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NONLINEAR STABILITY OF PLANAR STATIONARY WAVES FOR DAMPED WAVE EQUATION WITH NONLINEAR CONVECTION TERM IN 2-DIMENSIONAL SPACE WITH TWO BOUNDARIES

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Abstract

We investigate the nonlinear stability of increasing planar stationary wave of initial-boundary value problem for damped wave equation with nonlinear convection term in 2-dimensional space with two boundaries. For the non-degenerate case, under the smallness conditions on the initial-boundary perturbation, we use the standard energy method to obtain the global existence of solution and the algebraic convergence rate of solutions toward the planar stationary waves for such an initial-boundary value problem.

1. Introduction

In this paper, we consider the following damped wave equation with a nonlinear convection term in two-dimensional space:

$$u_{tt} - \Delta u + u_t + f(u)_x + g(u)_y = 0, \quad t > 0, \quad (x, y) \in (0, 1) \times R, \quad (1.1)$$

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with the initial condition

$$u(0, x, y) = u_0(x, y), \quad u_t(0, x, y) = u_1(x, y), \quad (x, y) \in [0, 1] \times R, \quad (1.2)$$

and the boundary condition

$$u(t, 0, y) = u_-(t, y), \quad u(t, 1, y) = u_+(t, y), \quad t \geq 0, \quad y \in R, \quad (1.3)$$

which satisfy $u_-(t, y) \to u_-$, $u_+(t, y) \to u_+$ as $t \to +\infty$ for any fixed $y \in R$, where $u_\pm$ is given constant, which satisfy $u_- < u_+$, and $u_0(0, y) = u_-(0, y)$, $u_0(1, y) = u_+(0, y)$, $u_1(0, y) = \partial_t u_-(0, y)$, $u_1(1, y) = \partial_t u_+(0, y)$ for compatibility. We assume that f, g are smooth, and f is convex, i.e., $f''(u) > 0$ for any u under consideration, and $(f(u), g(u))$ satisfies the following condition, which is called the sub-characteristic condition in Ueda's paper [1], i.e.,

$$\left(|f'(u)|^2 + |g'(u)|^2\right)^{\frac{1}{2}} < 1, \quad (1.4)$$

for any $u \in I[u_-, u_+] \triangleq [\min\{u_-, u_+\} - 1, \max\{u_-, u_+\} + 1]$.

The main purpose of this paper is devoted to studying the nonlinear stability of planar stationary wave $\phi(x)$, which is independent of t and y, and to deducing the decay rates of solution $u(t, x, y)$ toward the planar stationary wave of the initial-boundary value problem (1.1)-(1.3). By the way, we only consider the non-degenerate case $f'(u_+) < 0$, we will use the tricks in [1] to simplify the calculation of energy integral and the method in [3] to deal with the boundary energy integral.

We recall that the stationary wave $\phi(x)$ of (1.1)-(1.3) is the unique solution of the following problem:

$$f(\phi)_x = \phi_{xx}, \quad x \in (0, 1), \quad (1.5)$$

$$\phi(0) = u_-, \quad \phi(1) = u_+. \quad (1.6)$$

According to [1, 3, 4, 7], we give the corresponding existence result and related properties of $\phi(x)$ for the non-degenerate case $f'(u_+) < 0$ as follow.

Lemma 1.1. *A necessary condition for the existence of solutions to the boundary value problem* (1.5)-(1.6) *is* $f'(u_+) \le 0$, *we only consider the case* $f'(u_+) < 0$. *If* $u_- < u_+$, *there exists a monotone increasing solution* $\phi(x)$, *if* $u_- > u_+$ *and* $f(u) < f(u_+)$ *for* $u \in (u_+, u_-)$, *there exists a monotone decreasing solution* $\phi(x)$. *In both cases, the solution* $\phi(x)$ *satisfies*

$$|\partial_x^k \phi(x)| \le C|u_+ - u_-|, \tag{1.7}$$

for $x \in [0, 1]$, *where* $k \ge 0$ *is an integer.*

In recent years, the study of the initial-boundary value problem for damped equations with a nonlinear convection term in multi-dimensional half space

$$\begin{cases} u_{tt} - \Delta u + u_t + \operatorname{div}(f(u)) = 0, \quad t > 0, \quad x = (x_1, x') \in R_+^n (:= R_+ \times R^{n-1}), \\ u(0, x) = u_0(x) \to u_+, \text{ as } x_1 \to +\infty, \\ u_t(0, x) = u_1(x), \quad u(t, 0, x') = u_b, \quad x' = (x_2, x_3, \cdots, x_n) \in R^{n-1}, \end{cases} \tag{1.8}$$

has made much progress, where $\Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}$, $f = (f_1, \cdots, f_n)$. For the non-degenerate case $f_1'(u_+) < 0$, Ueda in [1] obtained the time global existence of solutions of problem (1.8) in one-dimensional space in an H-neighbourhood of the stationary wave under smallness condition on initial perturbation, and by using the space-time weighted energy method initiated by Kawashima and Matsumura [9], the time convergence rate is algebraically (resp., exponentially) fast if the initial perturbation decays algebraically (resp., exponentially) as $x \to +\infty$. Furthermore, Ueda et al. in [2] generalized the results in one-dimensional half space [1] to the multi-dimensional half space for the case $f_1'(u_b) < 0$, and Fan et al. in [4] showed the similar results as in [2] still hold for the case $f_1'(u_b) \ge 0$.

We note that the analysis in [1, 2, 4] relied on the smallness condition on initial perturbation. Fan et al. in [5, 6] studied the global stability of weak boundary layer solution to the initial-boundary problem (1.8) in one-dimensional half space and two-dimensional half space, respectively, together with the corresponding convergence rates (both algebraic and exponential), provided the strength of the planar boundary layer solution is suitably small. All the results in [1, 2, 4-6] were obtain under the non-degenerate case. This paper is dedicate to investigate the nonlinear stability of solutions and their decay rates toward the increasing planar stationary wave of the initial-boundary value problem for damped wave equation with nonlinear convection term in 2-dimensional space with two boundaries under smallness condition on initial-boundary perturbation.

Remark 1.2. Denote the usual Lebesgues space by $L^2 = L^2([0,1]\times R)$ with norm $\|f\| = \|f\|_{L^2}$ and Sobolev space by $H^2 = H^2([0,1]\times R)$ with norm $\|f\|_2 = \|f\|_{L^2} + \|\nabla f\|_{L^2} + \|\nabla^2 f\|_{L^2}$, ∂_x denotes the partial differentiation with respect to x and ∂_{xy} denotes the second order partial differentiation with respect to x, y. Notice that $\partial_x u = u_x$, $\partial_x^2 u = u_{xx}$, $\partial_{xy} u = u_{xy}$. And let $L^2(t, y) = L^2(R_+; L^2(R))$, $L^2(x) = L^2[0,1]$, $L^2(y) = L^2(R)$, $L^2(t) = L^2(R_+)$, where

$$\nabla = (\partial_x, \partial_y), \quad \nabla^2 = (\partial_x^2, \partial_{xy}, \partial_y^2),$$

$$|\nabla u| = \left\{(\partial_x u)^2 + (\partial_y u)^2\right\}^{1/2}, \quad \left|\nabla^2 u\right| = \left\{(\partial_x^2 u)^2 + 2(\partial_{xy} u)^2 + (\partial_y^2 u)^2\right\}^{1/2},$$

$$\|u(t)\| = \left(\int_R \int_0^1 |u(t, x, y)|^2 dxdy\right)^{1/2}.$$

For $\alpha \in R$, $L^2_\alpha = L^2_\alpha([0,1]\times R)$ denotes the algebraically weighted L^2 space with the norm

$$\|f\|_{L^2_\alpha} = \left(\int_R \int_0^1 (1+|x|)^\alpha f^2(x, y) dxdy\right)^{1/2}.$$

Throughout this paper, C denotes a positive constant, it may be different in different place, but we use the same notation for simplicity.

2. Main Results

We state our main results of this paper as follow:

Theorem 2.1. *Let* $\phi(x)$ *be the stationary wave of the problem* (1.5) *and* (1.6), *which is monotone increasing in* $x \in [0, 1]$. *Assume that* $u_0 - \phi \in H^2$, $u_1 \in H^1$ *and the boundary data* $u_\pm(t, y)$ *satisfies*

$$u_\pm(t, y) - u_\pm \in H^3(R_+; L^2(R)), \quad \partial_y^2 u_\pm(t, y) \in H^1(R_+; L^2(R)). \tag{2.1}$$

Put

$$E_0 = \|u_0 - \phi\|_2 + \|u_1\|_1,$$

$$R_0 = \sum_{k=0}^{3} \left(\|\partial_t^k (u_+(t, y) - u_+)\|_{L^2_{(t, y)}} + \|\partial_t^k (u_-(t, y) - u_-)\|_{L^2_{(t, y)}} \right)$$

$$+ \sum_{k=0,1} \left(\|\partial_t^k \partial_y^2 u_+(t, y)\|_{L^2_{(t, y)}} + \|\partial_t^k \partial_y^2 u_-(t, y)\|_{L^2_{(t, y)}} \right),$$

there is a positive constant δ_0 *such that if* $E_0 + R_0 \le \delta_0$, *then the initial-boundary problem* (1.1)-(1.3) *has a unique global solution* $u(t, x, y)$ *with*

$$u - \phi \in C(R_+; H^2), \quad u_t \in L^2(R_+; H^1),$$

and

$$\lim_{t \to \infty} \sup_{(x, y) \in [0, 1] \times R} |u(t, x, y) - \phi(x)| = 0,$$

$$\lim_{t \to \infty} \sup_{(x, y) \in [0, 1] \times R} |\nabla(u(t, x, y) - \phi(x))| = 0. \tag{2.2}$$

In this paper, we only consider the algebraic stability case since the exponential stability result can be obtain similarly. Now we give the algebraic stability result.

Theorem 2.2. *Let* $\phi(x)$ *be the stationary wave of the problem* (1.5) *and* (1.6), *which is monotone increasing in* $x \in [0, 1]$, *and let* $u(t, x, y)$ *be the global solution of the problem* (1.1)-(1.3), *which is constructed in Theorem* 2.1 *under the smallness condition on* $E_0 + R_0$. *If* $u_0 - \phi \in H^2_\alpha$, $u_1 \in H^1_\alpha$ *for* $\alpha \geq 0$ *and the boundary data* $u_\pm(t, y)$ *satisfies*

$$(1+t)^\gamma (u_\pm(t, y) - u_\pm) \in H^3(R_+; L^2(R)), (1+t)^\gamma \partial_y^2 u_\pm(t, y) \in H^1(R_+; L^2(R)), \tag{2.3}$$

for any γ *with* $\gamma > \alpha$, *then we have*

$$\|u(t, x, y) - \phi(x)\|_{H^2} \leq C(E_\alpha + R_\gamma)(1+t)^{-\alpha/2}, \tag{2.4}$$

for $t \geq 0$, *where* $E_\alpha = \|u_0 - \phi\|_{H^2_\alpha} + \|u_1\|_{H^1_\alpha}$ *and*

$$R_\gamma = \sum_{k=0}^{3} \left(\int_0^\infty \int_R (1+\tau)^\gamma \left(|\partial_t^k (u_+(\tau, y) - u_+)|^2 + |\partial_t^k (u_-(\tau, y) - u_-)|^2 \right) dy d\tau \right)^{\frac{1}{2}}$$

$$+ \sum_{k=0,1} \left(\int_0^\infty \int_R (1+\tau)^\gamma \left(|\partial_t^k \partial_y^2 u_+(\tau, y)|^2 + |\partial_t^k \partial_y^2 u_-(\tau, y)|^2 \right) dy d\tau \right)^{\frac{1}{2}}.$$

3. Uniform Energy Estimates

In this section, we introduce a new unknown function $v(t, x, y)$ by

$$u(t, x, y) = \phi(x) + v(t, x, y),$$

where $\phi(x)$ is the stationary solution of problem (1.5) and (1.6), then the initial-boundary problem (1.1)-(1.3) can be reformulated as

$$v_{tt} - \Delta v + v_t + (f(v+\phi) - f(\phi))_x + g(v+\phi)_y = 0, \quad t > 0, (x, y) \in (0, 1) \times R, \tag{3.1}$$

$$\left.\begin{aligned} v(0,\, x,\, y) &= v_0(x,\, y) \triangleq u_0(x,\, y) - \phi(x) \\ v_t(0,\, x,\, y) &= v_1(x,\, y) \triangleq u_1(x,\, y) \end{aligned}\right\}, \quad (x,\, y) \in [0,\, 1] \times R, \tag{3.2}$$

$$v(t,\, 0,\, y) = u_-(t,\, y) - u_-,\ v(t,\, 1,\, y) = u_+(t,\, y) - u_+, \quad t \geq 0,\ y \in R. \tag{3.3}$$

Theorem 3.1. *Assume that the same conditions as those in Theorem* 2.1 *hold and that* $v_0 \in H^2$, $v_1 \in H^1$ *and the boundary data satisfies* (2.1). *If there is a positive constant* δ_0 *such that if* $E_0 + R_0 \leq \delta_0$, *then the initial-boundary problem* (3.1)-(3.3) *has a unique global solution* $v(t,\, x,\, y)$ *with*

$$v \in C(R_+;\, H^2), \quad v_t \in L^2(R_+;\, H^1),$$

and

$$\sup_{(x,\, y) \in [0,\, 1] \times R} |v(t,\, x,\, y)| \to 0, \quad \sup_{(x,\, y) \in [0,\, 1] \times R} |\nabla v(t,\, x,\, y)| \to 0, \text{ as } t \to +\infty. \tag{3.4}$$

Theorem 2.1 is a direct consequence of Theorem 3.1, and the local existence combining with the priori estimates can prove Theorem 3.1. For any $T > 0$, we define the solution space by

$$X(0,\, T) = \{v \in C([0,\, T];\, H^2),\, v_t \in L^2([0,\, T];\, H^1),$$

$$\text{with } \partial^m v(t,\, x,\, y)\big|_{x=0,1} < +\infty, \text{ for } t \in (0,\, T], \quad m \in Z_+\}.$$

We can show the local existence of the solution for problem (3.1)-(3.3) in a standard way, so we omit the details. The following proposition gives a priori estimate.

Proposition 3.2 (Priori estimate)**.** *Suppose that* v *is a solution of* (3.1)-(3.3) *in* $X(0,\, T)$ *for a positive constant T. Put*

$$E_0 = \|v_0\|_2 + \|v_1\|_1, \quad E(T) = \sup_{0 \leq t \leq T} (\|v(t)\|_2 + \|v_t\|_1),$$

then there exists a positive constant δ *independent of T, such that if* $E(T) \leq \delta$, *then v satisfies the following energy estimate*:

$$\|v(t)\|_2^2 + \|v_t(t)\|_1^2 + \int_0^t \left(\left\| \sqrt{\phi_x} v(\tau) \right\|^2 + \|\nabla v(\tau)\|_1^2 + \|v_t(\tau)\|_1^2 \right) d\tau \le C(E_0^2 + R_0^2), \tag{3.5}$$

for $t \in [0, T]$, *where* C *is a positive constant*.

Proof. Calculate $2 \times (3.1)v_t + (3.1)v$, we obtain that

$$\partial_t I + \partial_x (J_1 - J_2) + \partial_y K + L = 0, \tag{3.6}$$

where

$$\begin{aligned}
I &= \frac{1}{2} v^2 + v_t^2 + |\nabla v|^2 + v v_t + 2\phi_x \int_0^v (f'(\phi + \eta) - f'(\phi)) d\eta, \\
J_1 &= (f(\phi + v) - f(\phi)) v - \int_0^v (f(\phi + \eta) - f(\phi)) d\eta, \\
J_2 &= 2 v_t v_x + v v_x, \\
K &= g(\phi + v) v - \int_\phi^{v+\phi} g(s) ds - 2 v_t v_y - v v_y, \\
L &= v_t^2 + |\nabla v|^2 + 2 f'(\phi + v) v_t v_x + 2 g'(\phi + v) v_y v_t \\
&\quad + \phi_x \int_0^v (f'(\phi + \eta) - f'(\phi)) d\eta.
\end{aligned} \tag{3.7}$$

According to the smallness condition $E(T) \le \delta$ and the Sobolev's inequality, $|v|$ and $|\nabla v|$ are bounded and by Ueda's paper [1], we have the following estimate:

$$\begin{aligned}
c\left(v_t^2 + |\nabla v|^2 + v^2 + \phi_x v^2\right) \le I \le C\left(v_t^2 + |\nabla v|^2 + v^2 + \phi_x v^2\right), \\
J_1 \le -c v^2, \quad |J_2| \le C\left(v_t^2 + |\nabla v|^2 + |v||\nabla v|\right), \\
c\left(v_t^2 + |\nabla v|^2 + \phi_x v^2\right) \le L \le C\left(v_t^2 + |\nabla v|^2 + \phi_x v^2\right),
\end{aligned} \tag{3.8}$$

where c and C are positive constants independent of T.

Now we estimate the integral $\int_0^t\int_R\int_0^1 \partial_x(J_1 - J_2)dxdyd\tau$ as follow:

$$\int_0^t\int_R\int_0^1 \partial_x J_1 dxdyd\tau \le C\int_0^{+\infty}\int_R \Big(|u_+(\tau,\ y) - u_+|^2 + |u_-(\tau,\ y) - u_-|^2\Big)dyd\tau, \tag{3.9}$$

$$\begin{aligned}\int_0^t\int_R\int_0^1 \partial_x J_2 dxdyd\tau &= \int_0^t\int_R \{2\partial_t u_+(\tau,\ y) + (u_+(\tau,\ y) - u_+)\}v_x(\tau,\ 1,\ y)dyd\tau\\ &\quad -\int_0^t\int_R \{2\partial_t u_-(\tau, y) + (u_-(\tau, y) - u_-)\}v_x(\tau, 0, y)dyd\tau\\ &\le C_{\mu_1}\int_0^{+\infty}\int_R \Big(|u_+(\tau, y) - u_+|^2 + |u_-(\tau, y) - u_-|^2\Big)dyd\tau\\ &\quad + C_{\mu_1}\int_0^{+\infty}\int_R \Big\{|\partial_t u_+(\tau,\ y)|^2 + |\partial_t u_-(\tau,\ y)|^2\Big\}dyd\tau\\ &\quad + \mu_1\int_0^t\int_R (v_x^2(\tau,\ 1,\ y) + v_x^2(\tau,\ 0,\ y))dyd\tau, \end{aligned}\tag{3.10}$$

where $\mu_1 > 0$ is a sufficient small constant and C_{μ_1} is positive constant.

$$\begin{aligned}\int_0^t\int_R (v_x^2(\tau,\ 1,\ y) + v_x^2(\tau,\ 0,\ y))dyd\tau &\le \int_0^t\int_R 2\|v_x\|_{L^2(x)}\|v_{xx}\|_{L^2(x)}dyd\tau\\ &\le \int_0^t\int_R \Big\{\|v_x\|^2_{L^2(x)} + \|v_{xx}\|^2_{L^2(x)}\Big\}dyd\tau\\ &\le \int_0^t \Big(\|v_x(\tau)\|^2 + \|v_{xx}(\tau)\|^2\Big)d\tau. \end{aligned}\tag{3.11}$$

Integrating (3.6) over $(0,\ t)\times[0,\ 1]\times R$ with respect to $(t,\ x,\ y)$, and combining (3.7)-(3.11), we have that

$$\|v(t)\|_1^2 + \|v_t(t)\|^2 + \int_0^t \left(\left\|\sqrt{\phi_x} v(\tau)\right\|^2 + \|\nabla v(\tau)\|^2 + \|v_t(\tau)\|^2 \right) d\tau$$

$$\le \mu_1 \int_0^t \|v_{xx}(\tau)\|^2 d\tau + C \Big\{ \|v_0\|_1^2 + \|v_1\|^2 + \|u_+(t,\, y) - u_+\|^2_{L^2(t,\, y)}$$

$$+ \|u_-(t,\, y) - u_-\|^2_{L^2(t,\, y)} + \|\partial_t u_+(t,\, y)\|^2_{L^2(t,\, y)} + \|\partial_t u_-(t,\, y)\|^2_{L^2(t,\, y)} \Big\}. \quad (3.12)$$

Next, calculating $2 \times \nabla(3.1)\nabla v_t + \nabla(3.1)\nabla v$, we obtain that

$$\partial_t A + \partial_x (B_1 + B_2) + \partial_y B_3 + (D_1 - D_2 - D_3) = 0, \quad (3.13)$$

where

$$\begin{aligned}
A &= \frac{1}{2} |\nabla v|^2 + |\nabla v_t|^2 + |\nabla^2 v|^2 + \nabla v \nabla v_t, \\
B_1 &= v_x (f(\phi + v) - f(\phi))_x + v_x g(\phi + v)_y - v_x v_{xx} - v_{xy} v_y, \\
B_2 &= 2v_{xt} (f(\phi + v) - f(\phi))_x + 2v_{xt} g(\phi + v)_y - 2v_{xt} v_{xx} - 2v_{xy} v_{yt}, \\
B_3 &= \{ (f(\phi + v) - f(\phi))_x + g(\phi + v)_y - v_{xx} - v_{xy} \} (v_y + 2v_{yt}), \quad (3.14) \\
D_1 &= |\nabla^2 v|^2 + |\nabla v_t|^2, \\
D_2 &= (v_{xx} + v_{yy}) \{ (f(\phi + v) - f(\phi))_x + g(\phi + v)_y \}, \\
D_3 &= (v_{xxt} + v_{yyt}) \{ (f(\phi + v) - f(\phi))_x + g(\phi + v)_y \}.
\end{aligned}$$

Similar to (3.8), one gets

$$c \left(|\nabla v_t|^2 + |\nabla v|^2 + |\nabla^2 v|^2 \right) \le A \le C \left(|\nabla v_t|^2 + |\nabla v|^2 + |\nabla^2 v|^2 \right), \quad (3.15)$$

where c and C are positive constants independent of T.

Now estimates the integral of (3.14)

$$\int_0^t \int_R \int_0^1 D_2 dx dy d\tau \le \mu_2 \int_0^t \|\nabla^2 v(\tau)\|^2 d\tau + C_{\mu_2} \int_0^t \left(\left\|\sqrt{\phi_x} v(\tau)\right\|^2 + \|\nabla v(\tau)\|^2 \right) d\tau, \quad (3.16)$$

where $\mu_2 > 0$ is a sufficient small constant and C_{μ_2} is a positive constant.

$$\int_0^t\int_R\int_0^1 D_3 dxdyd\tau = \int_R\int_0^1 D_2|_{\tau=0}^{\tau=t} dxdy$$

$$-\int_0^t\int_R\int_0^1 (v_{xx}+v_{yy})\{(f(\phi+v)-f(\phi))_{xt}+g(\phi+v)_{yt}\}dxdyd\tau. \qquad (3.17)$$

Let $c_f = \max\limits_{I[u_-,u_+]}|f'(u)|$, $c_g = \max\limits_{I[u_-,u_+]}|g'(u)|$, and $c_0 = \max\{c_f, c_g\}$, according to the sub-characteristic condition, we have $c_0 < 1$. By the boundedness of $|\nabla v|$, $|\phi_x|$ gets

$$\int_R\int_0^1 D_2|_{\tau=0} dxdy = \int_R\int_0^1 (v_{0xx}+v_{0yy})(f'(\phi+v_0)-f'(\phi))\phi_x dxdy$$

$$+\int_R\int_0^1 (v_{0xx}+v_{0yy})(f'(\phi+v_0)v_{0x}+g'(\phi+v_0)v_{0y})dxdy$$

$$\le C\int_R\int_0^1 \left(|v_0|^2+|\nabla v_0|^2+|\nabla^2 v_0|^2\right)dxdy, \qquad (3.18)$$

$$\int_R\int_0^1 D_2|_{\tau=t} dxdy$$

$$= \int_R\int_0^1 (v_{xx}(t,x,y)+v_{yy}(t,x,y))(f'(\phi+v(t,x,y))-f'(\phi))\phi_x dxdy$$

$$+\int_R\int_0^1 (v_{xx}(t,x,y)+v_{yy}(t,x,y))f'(\phi+v(t,x,y))v_x(t,x,y)dxdy$$

$$+\int_R\int_0^1 (v_{xx}(t,x,y)+v_{yy}(t,x,y))g'(\phi+v(t,x,y))v_y(t,x,y)dxdy$$

$$\le \mu_3\int_R\int_0^1 |\nabla^2 v(t)|^2 dxdy + C_{\mu_3}\int_R\int_0^1 \left(|v(t)|^2+|\nabla v(t)|^2\right)dxdy$$

$$\le \mu_3\|\nabla^2 v(t)\|^2 + C_{\mu_3}\|v(t)\|_1^2, \qquad (3.19)$$

$$\int_0^t\int_R\int_0^1 (v_{xx}+v_{yy})\{(f(\phi+v)-f(\phi))_{xt}+g(\phi+v)_{yt}\}dxdyd\tau$$

$$=\int_0^t\int_R\int_0^1 (v_{xx}+v_{yy})\{f''(\phi+v)v_t(\phi_x+v_x)+g''(\phi+v)v_yv_t\}dxdyd\tau$$

$$+\int_0^t\int_R\int_0^1 (v_{xx}+v_{yy})\{f'(\phi+v)v_{xt}+g'(\phi+v)v_{yt}\}dxdyd\tau$$

$$\le(\mu_3+c_0)\int_0^t\|\nabla^2 v(\tau)\|^2d\tau+c_0\int_0^t\|\nabla v_t(\tau)\|^2d\tau+C_{\mu_3}\int_0^t\|v_t(\tau)\|^2d\tau,$$

(3.20)

where $\mu_3>0$ is a sufficient small constant, which satisfy $\mu_3\ll 1-c_0$, and C_{μ_3} is a positive constant.

Notice that

$$\int_0^t\int_R -v_{xy}v_y dyd\tau=\int_0^t\left(-v_xv_y|_{y=-\infty}^{y=+\infty}+\int_R v_xv_{yy}dy\right)d\tau=\int_0^t\int_R v_xv_{yy}dyd\tau,$$

then by (3.1) and Sobolev's inequality, we obtain that

$$\int_0^t\int_R\int_0^1 \partial_x B_1 dxdyd\tau$$

$$=\int_0^t\int_R\{v_x(f(\phi+v)-f(\phi))_x-v_xv_{xx}+v_xv_{yy}+v_xg(\phi+v)_y\}\Big|_{x=0}^{x=1}dyd\tau$$

$$=\int_0^t\int_R v_x(-v_t-v_{tt}+2v_{yy})|_{x=0}^{x=1}dyd\tau$$

$$\le\frac{1}{2}\int_0^t\int_R\left(|v_x(\tau,1,y)|^2+|v_x(\tau,0,y)|^2\right)dyd\tau$$

$$+\frac{1}{2}\int_0^t\int_R\left\{|\partial_t u_+(\tau,y)|^2+|\partial_t^2 u_+(\tau,y)|^2+4|\partial_y^2 u_+(\tau,y)|^2\right\}dyd\tau$$

$$+\frac{1}{2}\int_0^t\int_R\left\{|\partial_t u_-(\tau,y)|^2+|\partial_t^2 u_-(\tau,y)|^2+4|\partial_y^2 u_-(\tau,y)|^2\right\}dyd\tau$$

$$\le \mu_4 \int_0^t \|v_{xx}(\tau)\|^2 d\tau + C_{\mu_4} \int_0^t \|v_x(\tau)\|^2 d\tau + C\Big\{\|\partial_t u_+(t,\ y)\|^2_{L^2(t,\,y)}$$

$$+ \|\partial_t^2 u_+(t,\ y)\|^2_{L^2(t,\,y)} + \|\partial_y^2 u_+(t,\ y)\|^2_{L^2(t,\,y)} + \|\partial_t u_-(t,\ y)\|^2_{L^2(t,\,y)}$$

$$+ \|\partial_t^2 u_-(t,\ y)\|^2_{L^2(t,\,y)} + \|\partial_y^2 u_-(t,\ y)\|^2_{L^2(t,\,y)}\Big\}, \tag{3.21}$$

where $\mu_4 > 0$ is a sufficient small constant, and C_{μ_4}, C are positive constants.

$$\int_0^t \int_R \int_0^1 \partial_x B_1 dx dy d\tau$$

$$= \int_0^t \int_R (-v_{tt} - v_t) v_{xt}|_{x=0}^{x=1} dy d\tau$$

$$+ \int_R \{v_x(t,\ x,\ y) v_{yy}(t,\ x,\ y) - v_x(0,\ x,\ y) v_{yy}(0,\ x,\ y)\}|_{x=0}^{x=1} dy$$

$$\le \int_0^t \int_R |v_x(\tau,\ 1,\ y)(\partial_t^2 u_+(\tau,\ y) + \partial_t^3 u_+(\tau,\ y))| dy d\tau$$

$$+ \int_0^t \int_R |v_x(\tau,\ 0,\ y)(\partial_t^2 u_-(\tau,\ y) + \partial_t^3 u_-(\tau,\ y))| dy d\tau$$

$$+ P_{1,1} + P_{1,2} + P_{2,1} + P_{2,2}, \tag{3.22}$$

where

$$P_{1,1} = \int_R \{|v_x(t,\ 1,\ y)|\,|\partial_y^2 u_+(t,\ y)| + |v_x(t,\ 0,\ y)|\,|\partial_y^2 u_-(t,\ y)|\} dy, \tag{3.23}$$

$$P_{1,2} = \int_R |v_x(t,\ 1,\ y)|\,(|\partial_t u_+(t,\ y)| + |\partial_t^2 u_+(t,\ y)|) dy$$

$$+ \int_R |v_x(t,\ 0,\ y)|\,(|\partial_t u_-(t,\ y)| + |\partial_t^2 u_-(t,\ y)|) dy, \tag{3.24}$$

$$P_{2,1} = \int_R \{|v_x(0,\,1,\,y)|\,|\partial_t u_+(0,\,y)| + |v_x(0,\,0,\,y)|\,|\partial_y^2 u_-(0,\,y)|\}dy, \tag{3.25}$$

$$P_{2,2} = \int_R |v_x(0,\,1,\,y)|\,(|\partial_t u_+(0,\,y)| + |\partial_t^2 u_+(0,\,y)|)dy$$

$$+ \int_R |v_x(0,\,0,\,y)|\,(|\partial_t u_-(0,\,y)| + |\partial_t^2 u_-(0,\,y)|)dy. \tag{3.26}$$

Similar to (3.21), we can easily get that

$$\int_0^t \int_R |v_x(\tau,\,1,\,y)\,(\partial_t^2 u_+(\tau,\,y) + \partial_t^3 u_+(\tau,\,y))|dyd\tau$$

$$\le \mu_5 \int_0^t \|v_{xx}(\tau)\|^2 d\tau + C_{\mu_5} \int_0^t \|v_x(\tau)\|^2 d\tau$$

$$+ C\left\{ \|\partial_t u_+(t,\,y)\|^2_{L^2(t,\,y)} + \|\partial_t^2 u_+(t,\,y)\|^2_{L^2(t,\,y)} + \|\partial_y^2 u_+(t,\,y)\|^2_{L^2(t,\,y)} \right\}, \tag{3.27}$$

$$\int_0^t \int_R |v_x(\tau,\,0,\,y)\,(\partial_t^2 u_-(\tau,\,y) + \partial_t^3 u_-(\tau,\,y))|dyd\tau$$

$$\le \mu_5 \int_0^t \|v_{xx}(\tau)\|^2 d\tau + C_{\mu_5} \int_0^t \|v_x(\tau)\|^2 d\tau$$

$$+ C\left\{ \|\partial_t u_-(t,\,y)\|^2_{L^2(t,\,y)} + \|\partial_t^2 u_-(t,\,y)\|^2_{L^2(t,\,y)} + \|\partial_y^2 u_-(t,\,y)\|^2_{L^2(t,\,y)} \right\}, \tag{3.28}$$

where $\mu_5 > 0$ is a sufficient small constant, and C_{μ_5}, C are positive constants.

By Sobolev's inequality and Holder's inequality, we obtain that

$$P_{1,1} \le \int_R C\|v_x(t,\,x,\,y)\|^{\frac{1}{2}}_{L^2(x)} \|v_{xx}(t,\,x,\,y)\|^{\frac{1}{2}}_{L^2(x)} (|\partial_y^2 u_+(t,\,y)| + |\partial_y^2 u_-(t,\,y)|)dy$$

$$\le C\left(\int_R \|v_x(t,\,x,\,y)\|_{L^2(x)}\|v_{xx}(t,\,x,\,y)\|_{L^2(x)}dy\right)^{\frac{1}{2}}\left(\int_R |\partial_y^2 u_+(t,\,y)|^2 dy\right)^{\frac{1}{2}}$$

$$+ C\left(\int_R \|v_x(t,\,x,\,y)\|_{L^2(x)}\|v_{xx}(t,\,x,\,y)\|_{L^2(x)}dy\right)^{\frac{1}{2}}\left(\int_R |\partial_y^2 u_-(t,\,y)|^2 dy\right)^{\frac{1}{2}}$$

$$\le \int_R \|v_x(t,\,x,\,y)\|_{L^2(x)}\|v_{xx}(t,\,x,\,y)\|_{L^2(x)}dy$$

$$+ C\left\{\int_R |\partial_y^2 u_+(t,\,y)|^2 dy + \int_R |\partial_y^2 u_-(t,\,y)|^2 dy\right\}, \tag{3.29}$$

$$\int_R \|v_x(t,\,x,\,y)\|_{L^2(x)}\|v_{xx}(t,\,x,\,y)\|_{L^2(x)}dy \le \mu_6\|v_{xx}(t)\|^2 + C\mu_6\|v_x(t)\|^2, \tag{3.30}$$

$$\int_R |\partial_y^2 u_+(t,\,y)|^2 dy \le C\int_R \|\partial_y^2 u_+(t,\,y)\|_{L^2(t)}\|\partial_t\partial_y^2 u_+(t,\,y)\|_{L^2(t)}dy$$

$$\le C\sum_{k=0,1}\|\partial_t^k\partial_y^2 u_+(t,\,y)\|^2_{L^2(t,\,y)}, \tag{3.31}$$

$$\int_R |\partial_y^2 u_-(t,\,y)|^2 dy \le C\int_R \|\partial_y^2 u_-(t,\,y)\|_{L^2(t)}\|\partial_t\partial_y^2 u_-(t,\,y)\|_{L^2(t)}dy$$

$$\le C\sum_{k=0,1}\|\partial_t^k\partial_y^2 u_-(t,\,y)\|^2_{L^2(t,\,y)}, \tag{3.32}$$

where $\mu_6 > 0$ is a sufficient small constant, and C_{μ_6}, C are positive constants.

Similar to the process of (3.29)-(3.32), we have

$$P_{1,2} \le \mu_7\|v_{xx}(t)\|^2 + C\mu_7\|v_x(t)\|^2 + C\sum_{k=1}^{3}\left\{\|\partial_t^k u_+(t,\,y)\|^2_{L^2(t,\,y)} + \|\partial_t^k u_-(t,\,y)\|^2_{L^2(t,\,y)}\right\}, \tag{3.33}$$

where $\mu_7 > 0$ is a sufficient small constant, and C_{μ_7}, C are positive constants.

$$P_{2,1} \le \int_R \|v_{0x}(x, y)\|_{L^2(x)} \|v_{0xx}(x, y)\|_{L^2(x)} dy + C\int_R |\partial_y^2 u_+(0, y)|^2 dy$$

$$+ C\int_R |\partial_y^2 u_-(0, y)|^2 dy$$

$$\le C\Big\{ \|\nabla v_0\|^2 + \|\nabla^2 v_0\|^2 + \|\partial_y^2 u_+(t, y)\|^2_{L^2(t,y)} + \|\partial_t \partial_y^2 u_+(t, y)\|^2_{L^2(t,y)}$$

$$+ \|\partial_y^2 u_-(t, y)\|^2_{L^2(t,y)} + \|\partial_t \partial_y^2 u_-(t, y)\|^2_{L^2(t,y)} \Big\}, \tag{3.34}$$

$$P_{2,2} \le C\|\nabla v_0\|_1^2 + C\sum_{k=1}^{3} \Big\{ \|\partial_t^k u_+(t, y)\|^2_{L^2(t,y)} + \|\partial_t^k u_-(t, y)\|^2_{L^2(t,y)} \Big\}, \tag{3.35}$$

where C is a positive constant.

Combining (3.22)-(3.35) gets

$$\int_0^t \int_R \int_0^t \partial_x B_2 dx dy d\tau$$

$$\le \mu_8 \Big(\|v_{xx}(t)\|^2 + \int_0^t \|v_{xx}(\tau)\|^2 d\tau \Big) + C_{\mu_8} \Big(\|v_x(t)\|^2 + \int_0^t \|v_x(\tau)\|^2 d\tau \Big)$$

$$+ C\Big\{ \|\nabla v_0\|^2 + \|\nabla^2 v_0\|^2 + R_0^2 \Big\}, \tag{3.36}$$

where $\mu_8 > 0$ is a sufficient small constant, and C_{μ_8}, C are positive constants.

Integrating (3.13) over $(0, t) \times [0, 1] \times R$ with respect to (t, x, y), and combining (3.14)-(3.21), (3.36), we obtain that

$$\|\nabla^2 v(t)\|^2 + \|\nabla v_t(t)\|^2 + \int_0^t \Big(\|\nabla^2 v(\tau)\|^2 + \|\nabla v_t(\tau)\|^2 \Big) d\tau$$

$$\le C\Big(E_0^2 + R_0^2 + \|\nabla v(t)\|^2 + \int_0^t \Big(\big\|\sqrt{\phi_x}\, v(\tau)\big\|^2 + \|\nabla v(\tau)\|^2 + \|v_t(\tau)\|^2 \Big) d\tau \Big). \tag{3.37}$$

Hence (3.12) and (3.37) imply (3.5).

Moreover, we can proof (3.4) the exact same way as in [1], we omit the detail here, then this complete the proof of Theorem 3.1.

4. Algebraically Weighted Energy Estimates

Theorem 4.1. *Let* $v(t, x, y)$ *be the global solution to the problem* (3.1)-(3.3), *which is constructed in Theorem* 3.1 *under the smallness condition on* $E_0 + R_0$. *If* $v_0 \in H^2_\alpha$, $v_1 \in H^1_\alpha$ *for* $\alpha \geq 0$ *and the boundary data* $u_\pm(t, y)$ *satisfies* (2.3), *then we have*

$$\|v(t, x, y)\|_{H^2} \leq C(E_\alpha + R_\gamma)(1 + t)^{-\alpha/2}, \tag{4.1}$$

for any γ *with* $\gamma > \alpha$, $t \geq 0$.

Obviously, Theorem 2.2 is a direct consequence of Theorem 4.1, in order to prove Theorem 4.1, we need to prove the following priori estimates:

Proposition 4.2. *Let* $v(t, x, y)$ *be the solution of problem* (3.1)-(3.3) *satisfies the same condition as given in Theorem* 4.1, *and* $\alpha \geq 0$, *then for any* β, γ *with* $0 \leq \gamma, \beta \leq \alpha$, *we get*

$$(1 + t)^\gamma \|(v_t, \nabla v_t, v, \nabla v, \nabla^2 v)(t)\|^2_{L^2_\beta} + \int_0^t (1 + \tau)^\gamma \|(v_t, \nabla v_t, \nabla v, \nabla^2 v)(\tau)\|^2_{L^2_\beta} d\tau$$

$$+ \int_0^t (1 + \tau)^\gamma \left\|\sqrt{\phi_x} v(\tau)\right\|^2_{L^2_\beta} d\tau + \beta \int_0^t (1 + \tau)^\gamma \|v(\tau)\|^2_{L^2_{\beta-1}} d\tau$$

$$\leq C(E^2_\beta + R^2_\gamma) + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|(v_t, \nabla v_t, v, \nabla v, \nabla^2 v)(\tau)\|^2_{L^2_\beta} d\tau$$

$$+ \beta C \int_0^t (1 + \tau)^\gamma \|(v_t, \nabla v_t, \nabla v, \nabla^2 v)(\tau)\|^2_{L^2_{\beta-1}} d\tau, \tag{4.2}$$

where C *is a positive constant independent of* β *and* γ.

Proof. Multiply (3.6) by $(1+t)^{\gamma}(1+x)^{\beta}$, we obtain

$$\partial_t\{(1+t)^{\gamma}(1+x)^{\beta}I\}-\gamma(1+t)^{\gamma-1}(1+x)^{\beta}I+\partial_x\{(1+t)^{\gamma}(1+x)^{\beta}(J_1-J_2)\}$$
$$-\beta(1+t)^{\gamma}(1+x)^{\beta-1}(J_1-J_2)+\partial_y\{(1+t)^{\gamma}(1+x)^{\beta}K\}+(1+t)^{\gamma}(1+x)^{\beta}L=0. \tag{4.3}$$

Integrating (4.3) over $(0, t)\times[0, 1]\times R$ with respect to (t, x, y), and similar to the calculation of (3.12), we have that

$$(1+t)^{\gamma}\|(v_t, v, \nabla v)(t)\|^2_{L^2_{\beta}}+\int_0^t(1+\tau)^{\gamma}\|(v_t, \nabla v)(\tau)\|^2_{L^2_{\beta}}d\tau$$
$$+\int_0^t(1+\tau)^{\gamma}\|\sqrt{\phi_x}v(\tau)\|^2_{L^2_{\beta}}d\tau+\beta\int_0^t(1+\tau)^{\gamma}\|v(\tau)\|^2_{L^2_{\beta-1}}d\tau$$
$$\le C\|(v_1, v_0, \nabla v_0)\|^2_{L^2_{\beta}}+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|(v_t, v, \nabla v)(\tau)\|^2_{L^2_{\beta}}d\tau$$
$$+C\beta\int_0^t(1+\tau)^{\gamma}\|(v_t, \nabla v)(\tau)\|^2_{L^2_{\beta-1}}d\tau+CR_{\gamma}^2+\mu_6\int_0^t(1+\tau)^{\gamma}\|\nabla^2 v(\tau)\|^2d\tau, \tag{4.4}$$

where μ_9 is a sufficient small constant and C is a positive constant independent of β and γ.

Next, multiplying (3.13) by $(1+t)^{\gamma}(1+x)^{\beta}$, one gets

$$\partial_t\{(1+t)^{\gamma}(1+x)^{\beta}A\}-\gamma(1+t)^{\gamma-1}(1+x)^{\beta}A$$
$$+\partial_x\{(1+t)^{\gamma}(1+x)^{\beta}(B_1+B_2)\}-\beta(1+t)^{\gamma}(1+x)^{\beta-1}(B_1+B_2)$$
$$+\partial_y\{(1+t)^{\gamma}(1+x)^{\beta}B_3\}+(1+t)^{\gamma}(1+x)^{\beta}(D_1-D_2-D_3)=0. \tag{4.5}$$

In what follows, we give the estimates of some terms of (4.5) one by one. Since $\phi_x>0$, $f''>0$ and ϕ_x, f'' are bounded, then we obtain that

$$\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^\beta D_2 dxdyd\tau$$

$$\le C\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^\beta(v_{xx}+v_{yy})vdxdyd\tau\right|+Q, \tag{4.6}$$

where

$$Q=\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^\beta(v_{xx}+v_{yy})(f'(v+\phi)v_x+g'(v+\phi)v_y)dxdyd\tau\right|$$

$$\le \mu_{10}\int_0^t(1+\tau)^\gamma\|\nabla^2 v(\tau)\|^2_{L^2_\beta}d\tau+C_{\mu_{10}}\int_0^t(1+\tau)^\gamma\|\nabla v(\tau)\|^2_{L^2_{\beta-1}}d\tau. \tag{4.7}$$

$$C\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^\beta(v_{xx}+v_{yy})vdxdyd\tau\right|$$

$$\le C\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^\beta|\nabla v|^2dxdyd\tau\right|+C\beta\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^{\beta-1}v_xvdxdyd\tau\right|$$

$$+C\left|\int_0^t\int_R(1+\tau)^\gamma(1+x)^\beta v_xv\Big|_{x=0}^{x=1}dyd\tau+\int_0^t\int_0^1(1+\tau)^\gamma(1+x)^\beta v_yv\Big|_{y=-\infty}^{y=+\infty}dxd\tau\right|, \tag{4.8}$$

$$C\beta\left|\int_0^t\int_R\int_0^1(1+\tau)^\gamma(1+x)^{\beta-1}v_xvdxdyd\tau\right|$$

$$\le \mu_{10}\beta\int_0^t(1+\tau)^\gamma\|v(\tau)\|^2_{L^2_{\beta-1}}d\tau+C_{\mu_{10}}\beta\int_0^t(1+\tau)^\gamma\|\nabla v(\tau)\|^2_{L^2_{\beta-1}}d\tau, \tag{4.9}$$

$$C\left|\int_0^t\int_R(1+\tau)^\gamma(1+x)^\beta v_xv\Big|_{x=0}^{x=1}dyd\tau\right|$$

$$\le \mu_{10}\int_0^t(1+\tau)^\gamma\|\nabla^2 v(\tau)\|^2d\tau+C_{\mu_{10}}\int_0^t(1+\tau)^\gamma\|\nabla v(\tau)\|^2d\tau$$

$$+C\int_0^{+\infty}\int_R(1+\tau)^\gamma\left(|u_+(\tau,y)-u_+|^2+|u_-(\tau,y)-u_-|^2\right)dyd\tau, \tag{4.10}$$

where $\mu_{10} > 0$ is a sufficient small constant, and $C_{\mu_{10}}$, C are positive constants.

Using (3.18)-(3.20) gets

$$\int_0^t\int_R\int_0^1 (1+\tau)^\gamma (1+x)^\beta D_3 dxdyd\tau$$

$$= \int_R\int_0^1 \Big\{(1+\tau)^\gamma (1+x)^\beta D2\Big\}\Big|_{r=0}^{r=t} dxdy - \gamma\int_R\int_0^1\int_0^t (1+\tau)^{\gamma-1}(1+x)^\beta D_2 d\tau dxdy$$

$$- \int_R\int_0^1\int_0^t (1+\tau)^\gamma (1+x)^\beta (v_{xx}+v_{yy})\Big\{(f(\phi+v)-f(\phi))_{xt} + g(\phi+v)_{yt}\Big\}d\tau dxdy$$

$$\le \mu_{11}(1+t)^\gamma \|\nabla^2 v(t)\|^2_{L^2_\beta} + C_{\mu_{11}}(1+t)^\gamma \|(v,\nabla v)(t)\|^2_{L^2_\beta} + C\|(v_0,\nabla v_0,\nabla^2 v_0)\|^2_{L^2_\beta}$$

$$+ C\gamma\int_0^t (1+\tau)^{\gamma-1}\|(v,\nabla v,\nabla^2 v)(\tau)\|^2_{L^2_\beta} d\tau + C_{\mu_{11}}\int_0^t (1+\tau)^\gamma \|v_t(\tau)\|^2_{L^2_\beta} d\tau$$

$$+ (\mu_{11}+c_0)\int_0^t (1+\tau)^\gamma \|\nabla^2 v(\tau)\|^2_{L^2_\beta} d\tau + c_0\int_0^t (1+\tau)^\gamma \|\nabla v_t(\tau)\|^2_{L^2_\beta} d\tau, \qquad (4.11)$$

where $\mu_{11} > 0$ is a sufficient small constant, which satisfy $\mu_{11} \ll 1 - c_0$, and $C_{\mu_{11}}$, C are positive constants.

By the boundedness of ϕ_x, f', g', f'', we have

$$|B_1| \le \mu v^2 + C_\mu\Big(|\nabla v|^2 + |\nabla^2 v|^2\Big),$$

$$|B_2| \le \mu v^2 + C_\mu\Big(|\nabla v_t|^2 + |\nabla v|^2 + |\nabla^2 v|^2\Big),$$

then it is easy to get

$$\int_0^t\int_R\int_0^1 \beta(1+\tau)^\gamma (1+x)^{\beta-1} B_1 dxdyd\tau$$

$$\le \mu_{12}\beta\int_0^t (1+\tau)^\gamma \|v(\tau)\|^2_{L^2_{\beta-1}} d\tau + C_{\mu_{12}}\beta\int_0^t (1+\tau)^\gamma \|(\nabla v,\nabla^2 v)(\tau)\|^2_{L^2_{\beta-1}} d\tau,$$

(4.12)

$$\int_0^t\int_R\int_0^1 \beta(1+\tau)^\gamma(1+x)^{\beta-1}B_2 dxdyd\tau$$

$$\le \mu_{12}\beta\int_0^t(1+\tau)^\gamma\|v(\tau)\|^2_{L^2_{\beta-1}}d\tau + C_{\mu_{12}}\beta\int_0^t(1+\tau)^\gamma\|(\nabla v_t,\nabla v,\nabla^2 v)(\tau)\|^2_{L^2_{\beta-1}}d\tau, \tag{4.13}$$

where $\mu_{12}>0$ is a sufficient small constant and $C_{\mu_{12}}$ is a positive constant.

Similar to the calculation of (3.21) and (3.36), we obtain that

$$\begin{aligned}
&\int_0^t\int_R\int_0^1 \partial_x\left\{(1+\tau)^\gamma(1+x)^\beta B_1\right\}dxdyd\tau\\
&= \int_0^t\int_R(1+\tau)^\gamma(1+x)^\beta v_x(-v_t-v_{tt}+2v_{yy})\Big|_{x=0}^{x=1}dyd\tau\\
&\le \mu_{13}\int_0^t(1+\tau)^\gamma\|v_{xx}(\tau)\|^2d\tau + C_{\mu_{13}}\int_0^t(1+\tau)^\gamma\|v_x(\tau)\|^2d\tau\\
&\quad + C\int_0^{+\infty}\int_R(1+\tau)^\gamma\left(|\partial_t u_+(\tau,y)|^2+|\partial_t^2 u_+(\tau,y)|^2+|\partial_y^2 u_+(\tau,y)|^2\right)dyd\tau\\
&\quad + C\int_0^{+\infty}\int_R(1+\tau)^\gamma\left(|\partial_t u_-(\tau,y)|^2+|\partial_t^2 u_-(\tau,y)|^2+|\partial_y^2 u_-(\tau,y)|^2\right)dyd\tau,
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_0^t\int_R\int_0^1 \partial_x\left\{(1+\tau)^\gamma(1+x)^\beta B_2\right\}dxdyd\tau\\
&= \int_0^t\int_R(1+\tau)^\gamma(1+x)^\beta(-v_{tt}-v_t)v_{xt}\Big|_{x=0}^{x=1}dyd\tau\\
&\quad -(1+x)^\beta\left\{\int_0^t\int_R\gamma(1+\tau)^{\gamma-1}v_xv_{yy}dyd\tau-\int_R(1+\tau)^\gamma v_xv_{yy}\Big|_{\tau=0}^{\tau=t}dy\right\}\Big|_{x=0}^{x=1},
\end{aligned} \tag{4.15}$$

$$\int_0^t\int_R (1+\tau)^\gamma (1+x)^\beta (-v_{tt}-v_t)v_{xt}\Big|_{x=0}^{x=1}dyd\tau \le C(1+t)^\gamma P_{1,2}+CP_{2,2}$$

$$+C\int_0^t\int_R \Big\{(1+\tau)^\gamma |(v_{ttt}+v_{tt})v_x|+\gamma(1+\tau)^{\gamma-1}|(v_{tt}+v_t)v_x|\Big\}\Big|_{x=1}dyd\tau$$

$$+C\int_0^t\int_R \Big\{(1+\tau)^\gamma |(v_{ttt}+v_{tt})v_x|+\gamma(1+\tau)^{\gamma-1}|(v_{tt}+v_t)v_x|\Big\}\Big|_{x=0}dyd\tau$$

$$\le \mu_{13}\Big\{\|v_{xx}(t)\|^2+\int_0^t(1+\tau)^\gamma\|v_{xx}(\tau)\|^2d\tau\Big\}$$

$$+C_{\mu_{13}}\Big\{\|v_x(t)\|^2+\int_0^t(1+\tau)^\gamma\|v_x(\tau)\|^2d\tau\Big\}+C\Big(\|\nabla v_0\|^2+\|\nabla^2 v_0\|^2+R_\gamma^2\Big), \tag{4.16}$$

$$(1+x)^\beta\Big\{\int_0^t\int_R \gamma(1+\tau)^{\gamma-1}v_xv_{yy}dyd\tau-\int_R(1+\tau)^\gamma v_xv_{yy}\Big|_{\tau=0}^{\tau=t}dy\Big\}\Big|_{x=0}^{x=1}$$

$$\le C\int_0^t\int_R \gamma(1+\tau)^{\gamma-1}|v_xv_{yy}|\Big|_{x=1}dyd\tau+C\int_0^t\int_R\gamma(1+\tau)^{\gamma-1}|v_xv_{yy}|\Big|_{x=0}dyd\tau$$

$$+C((1+t)^\gamma P_{1,1}+P_{2,1})$$

$$\le C\gamma\int_0^t(1+\tau)^{\gamma-1}\|(v_x,v_{xx})(\tau)\|^2d\tau+C\Big(\|\nabla v_0\|^2+\|\nabla^2v_0\|^2+R_\gamma^2\Big), \tag{4.17}$$

where $\mu_{13}>0$ is a sufficient small constant, and $C_{\mu_{13}}$, C are positive constants.

Integrating (4.5) over $(0,t)\times[0,1]\times R$ with respect to (t,x,y), and combining (3.15), (4.6)-(4.17) get

$$(1+t)^\gamma\|(\nabla v_t,\nabla v,\nabla^2 v)(t)\|_{L^2_\beta}^2+\int_0^t(1+\tau)^\gamma\|(\nabla v_t,\nabla^2v)(\tau)\|_{L^2_\beta}^2d\tau$$

$$\le C(E_\beta^2+R_\gamma^2)+C\gamma\int_0^t(1+\tau)^{\gamma-1}\|(\nabla v_t,v,\nabla v,\nabla^2v)(\tau)\|_{L^2_\beta}^2d\tau$$

$$+ C\beta\int_0^t (1+\tau)^{\gamma}\|(\nabla v_t, \nabla v, \nabla^2 v)(\tau)\|^2_{L^2_{\beta-1}} d\tau + C(1+t)^{\gamma}\|(v, \nabla v)(t)\|^2_{L^2_{\beta}}$$

$$+ C\int_0^t (1+\tau)^{\gamma}\|(v_t, \nabla v)(\tau)\|^2_{L^2_{\beta}} d\tau. \tag{4.18}$$

Notice that (4.4) and (4.18) imply (4.2), by (4.2) and use the exactly same method in [1], we have that

$$(1+t)^{\gamma}\|(v_t, \nabla v_t, v, \nabla v, \nabla^2 v)(t)\|^2_{L^2} + \int_0^t (1+\tau)^{\gamma}\|(v_t, \nabla v_t, \nabla v, \nabla^2 v)(\tau)\|^2_{L^2} d\tau$$

$$+ \int_0^t (1+\tau)^{\gamma}\left\|\sqrt{\phi_x}\, v\ (\tau)\right\|^2_{L^2} d\tau \le C(E_\alpha^2 + R_\gamma^2)(1+t)^{\gamma-\alpha}, \tag{4.19}$$

for any γ with $\gamma > \alpha$.

And (4.19) imply (4.1), then we complete the proof of Theorem 4.1.

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