]$. The parameters α, β_1, and β_2 are real finite constants. Such type of systems arise in the study of obstacle, unilateral, moving and free boundary value problems and has important applications in other branches of pure and applied sciences, see [1-12]. In general, it is not possible to obtain the analytical solution of (1.1) for arbitrary choices of $f(x)$, $g(x)$, and $r(x)$. We usually resort to some numerical methods for obtaining an approximate solution of (1.1). Many scholars have made outstanding contributions on this issue. Noor and Khalifa [1] have used collocation method with quintic B-splines as basis functions to solve a special form of (1.1), namely,

$$y''' = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ y(x) - 1, & \frac{1}{5} \le x \le \frac{3}{4}, \\ 0, & \frac{3}{4} \le x \le 1, \end{cases} \tag{1.3}$$

with the boundary conditions:

$$y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 0, \tag{1.4}$$

and the continuity conditions of y, y', and y'' at $\frac{1}{4}$ and $\frac{3}{4}$. After this, Al-Said and co-workers [2-5] have developed first- and second-order methods for solving problems (1.3) and (1.4). Then Al-Said and Noor have used cubic spline functions to solve the problems (1.3) and (1.4).

In the present paper, we will use non-polynomial spline method to derive some consistency relations which are then used to develop a numerical technique for solving problem (1.1). In Section 2, we develop the new non-polynomial spline method for solving (1.1). The convergence analysis of the method is considered in Section 3. Section 4 is devoted to the application to a system of third-order boundary value problems. Conclusion is given in Section 5.

2. Numerical Method

For simplicity, we take $c = \frac{3a + b}{4}$ and $d = \frac{a + 3b}{4}$ in order to develop the numerical method for approximating solution of a system of differential equations (1.1). For this purpose, we define a grid of $n + 1$ equally spaced points $x_i = a + ih$, $i = 0, 1, \cdots, n$, where by $h = \frac{b - a}{n + 1}$. For each segment, the non-polynomial spline function $P_i(x)$ has the following form:

$$P_i(x) = a_i \sin k(x - x_i) + b_i e^{k(x - x_i)} + c_i(x - x_i)^2 + d_i(x - x_i) + e_i,$$
$$i = 0, 1, \cdots, n, \tag{2.1}$$

where a_i, b_i, c_i, d_i, and e_i are constants and k is free parameter. The function $P_i(x)$, which interpolates $y(x)$ at the mesh points x_i depends on k and reduces to quartic spline in $[a, b]$ as $k \to 0$.

The function $P_i(x)$ is continuous at x_i and its second derivative is continuous, in the point of each interval period $[x_i, x_{i+1}]$, we can derive five undermined coefficients of (2.1) by using the property of $j(j = 1, 2, 3)$ derivatives's continuity at the common points (x_i, y_i) of $P_i(x)$. The first, second, and third derivatives as follows:

$$P_i'(x) = ka_i \cos k(x - x_i) + kb_i e^{k(x-x_i)} + 2c_i(x - x_i)^2 + d_i,$$

$$P_i''(x) = -k^2 a_i \sin k(x - x_i) + k^2 b_i e^{k(x-x_i)} + 2c_i,$$

$$P_i'''(x) = -k^3 a_i \cos k(x - x_i) + k^3 b_i e^{k(x-x_i)}. \tag{2.2}$$

To derive expression for the coefficients of (2.1), we first define

$$\begin{aligned} &P_i(x_i) = y_i, \quad P_i(x_{i+1}) = y_{i+1}, \\ &P_i'(x_i) = D_i, \quad P_i'(x_{i+1}) = D_{i+1}, \\ &P_i''(x_i) = S_i, \quad P_i''(x_{i+1}) = S_{i+1}. \end{aligned} \tag{2.3}$$

In combination with (2.1), (2.2), and (2.3) at $x = x_i$, $x = x_{i+1}$, $i = 0, 1, \cdots, n - 1$, we have

$$\begin{cases} b_i + e_i = y_i, & a_i \sin kh + b_i e^{kh} + c_i h^2 + d_i h + e_i = y_{i+1}, \\ ka_i + kb_i + d_i = D_i, & ka_i \cos kh + kb_1 e^{kh} + 2c_i h + d_i = D_{i+1}, \\ -k^3 a_i + k^3 b_i = S_i, & -k^3 a_i \cos kh + k^3 b_i e^{kh} = S_{i+1}. \end{cases} \tag{2.4}$$

Let $\theta = kh$, solving (2.4), we have

$$a_i = h^3 \frac{S_i e^{\theta} - S_{i+1}}{\theta^3(\cos\theta - e^{\theta})},$$

$$b_i = h^3 \frac{S_i \cos\theta - S_{i+1}}{\theta^3(\cos\theta - e^{\theta})},$$

$$c_i = \frac{y_{i+1} - y_i}{h^2} - \frac{D_i}{h} + h\,\frac{(S_i e^{\theta} + S_i \cos\theta - 2S_{i+1})}{\theta^2(\cos\theta - e^{\theta})}$$

$$- h\,\frac{(S_i e^{\theta} - S_{i+1})\sin\theta + (S_i\cos\theta - S_{i+1})(e^{\theta} - 1)}{\theta^3(\cos\theta - e^{\theta})},$$

$$d_i = D_i - h^2\,\frac{S_i e^{\theta} + S_i\cos\theta - 2S_{i+1}}{\theta^2(\cos\theta - e^{\theta})},$$

$$e_i = y_i - h^3\,\frac{S_i\cos\theta - S_{i+1}}{\theta^3(\cos\theta - e^{\theta})}.$$

Using the continuity condition of the first and second derivatives at (x_i, y_i), that is, $P_{i-1}^{(n)}(x_i) = P_i^{(n)}(x_i)$, where $n = 1$ and $n = 2$, we get the following consistency relations for $i = 1, 2, \cdots, n$:

$$D_i + D_{i-1} = 2\,\frac{y_i - y_{i-1}}{h} + h^2\,\frac{2S_{i-1}e^{\theta}\cos\theta - 2S_i - (S_i - S_{i-1})(\cos\theta + e^{\theta})}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ 2h^2\,\frac{S_{i-1}(\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta) + S_i(e^{\theta} + \sin\theta - 1)}{\theta^3(\cos\theta - e^{\theta})}, \quad (2.5)$$

and

$$D_i - D_{i-1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h} + h^2\,\frac{(S_i - S_{i-1}e^{\theta})(\cos\theta - \sin\theta) + S_i e^{\theta} - S_{i+1}}{2\theta(\cos\theta - e^{\theta})}$$

$$+ h^2\,\frac{(S_i - S_{i-1})(\cos\theta + e^{\theta}) + 2(S_i - S_{i-1})}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ h^2\,\frac{(S_i - S_{i-1})(\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta) + (S_i - S_{i+1})(1 - e^{\theta} - \sin\theta)}{\theta^3(\cos\theta - e^{\theta})}.$$

(2.6)

Adding Equations (2.5) and (2.6), we get

$$D_i = \frac{y_{i+1} - y_{i-1}}{2h} + h^2 \frac{(S_i - S_{i-1}e^{\theta})(\cos\theta - \sin\theta) + S_i e^{\theta} - S_{i+1}}{4\theta(\cos\theta - e^{\theta})}$$

$$+ h^2 \frac{S_{i-1}\cos\theta e^{\theta} - S_{i+1}}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ h^2 \frac{(S_i + S_{i-1})(\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta) + (S_i + S_{i+1})(e^{\theta} + \sin\theta - 1)}{2\theta^3(\cos\theta - e^{\theta})}. \tag{2.7}$$

Similarly,

$$D_{i-1} = \frac{y_i - y_{i-2}}{2h} + h^2 \frac{(S_{i-1} - S_{i-2}e^{\theta})(\cos\theta - \sin\theta) + S_{i-1}e^{\theta} - S_i}{4\theta(\cos\theta - e^{\theta})}$$

$$+ h^2 \frac{S_{i-2}\cos\theta e^{\theta} - S_i}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ h^2 \frac{(S_{i-1} + S_{i-2})(\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta) + (S_{i-1} + S_i)(e^{\theta} + \sin\theta - 1)}{2\theta^3(\cos\theta - e^{\theta})}. \tag{2.8}$$

D_{i-1} and D_i are eliminated from Equation (2.5) with the help of Equations (2.7) and (2.8), as a result, we get the following scheme:

$$-y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} + h^3\Big[\frac{(\cos\theta - \sin\theta)e^{\theta}}{2\theta(\cos\theta - e^{\theta})} + \frac{-2\cos\theta e^{\theta}}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ \frac{\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta}{\theta^3(\cos\theta - e^{\theta})}\Big]S_{i-2}$$

$$+ h^3\Big[\frac{e^{\theta} + \cos\theta - \sin\theta - e^{\theta}\cos\theta + e^{\theta}\sin\theta}{2\theta(\cos\theta - e^{\theta})} + \frac{-2(e^{\theta}\cos\theta + \cos\theta + e^{\theta})}{\theta^2(\cos\theta - e^{\theta})}$$

$$+ \frac{-2(\cos\theta - e^{\theta}\cos\theta - e^{\theta}\sin\theta) + e^{\theta} + \sin\theta - 1}{\theta^3(\cos\theta - e^{\theta})}\Big]S_{i-1}$$

$$+ h^3\Big[\frac{\cos\theta - \sin\theta + e^\theta - 1}{2\theta(\cos\theta - e^\theta)} + \frac{2(\cos\theta + e^\theta + 1)}{\theta^2(\cos\theta - e^\theta)}$$

$$+ \frac{(\cos\theta - e^\theta\cos\theta - e^\theta\sin\theta) - 2(e^\theta + \sin\theta - 1)}{\theta^3(\cos\theta - e^\theta)}\Big]S_i$$

$$+ h^3\Big[\frac{-1}{2\theta(\cos\theta - e^\theta)} + \frac{-2}{\theta^2(\cos\theta - e^\theta)} + \frac{(e^\theta + \sin\theta - 1)}{\theta^3(\cos\theta - e^\theta)}\Big]S_{i+1} = 0,$$

$$i = 2, \cdots, n-1, \tag{2.9}$$

where

$$S_i = \begin{cases} f_i, & \text{for } 0 \le i \le \frac{n}{4} \text{ and } \frac{3n}{4} \le i \le n, \\ g_i y_i + f_i + r_i, & \text{for } \frac{n}{4} \le i \le \frac{3n}{4}, \end{cases}$$

and $i = 0, 1, \cdots, n-1$.

For simplicity, we rewrite Equation (2.9)

$$-y_{i-2} + h^3\alpha_1 S_{i-2} + 3y_{i-1} + h^3\beta_1 S_{i-1} - 3y_i + h^3\alpha_2 S_i + y_{i+1}$$

$$+ h^3\beta_2 S_{i+1} = 0, \tag{2.10}$$

where

$$\alpha_1 = \frac{(\cos\theta - \sin\theta)e^\theta}{2\theta(\cos\theta - e^\theta)} + \frac{-2\cos\theta e^\theta}{\theta^2(\cos\theta - e^\theta)} + \frac{\cos\theta - e^\theta\cos\theta - e^\theta\sin\theta}{\theta^3(\cos\theta - e^\theta)},$$

$$\beta_1 = \frac{(e^\theta + \cos\theta - \sin\theta - e^\theta\cos\theta + e^\theta\sin\theta)}{2\theta(\cos\theta - e^\theta)} + \frac{-2(e^\theta\cos\theta + \cos\theta + e^\theta)}{\theta^2(\cos\theta - e^\theta)}$$

$$+ \frac{-2(\cos\theta - e^\theta\cos\theta - e^\theta\sin\theta) + e^\theta + \sin\theta - 1}{\theta^3(\cos\theta - e^\theta)},$$

$$\alpha_2 = \frac{\cos\theta - \sin\theta + e^\theta - 1}{2\theta(\cos\theta - e^\theta)} + \frac{2(\cos\theta + e^\theta + 1)}{\theta^2(\cos\theta - e^\theta)}$$

$$+ \frac{(\cos\theta - e^\theta\cos\theta - e^\theta\sin\theta) - 2(e^\theta + \sin\theta - 1)}{\theta^3(\cos\theta - e^\theta)},$$

$$\beta_2 = \frac{-1}{2\theta(\cos\theta - e^{\theta})} + \frac{-2}{\theta^2(\cos\theta - e^{\theta})} + \frac{e^{\theta} + \sin\theta - 1}{\theta^3(\cos\theta - e^{\theta})}, \quad i = 2, \cdots, n-1.$$

The recurrence relation (2.10) gives $n-2$ linear equations in n unknown y_i, $i = 1, \cdots, n$. We need two more equations at each end of the range of integration. These two equations are given by [6]

$$3y_0 - 4y_1 + y_2 = -2hD_0 + \frac{h^3}{12}(3S_0 + 4S_1 + S_2), \quad i = 1, \tag{2.11}$$

and

$$-3y_{n-2} + 8y_{n-1} - 5y_n = -2hD_{n+1} + \frac{h^3}{12}(3S_{n-2} + 10S_{n-1} + 31S_n), \quad i = n. \tag{2.12}$$

The local truncation errors L_i, $i = 1, \cdots, n$ associated with our scheme is

$$L_i = \begin{cases} t_1 + o(h^6), & i = 1, \\ t_i + o(h^8), & 2 \le i \le n-1, \\ t_n + o(h^6), & i = n, \end{cases} \tag{2.13}$$

where

$$\begin{aligned}
t_1 &= -\frac{1}{10}h^5 y^{(5)}(\zeta_1), && a < \zeta_1 < x_2, \\
t_i &= h^3(1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2)y_i'''(\zeta_i) \\
&\quad + h^4(\frac{1}{2} + 2\alpha_1 + \beta_1 - \beta_2)y_i^{(4)}(\zeta_i) \\
&\quad + h^5(\frac{1}{4} - 2\alpha_1 - \frac{1}{2}\beta_1 - \frac{1}{2}\alpha_2)y_i^{(5)}(\zeta_i) \\
&\quad + h^6(-\frac{1}{12} + \frac{8}{6}\alpha_1 + \frac{1}{6}\beta_1 - \frac{1}{6}\beta_2)y_i^{(6)}(\zeta_i) \\
&\quad + h^7(\frac{1}{40} - \frac{16}{24}\alpha_1 - \frac{1}{24}\beta_1 - \frac{1}{24}\beta_2)y_i^{(7)}(\zeta_i), && x_{i-2} < \zeta_i < x_{i+1}, \\
t_n &= -\frac{1}{10}h^5 y^{(5)}(\zeta_n), && x_{n-2} < \zeta_n < b.
\end{aligned} \tag{2.14}$$

In conclusion, the constructed process of the above numerical method as follows:

Algorithm 1: The algorithm of non-polynomial spline method.

Step 1. Construct non-polynomial spline function $P_i(x)$ in the point of each interval period $[x_i, x_{i+1}]$, that is, $P_i(x) = a_i \sin k(x - x_i) + b_i e^{k(x-x_i)} + c_i(x - x_i)^2 + d_i(x - x_i) + e_i$, $i = 0, 1, \cdots, n$.

Step 2. Derive five undermined coefficients a_i, b_i, c_i, d_i, and e_i by using the property of $j(j = 1, 2, 3)$ derivative's continuity at the common points (x_i, y_i) of $P_i(x)$ in the point of each interval period $[x_i, x_{i+1}]$.

Step 3. Derive the equation of the numerical method according to $j(j = 1, 2, 3)$ derivative's continuity at the common points (x_i, y_i) of $P_i(x)$, that is,

$$\begin{cases} 3y_0 - 4y_1 + y_2 = -2hD_0 + \frac{h^3}{12}[3S_0 + 4S_1 + S_2], & i = 1, \\ -y_{i-2} + h^3\alpha_1 S_{i-2} + 3y_{i-1} + h^3\beta_1 S_{i-1} - 3y_i + h^3\alpha_2 S_i + y_{i+1} + h^3\beta_2 S_{i+1} = 0, & 2 \le i \le n-1, \\ -3y_{n-2} + 8y_{n-1} - 5y_n = -2hD_{n+1} + \frac{h^3}{12}[3S_{n-2} + 10S_{n-1} + 31S_n], & i = n. \end{cases}$$

Step 4. Derive the local truncation errors L_i, $i = 1, \cdots, n$ applying the Taylor expansion, that is, (2.13) and (2.14).

Step 5. End.

3. Convergence Analysis

In this section, we study the convergence analysis of the method developed in Section 2. For this purpose, we first give the necessary lemma as follows:

Lemma 1 ([13]). *Assume* $Y = (y_i)$ *is the analytical solution of obstacle boundary value problem* (1.1), $Y^{(1)} = (y_i^{(1)})$, $i = 1, \cdots, n$ *is the approximate solution at each point and satisfies the boundary condition*

(1.2), $e_i = y_i - y_i^{(1)}$ *is the absolute error. If* $\|E\|_\infty \cong o(h^2)$, *then the numerical method developed in Section* 2 *is second-order convergent on the whole interval* $[a, b]$.

The method developed in Section 2 is described in matrix form in the following way. Let $A = (a_{ij})_{i,j=1}^n$ denotes the four diagonal matrix. Clearly, the system (2.10)-(2.12) can be expressed in matrix form as

$$AY = C + T, \quad AY^{(1)} = C, \quad AE = T, \tag{3.1}$$

where $Y = (y_i)$, $Y^{(1)} = (Y_i^{(1)})$, $C = (c_i)$, $E = (e_i) = (y_i - y_i^{(1)})$ be n-dimensional column vectors, (y_i), $(y_i^{(1)})$ denote the analytical solution and approximate solution, respectively, and $A = A_0 + \frac{1}{12}h^3BG$, $G = \text{diag}(g_i)$, $i = 1, \cdots, n$, where $g_i \neq 0$ for $\frac{n+1}{4} < i \leq \frac{3(n+1)}{4}$, and the matrices A_0 and B are defined by

$$A_0 = \begin{pmatrix} -4 & 1 & 0 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & 3 & -3 & 1 \\ 0 & 0 & 0 & -3 & 8 & 5 \end{pmatrix}, \tag{3.2}$$

$$B = \begin{pmatrix} \frac{-4}{12} & \frac{-1}{12} & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & \frac{-3}{12} & \frac{-10}{12} & \frac{-31}{12} \end{pmatrix}, \tag{3.3}$$

$$c_i = \begin{cases} -3m - 2hp + \frac{1}{12}h^3F_1, & i = 1, \\ A_1 + h^3F_2, & i = 2, \\ h^3F_i, & 3 \le i \le \frac{n}{4} - 1,\ \frac{3n}{4} + 3 \le i \le n - 1, \\ h^3[F_i + \alpha_i r], & i = \frac{n}{4},\ i = \frac{3n}{4} + 2, \\ h^3[F_i + \alpha_1 r + \beta_1 r], & i = \frac{n}{4} + 1,\ i = \frac{3n}{4} + 1, \\ h^3[F_i + \alpha_1 r + \beta_1 r + \alpha_2 r], & i = \frac{n}{4} + 2,\ i = \frac{3n}{4}, \\ h^3[F_i + \alpha_1 r + \beta_1 r + \alpha_2 r + \beta_2 r], & i = \frac{n}{4} + 3,\ i = \frac{3n}{4} - 1, \\ -2hq + \frac{1}{12}h^3F_i, & i = n, \end{cases} \tag{3.4}$$

where

$$F_i = \begin{cases} 3f_0 + 4f_1 + f_2, & i = 1, \\ \alpha_1 f_{i-2} + \beta_1 f_{i-1} + \beta_2 f_i + \alpha_2 f_{i+1}, & 2 \le i \le n - 1, \\ 3f_{i-2} + 10f_{i-1} + 31f_i, & i = n, \end{cases} \tag{3.5}$$

$m = y_0 = y(a)$, $D_0 = p = y'(a)$, $D_{n+1} = q = y'(b)$ are known. So, we have the following theorem:

Theorem 1. *The new numerical method that presented by* (2.10)-(2.12) *based on the non-polynomial spline functions is second-order convergent on the interval* $[a, b]$.

Proof. According to Lemma 1, our main purpose is to derive a bound on $\|E\|$. From Equation (3.1), we have

$$E = A^{-1}T = (A_0 + \frac{1}{12}h^3BG)^{-1}T = (I + \frac{1}{12}h^3A_0^{-1}BG)^{-1}A_0^{-1}T,$$

then

$$\|E\|_\infty = \|(I + \frac{1}{12}A_0^{-1}BG)^{-1}\|_\infty \|A_0^{-1}\|_\infty \|T\|_\infty,$$

where $\|\cdot\|_\infty$ represents the ∞-norm in matrix vector. Using the result $\|E\|_\infty = 1$ and $\|(I + A)^{-1}\|_\infty \le (1 - \|A\|_\infty)^{-1}$, we get the following expression:

$$\|E\|_\infty \le \frac{\|A_0^{-1}\|_\infty \cdot \|T\|_\infty}{1 - \frac{1}{12}h^3\|A_0^{-1}\|_\infty \|B\|_\infty \|G\|_\infty}, \tag{3.6}$$

provided that $\frac{1}{12}h^3\|A_0^{-1}\|_\infty \|B\|_\infty \|G\|_\infty < 1$.

Now from (2.13)-(2.14), we have

$$\|T\|_\infty = h^5(\frac{1}{4} - 2\alpha_1 - \frac{1}{2}\beta_1 - \frac{1}{2}\alpha_2)M_5, \quad M_5 = \max|y^{(5)}(x)|. \tag{3.7}$$

The following proof is to prove that the matrix A_0 is nonsingular, due to

$$|A_0| = \begin{cases} -4, & i = 1, \\ 9, & i = 2, \\ -40, & i = 3, \end{cases}$$

from $i = 4$, the value of $|A_0|$ has obvious laws. So when $i = 4$,

$$\begin{pmatrix} -4 & -1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ -1 & 3 & -3 & 1 \\ 0 & 3 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} -4 & 0 & 0 & 0 \\ 3 & -\frac{9}{4} & 0 & 0 \\ -1 & -\frac{4}{11} & -\frac{16}{9} & 0 \\ 0 & -3 & \frac{5}{3} & -\frac{5}{4} \end{pmatrix},$$

then $|A_0| = (-4)(-\frac{9}{4})(-\frac{16}{9})(-\frac{5}{4}) = 20$, thus, when $i = n$, we have $|A_0| = (-4)(-\frac{9}{4})(-\frac{16}{9})\cdots(\frac{b}{m})(\frac{a}{n}) \neq 0$, therefore, A_0 is nonsingular. According to [14], the matrix A_0^{-1} satisfies the inequality:

$$\|A_0^{-1}\|_\infty \leq \frac{3n^3 + 5n^2 - 8n + 4}{32}. \tag{3.8}$$

Then using $x_0 = a$, $x_n = b$, and $h = \frac{b-a}{n+1}$, we can write (3.8) in the following form:

$$\|A_0^{-1}\|_\infty \leq \frac{14h^3 - 9(b-a)h^2 - 4(b-a)^2h + 3(b-a)^3}{32h^3}. \tag{3.9}$$

Thus, using (3.6), (3.7), and (3.9), and the fact that $\|B\|_\infty = \frac{44}{12}$, $\|G\|_\infty < |g(x)|$, we get

$$\|E\|_\infty \leq \frac{\|A_0^{-1}\|_\infty M_5 h^5}{1 - \frac{1}{12}h^3\|A_0^{-1}\|_\infty\|B\|_\infty|g(x)|} = \frac{\lambda M_5 h^2}{12[3 - 11\lambda|g(x)|]} \cong o(h^2), \tag{3.10}$$

where

$$\lambda = \frac{14h^3 - 9(b-a)h^2 - 4(b-a)^2h + 3(b-a)^3}{32}.$$

This inequality shows that the method developed in this paper is a second-order convergent method for arbitrary values of α_1, α_2, β_1, and β_2, whose sum is less than or equal to $\frac{44}{12}$.

4. Application

To illustrate the application of the numerical method developed in the previous sections, we consider the third-order obstacle boundary value problem of finding y such that

$$-y''' \geq f, \qquad \text{on } \Omega = [0, 1], \tag{4.1}$$

$$y \geq \varphi, \qquad \text{on } \Omega = [0, 1], \tag{4.2}$$

$$[-y''' - f](y - \varphi) = 0, \text{ on } \Omega = [0, 1], \tag{4.3}$$

$$y(0) = 0, \quad y'(0) = 0, \quad y'(1) = 0, \tag{4.4}$$

where $f(x)$ is a continuous function and $\varphi(x)$ is the obstacle function. We study the problem (4.1)-(4.4) in the framework of the variational inequality approach. To do so, first we define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \geq \varphi, \text{ on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolov space, which is in fact a Hilbert space. For the definitions of the spaces $H_0^2(\Omega)$, see [15]. In passing, we remark that problem (4.1)-(4.4) without constraints has been studied in [10-12]. Thus, it is clear that problem (4.1)-(4.4) is a generalization of third-order boundary value problems, which have many important applications in engineering sciences. Now, using the techniques of Noor [7] and Filippov [16], one can show easily that the energy functional associated with problem (4.1)-(4.4) is

$$\begin{aligned} I(v) &= -\int_0^1 \left(\frac{d^3v}{dx^3}\right)\left(\frac{dv}{dx}\right)dx \\ &= \int_0^1 \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx \\ &= \langle T(v),\ g(v)\rangle - 2\langle f,\ g(v)\rangle, \end{aligned} \tag{4.5}$$

where

$$\langle T(u),\ g(v)\rangle = \int_0^1 \left(\frac{d^2v}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right)dx, \tag{4.6}$$

and $\frac{d}{dx}$ is a linear operator. It is clear that the operator T defined by (4.6) is linear. Using the technique of Noor [7], one can show easily that the minimum of the function $I(v)$ defined by (4.5) associated with the problem (4.1)-(4.4) on the closed convex set K can be characterized by the variational inequality

$$\langle T(y),\ g(v) - g(y)\rangle \geq \langle f,\ g(v) - g(y)\rangle, \text{ for all } g(v) \in K. \tag{4.7}$$

Now, using the ideas of Lewy and Stampacchia [17], problem (4.1)-(4.4) can be written as

$$-y''' + v\{(y - v)\}(y - v) = f, \quad 0 < x < 1, \tag{4.8}$$

$$y(0) = y'(0) = y'(1) = 0, \tag{4.9}$$

where

$$v(t) = \begin{cases} 1, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0, \end{cases} \tag{4.10}$$

is a discontinuous function, and where φ is the given obstacle function defined by

$$\varphi(x) = \begin{cases} -1, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1, \\ 1, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}. \end{cases} \tag{4.11}$$

From Equations (4.8)-(4.11), we obtain the following system of differential equations:

$$y''' = \begin{cases} f, & \text{for } 0 \leq x \leq \frac{1}{4} \text{ and } \frac{3}{4} \leq x \leq 1, \\ y + f + 1, & \text{for } \frac{1}{4} \leq x \leq \frac{3}{4}, \end{cases} \tag{4.12}$$

with the boundary conditions:

$$y(0) = y'(0) = y'(1) = 0, \tag{4.13}$$

and the condition of continuity of y, y', y'' at $x = \frac{1}{4}$ and $x = \frac{3}{4}$. Note that the system of differential equations (4.12) is a special form of the system (1.1) with $g(x) = 1$ and $r(x) = -1$.

For $f = 0$, the system of differential equations (4.12) reduces to

$$y''' = \begin{cases} 0, & \text{for } 0 \le x \le \frac{1}{4} \text{ and } \frac{3}{4} \le x \le 1, \\ y - 1, & \text{for } \frac{1}{4} \le x \le \frac{3}{4}, \end{cases} \tag{4.14}$$

with the boundary conditions (4.13). The analytical solution for this problem is

$$y(x) = \begin{cases} \frac{1}{2} a_1 x^2, & \text{for } 0 \le x \le \frac{1}{4}, \\ 1 + a_2 e^x + e^{-\frac{x}{2}}[a_3 \cos(\frac{\sqrt{3}}{2} x) + a_4 \sin(\frac{\sqrt{3}}{2} x)], & \text{for } \frac{1}{4} \le x \le \frac{3}{4}, \\ \frac{1}{2} a_5 x(x-2) + a_6, & \text{for } \frac{3}{4} \le x \le 1. \end{cases} \tag{4.15}$$

To find the constants, a_i, $i = 1, 2, \cdots, 6$, we apply the continuity conditions of y, y', y'' at $x = \frac{1}{4}$ and $x = \frac{3}{4}$, and we obtain the following system of linear equations:

$$\begin{pmatrix} \frac{1}{32} & -S_1 & -S_2CS_1 & -S_2SC_1 & 0 & 0 \\ \frac{1}{4} & -S_1 & \frac{1}{2}S_2\eta_1 & -\frac{1}{2}S_2\eta_3 & 0 & 0 \\ 1 & -S_1 & -\frac{1}{2}S_2\eta_2 & \frac{1}{2}S_2\eta_4 & 0 & 0 \\ 0 & S_3 & S_4CS_2 & S_4SC_2 & \frac{15}{32} & -1 \\ 0 & S_3 & -\frac{1}{2}S_4\lambda_1 & \frac{1}{2}S_4\lambda_3 & \frac{1}{4} & 0 \\ 0 & S_3 & \frac{1}{2}S_4\lambda_2 & -\frac{1}{2}S_4\lambda_4 & -1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

where

$$S_1 = e^{\frac{1}{4}}, \quad S_2 = e^{\frac{-1}{8}}, \quad S_3 = e^{\frac{3}{4}}, \quad S_4 = e^{\frac{-3}{8}},$$

$$\eta_1 = \sqrt{3}SC_1 + CS_1, \quad \eta_2 = \sqrt{3}SC_1 - CS_1, \quad \eta_3 = \sqrt{3}CS_1 - SC_1,$$

$$\eta_4 = \sqrt{3}CS_1 + SC_1,$$

$$\lambda_1 = \sqrt{3}SC_2 + CS_2, \quad \lambda_2 = \sqrt{3}SC_2 - CS_2, \quad \lambda_3 = \sqrt{3}CS_2 - SC_2,$$

$$\lambda_4 = \sqrt{3}CS_2 + SC_2,$$

$$CS_1 = \cos\frac{\sqrt{3}}{8}, \; SC_1 = \sin\frac{\sqrt{3}}{8}, \; CS_2 = \cos\frac{3\sqrt{3}}{8}, \; SC_2 = \sin\frac{3\sqrt{3}}{8}.$$

One can find the analytical solution of this system of linear equation by using Gaussian elimination.

For a variety values of h, the boundary value problem defined by (4.13) and (4.14) was solved using the numerical method developed in the previous sections and some results are given in Table 1. The system of differential equations (4.14) along with the boundary conditions (4.13) was also solved in [2] using a finite difference method based on the difference formula

$$h^3 y_i''' = -y_{i-2} + 3y_{i-1} - 3y_i + y_{i+1} + o(h^4), \tag{4.16}$$

and in [1] using a third-order collocation method with quintic B-spline as basis functions, and in [18] using a cubic spline method with polynomial functions as basis functions. Their numerical results are also given in Table 1.

Table 1. The observed maximum errors $\|E\|_\infty$

h	Our method	Cubic splines [18]	Colloc-quintic [1]	Scheme(4.16) [2]
$\frac{1}{16}$	1.20×10^{-3}	1.23×10^{-3}	1.26×10^{-3}	6.89×10^{-3}
$\frac{1}{32}$	6.74×10^{-4}	5.53×10^{-4}	5.60×10^{-4}	7.11×10^{-3}
$\frac{1}{64}$	4.82×10^{-4}	2.61×10^{-4}	3.10×10^{-4}	7.27×10^{-3}
$\frac{1}{128}$	2.98×10^{-4}	1.27×10^{-4}	1.61×10^{-4}	7.36×10^{-3}

It may be noted from Table 1, when the step size $h = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$, our non-polynomial spline method gives better results than the finite difference method developed in [2]; when the step size $h = \frac{1}{16}$, our method gives better result than the other three methods in Table 1. Therefore, our proposed method has high precision and good approximation effect.

From Table 1, we can also see that the maximum errors decrease gradually as the step size decreases and the maximum errors reach 10^{-4} when the step size $h = \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$.

In order to analyze our proposed method's property more intuitively, we will do the error analysis with the analytical solution. Figure 1 shows the compared results between the numerical solution of our proposed method and the analytical solution when the step size $h = \frac{1}{32}, \frac{1}{128}$. From Figure 1, we can see, when the step size $h = \frac{1}{128}$, the approximate solutions of our proposed method almost coincide with analytical solutions, and the approximative effect is much better than the step size $h = \frac{1}{32}$. In addition, we often reduce the step size in order to get high precision and the ideal approximative effect, but which may be consume

a lot of computing time of computer. Therefore, for the practical problems, as long as it can meet our needs, we tend to take a small step size simply, which reflects the superiority of our proposed method.

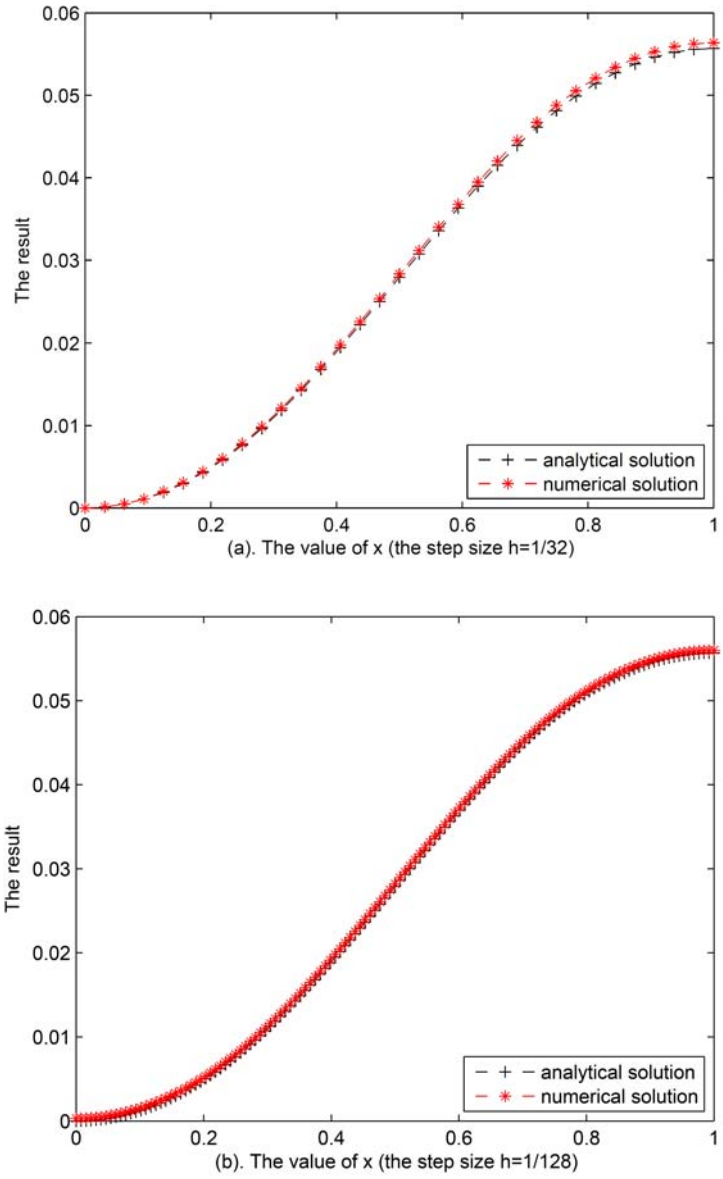

Figure 1. The analytical solution and numerical solution under different step sizes.

5. Conclusion

In this paper, first, we have developed a new numerical method: non-polynomial spline functions constructed with exponential function e^{kx}, trigonometric function and polynomial functions as basis functions for solving a system of third-order obstacle boundary problems. Then through convergence analysis and application, we see that our proposed method is of feasibility. Finally, a practical example is given to illustrate our proposed method's high precision and good approximation effect.

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