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SYMPLECTIC MATRICES AND STRONG STABILITY OF HAMILTONIAN SYSTEMS WITH PERIODIC COEFFICIENTS

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Abstract

A theory of the analysis of strong stability of Hamiltonian systems with periodic coefficients is presented. It is based on the strong stability of fundamental solution evaluated at its period. However, the matrix solution being symplectic, this analysis leads to the search of strong stability of a symplectic matrix. A method for the determination of the monodromy matrix is given and an application of this study on two Hamiltonian systems with periodic coefficients from the book of Yakubovich and Starzhinskii, is presented.

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1. Introduction

Let $J \in \mathbb{R}^{2N \times 2N}$ be an invertible and skew-symmetric matrix. A matrix $W \in \mathbb{R}^{2N \times 2N}$ is J-symplectic, if $W^T J W = J$ [2, 3, 4, 5, 8, 10, 14]. Throughout this paper, the identity matrix and null matrix of order k are, respectively, denoted by I_k and 0_k (or just I and 0 when the order is clear from context). The spectrum of W are symmetrically disposed (with corresponding multiplicities) with respect to both the unit circle and the real axis; thus if λ is an eigenvalue of W, $\bar{\lambda}, 1/\lambda,$ and $1/\bar{\lambda}$ are also eigenvalues of W. The symplectic matrices are sometimes obtained from Hamiltonian differential systems with periodic coefficients (see [14, Chapter 3]). These Hamiltonian systems are linear differential equations of the form

$$J \frac{dx(t)}{dt} = H(t)x(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $H(t) \in \mathbb{R}^{2N \times 2N}$ is symmetric and T-periodic, in the other words, $H(t+T) = H(t) = (H(t))^T$ and J is skew-symmetric matrix. From system (1.1), we consider the following matrix system:

$$J \frac{dX(t)}{dt} = H(t)X(t) \quad \text{with} \quad X(0) = I_{2N}. \tag{1.2}$$

We collect some important definitions of [14, Vol. 1, Chapter 2] as following:

Definition 1.1. (1) The $2N \times 2N$ matrix $X(t)$ satisfying Equation (1.2) is called the matrizant of Equation (1.1).

(2) A fundamental set of solutions of Equation (1.1) is any set of n solutions $x_1(t), x_2(t), \ldots, x_{2N}(t)$, which are linearly independent for any t. The square matrix $X(t)$ with columns $x_1(t), x_2(t), \ldots, x_{2N}(t)$ is called a fundamental matrix of (1.1). We will call $X(t)$ the fundamental solution of Equation (1.2).

(3) The value at the period T of the fundamental matrix $X(t)$ defined by the initial condition $X(0) = I_n$, is called the monodromy matrix and its eigenvalues are the multipliers of system (1.1). The set of all multipliers is called the spectrum of Equation (1.1).

We recall the following proposition of [14, Chapter 2, Section 3], which shows an important property of the fundamental solution of Equation (1.2).

Proposition 1.1. *The matrizant* $X(t) \in \mathbb{R}^{2N \times 2N}$ *of a Hamiltonian equation* (1.2) *satisfies the identity*

$$X(t)^T JX(t) = J. \tag{1.3}$$

Conversely, if a matrix $X(t)$ *satisfies condition* (1.3) *such that*: *it is continuous*, *has an integrable piecewise-continuous derivative, and* $X(0) = I_{2N}$, *then it is the matrizant of some Hamiltonian equation* (1.2).

Thus system (1.1) is said strongly stable if all its fundamental solutions are bounded on $\mathbb{R}$ and its property is preserved by any small perturbations preserving its Hamiltonian structure [14]. Therefore, the strong stability of Hamiltonian systems reduces to the analysis of their fundamental solutions. Thus, it suffices to apply the analysis of the strong stability of symplectic matrices to its fundamental solutions. The monodromy matrix of Equation (1.1) will indicate whether the system is (strongly) stable (see [14, Chapter 2, Section 2]).

The Hamiltonian systems appear in control theory [14, 13, 12, 1]: More specifically in optimal control theory [12] and parametric resonance [14]. Many problems in physics and engineering lead to systems of linear differential equations with periodic coefficients consequently to Hamiltonian systems with periodic coefficients.

In Section 2, we give some preliminaries on symplectic matrices useful in the study of the strong stability of Hamiltonian systems with periodic coefficients. In Section 3, we present the theory of the strong stability of Hamiltonian systems with periodic coefficients from the symplectic matrices. In this section, we present a method to extract the monodromy matrix of Hamiltonian systems with periodic coefficients and as an application, we determine the strong stability of two systems for some values of its parameters. These are a Hamiltonian system of order 6 with periodic coefficients and Mathieu system given in [14].

2. Preliminary on Strong Stability of Symplectic Matrices

Let W be a J-symplectic matrix. Classical definitions of stability are given below. Unless otherwise stated, the symbol $\| \ \|$ denotes the spectral norm (2-norm).

Definition 2.1. We say that W is stable, if $\|W^k\| < \infty$ for all $k \in \mathbb{N}$ and for all matrix norm $\| \ \|$.

As $W^T JW = J,$ the condition $\|W^k\| < \infty$ is equivalent to $\|W^{-k}\| < \infty,$ $\forall k \in \mathbb{N}.$

It is clear that a symplectic matrix is stable, if and only if all its eigenvalues are on the unit circle and are not defective. The following definition characterizes another type of stability: **The strong stability**.

Definition 2.2. We say that W is strongly stable, if there exists $\epsilon > 0$ such that all J-symplectic matrix $\widetilde{W}$ verifying $\|W - \widetilde{W}\| \leq \epsilon$ is stable.

In the other words, W is strongly stable if it is stable and remains when it is subjected to small perturbations conserving the symplecticity. This concept is robust than the stability. It is also more difficult to characterize. Obviously, the eigenvalues of such matrix should be on the unit circle and semi-simple, but it is not enough to characterize this type of stability. Indeed, the eigenvalues must, in addition, be either of first kind or second kind according to the following definition:

Definition 2.3. Let λ be an eigenvalue of modulus 1 of *W*. We say that λ is of first kind (respectively, of second kind) if any eigenvector x associated with λ verifies $(iJx, x) > 0$ (respectively, $(iJx, x) < 0$). If $(Jx, x) = 0$, we say that λ is mixed kind.

The above definition has a meaning since the matrix iJ is complex Hermitian. Krein, Gelfand and Lisdskii [14] introduced this classification and used it to characterize the strong stability of symplectic matrices. The characterization is summarized in the following theorem, we subsequently call: The criterion of Krein, Gelfand and Lisdskii (KGL criterion).

Theorem 2.1 (KGL criterion)**.** *The matrix W is strongly stable, if and only if all its eigenvalues are on the unit circle and either of first kind or second kind.*

Note that any eigenvector x of W associated with eigenvalue λ outside the unit circle, satisfies $(Jx, x) = 0$. This allows to extend the definition of an eigenvalue mixed kind with eigenvalues outside the unit circle. When W has not any eigenvalue of mixed kind, we have the following theorem:

Theorem 2.2. *Let λ be an eigenvalue of type not mixed kind of W, i.e., λ is on unit circle and is either of first kind or either of second kind. Then, there exist constants $\delta > 0$ and $\gamma > 0$ such that for any J-symplectic matrix $\widetilde{W}$ satisfying $\|W - \widetilde{W}\| < \delta$, the eigenvalues $\overline{\lambda}$ of $\widetilde{W}$ in the neighbourhood $|\lambda - \overline{\lambda}| < \gamma$ are on the unit circle and are semi-simple.*

Proof. It is found in [14, p. 161]. We reproduce it for clarity reasoning by contradiction: There exists a sequence of matrices $W_1, W_2, \ldots, \to W$ of eigenvalues, respectively, $\lambda_1, \lambda_2, \ldots, \lambda$ such that λ_m is an eigenvalue of W_m with $|\lambda_m| \neq 1$, where λ is of modulus 1 but defective (i.e., not semi-simple). Let x_m be an eigenvector corresponding to λ_m such that

$\|x_m\| = 1$. Then this gives us $(Jx_m, x_m) = 0$, $m = 1, 2, 3, \ldots$. The sequence (x_m) being bounded, we can extract a convergent subsequence to a vector x, and by passing to the limit, we obtained $Wx = \lambda x$, and $\|x\| = 1$, $(Jx, x) = 0$. The eigenvalue λ is of mixed kind. Hence a contradiction. □

However, considering the symmetric matrix (see [7, 8])

$$S_0 = \frac{1}{2} J(W - W^{-1}), \tag{2.1}$$

we define a new classification of the eigenvalues of W in accordance with the following definition:

Definition 2.4. Let λ be an eigenvalue of modulus 1 of W. We say that λ is an r-eigenvalue (respectively, g-eigenvalue) if any eigenvector x associated with λ verifies $(S_0 x, x) > 0$ (respectively, $(S_0 x, x) < 0$).

Thus, based on the Definition 2.4, we obtained the following proposition:

Proposition 2.1. *The spectrum of a symplectic matrix W is formed only of r- and/or g-eigenvalues, if and only if it is formed only of eigenvalues of first or second kind.*

Hence the following theorem gives another definition of the strong stability of a symplectic matrix W [4, 5]:

Theorem 2.3. *The matrix W is strongly stable if and only if*

(1) *all its eigenvalues are on the unit circle*;

(2) *the eigenvalues are either r-eigenvalues and/or either g-eigenvalues*.

Several methods based on this theorem allow the analysis of strong stability. Thus, Dosso et al. in [3, 4, 5] proposed analysis's methods of the strong stability of symplectic matrices. We have, for example, a spectral trichotomy method for symplectic matrices proposed by Dosso and Sadkane [3], which allows to compute a good approximation of the projectors $\mathbb{P}_0$, $\mathbb{P}_1$, and $\mathbb{P}_\infty$ onto the invariant subspaces of the symplectic matrix associated to the eigenvalues, respectively, inside, on and outside the unit circle. Another example presented in [5], determines with the projectors $\mathbb{P}_0$, $\mathbb{P}_1$, and $\mathbb{P}_\infty$, the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ associated with r-eigenvalues and g-eigenvalues, respectively. From these projectors, we have the following theorem [5].

Theorem 2.4. *The symplectic matrix W is strongly stable*, *if and only if*

$$\mathbb{P}_0 = \mathbb{P}_\infty = 0, \quad \mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I.$$

We recall the algorithms for the analysis of the strong stability by a spectral dichotomy method. Algorithm 2.1 computes a good approximation of the projector $\mathbb{P}$ onto the invariant subspace associated with eigenvalues of a matrix, inside the circle $\mathcal{C}(0, r)$ of center 0 and of radius r. For details, see [5, 9].

Algorithm 2.1. (Spectral dichotomy of matrix A of size N with respect to the circle $\mathcal{C}(0, r)$):

(1) Initialize

(a) $A_0 = -\dfrac{A}{r}$;

(b) Resolve

$$\begin{pmatrix} A_0 & I_N \\ I_N & A_0 \end{pmatrix} \begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_N \end{pmatrix};$$

(c) Put $Z_1^{(2)} = K_1, Z_2^{(2)} = L_1$ and compute $H_1 = \left(Z_1^{(2)}\right)^* Z_1^{(2)} + \left(Z_2^{(2)}\right)^* Z_2^{(2)}$.

(2) Iterate: for $j = 1, 2, \ldots, n$

(a) Put

$$A_j = A_0 Z_1^{(2^j)}, \quad B_j = Z_{2^j}^{(2^j)};$$

(b) Resolve

$$\begin{pmatrix} B_j & A_j \\ A_j & B_j \end{pmatrix} \begin{pmatrix} K_1 \\ L_1 \end{pmatrix} = \begin{pmatrix} 0 \\ I_N \end{pmatrix};$$

(c) Compute

$$Z_1^{(2^{j+1})} = Z_1^{(2^j)} K_{j+1},$$

$$Z_{2^{j+1}}^{(2^{j+1})} = Z_{2^j}^{(2^j)},$$

$$H_{j+1} = K_{j+1}^* H_j K_{j+1} + L_{j+1}^* H_j L_{j+1}.$$

(3) Put

$$\mathbb{P} = Z_1^{(2^{n+1})} \quad \text{and} \quad \mathbb{H} = H_{n+1}.$$

Note that the spectral portrait of the matrix A is the graph of the function

$$r \mapsto f(r) = \|H(r)\|,$$

where

$$\mathbb{H}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(I - e^{i\theta} \frac{A^*}{r} \right)^{-1} \left(I - e^{i\theta} \frac{A}{r} \right)^{-1} d\theta,$$

computed by Algorithm 2.1. The algorithm which allows to determine the strong stability of a J-symplectic matrix W, is the following [5, 7].

Algorithm 2.2. (1) Using Algorithm 2.1 and appropriate choice of the parameter r_0, to determine the projectors $\mathbb{P}_0$, $\mathbb{P}_1$, and $\mathbb{P}_\infty$.

(2) If $\mathbb{P}_0 \neq 0$ $(\text{or } \mathbb{P}_\infty \neq 0)$, there is not strong stability. Else compute

$$S_0 = (1/2)((JW) + (JW)^T).$$

If S_0 is singular (poorly conditioned), there is not stability. Else compute

$$A = (W - I)(W + I)^{-1}.$$

From the spectral portrait of A, determine the scalars $(a_k)_{1 \leq k \leq m+1}$ such that the eigenvalues $(l_k)_{1 \leq k < m}$ of matrix A verify

$$0 < a_k < |l_k| < a_{k+1} < \infty, \quad \forall k \in \{1, 2, \dots, m\}.$$

(3) Determine the projectors $P_1 = 0, P_2, \dots, P_{m+1} = I$ by the spectral dichotomy applied to A and circle $\mathcal{C}(0, a_k)$, $k = 1, 2, \dots, m + 1$.

(4) For $k = 1, 2, \dots, m$, compute

$$Q_k = P_{k+1} - P_k \quad \text{and} \quad S_k = Q_k^T S_0 Q_k.$$

If S_k is not definite, then is not strong stability. If all the matrices S_k are semi-definite, then there is strong stability. In this case,

$$\mathbb{P}_r = \sum_{S_k \geq 0} Q_k \quad \text{and} \quad \mathbb{P}_g = \sum_{S_k \leq 0} Q_k.$$

3. Hamiltonian Systems with Periodic Coefficients: Strong Stability Analysis

Consider the Hamiltonian system with periodic coefficients (1.2). For all $t \in \mathbb{R}$, the fundamental solution $X(t)$ satisfies the relation

$$X(t + T) = X(t)X(T)(\neq X(T)X(t)). \tag{3.1}$$

About the stability of such systems, we first have the following classical definition:

Definition 3.1. System (1.1) is stable if each of all its solution $x(t)$ remains bounded for all $t \in \mathbb{R}$. From (3.1), we have for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$

$$X(t + nT) = X(t)X^n(T). \tag{3.2}$$

Therefore, any solution $x(t) = X(t)x_0$ of (1.1) verifies

$$\sup_{t\in\mathbb{R}} \|x(t)\| \leq \sup_{0\leq t\leq T} \|X(t)\| \sup_{n\in\mathbb{N}} \|X^n(T)x_0\|. \tag{3.3}$$

On the other hand,

$$\sup_{n\in\mathbb{N}} \|X^n(T)x_0\| = \sup_{n\in\mathbb{N}} \|X(nT)x_0\|$$

$$= \sup_{n\in\mathbb{N}} \|x(nT)\| \leq \sup_{t\in\mathbb{R}} \|x(t)\|.$$

In the other words, a necessary and sufficient condition for stability of (1.1) is that the powers $X(T)$ remain bounded; and since $X(T)$ is J-symplectic, this implies that there exists a constant C such that

$$\sup_{n\in\mathbb{Z}} \|X^n(T)\| \leq C.$$

Furthermore, system (1.1) is strongly stable if there exists $\varepsilon > 0$ such that any Hamiltonian system with T-periodic coefficients of the form

$$J\frac{dx(t)}{dt} = \widetilde{H}(t)x(t),$$

and satisfying

$$\|H - \widetilde{H}\| \equiv \int_0^T \|H(t) - \widetilde{H}(t)\| dt < \varepsilon,$$

is stable. We have the following important theorem (see [14, p. 196]):

Theorem 3.1. *System* (1.1) *is strongly stable if and only if the J-symplectic matrix* $X(T)$ *is strongly stable.*

Thus, the stability (respectively, strong stability) of (1.1) is reduced to the stability (respectively, strong stability) of the J-symplectic matrix $W = X(T)$, which is the monodromy matrix of system (1.2). The following subsection gives a method to compute $X(T)$.

3.1. Resolution by the method of double orthogonal sweeps

This method consists in subdividing the integration interval into $2n$ subintervals. In the other words, we subdivide each half-period in n intervals. Thus, we use the orthogonal sweep method on each half period setting the starting time, $t = 0$ if the integration is over the interval $[-\pi, \pi]$ or $t = T / 2$ if the integration is over the interval $[0, 2\pi]$ (see [6] for the notion of orthogonal sweeps). We call this method: the method of double orthogonal sweep. In our case, we take the initial time $t = T / 2$. Thus, putting $A(t) = J^{-1}H(t)$, we consider the following linear system:

$$\begin{cases} \dfrac{dU(t)}{dt} = A(t)U(t), & \dfrac{T}{2} \le t \le T, \\ \dfrac{dV(t)}{dt} = -A(t)V(t), & 0 \le t \le \dfrac{T}{2}, \end{cases} \tag{3.4}$$

of initial conditions

$$V\left(\frac{T}{2}\right) = U\left(\frac{T}{2}\right) = \frac{I_{2N}}{\sqrt{2}}.$$

Lemma 3.1. *The solutions* $U(t)$ *and* $V(t)$ *of the equations of system* (3.4) *verify*

$$(U(t))^T JU(t) = (1/2)J, \quad \frac{T}{2} \le t \le T, \tag{3.5}$$

$$(V(t))^T JV(t) = (1/2)J, \quad 0 \le t \le \frac{T}{2}. \tag{3.6}$$

Proof. We have

$$\begin{aligned} \frac{d}{dt}(U(t)^T JU(t)) &= \frac{dU(t)^T}{dt} JU(t) + U(t)^T J \frac{dU(t)}{dt} \\ &= U(t)^T (A(t))^T JU(t) + U(t)^T JA(t)U(t) \\ &= -U(t)^T H(t)U(t) + U(t)^T H(t)U(t), \end{aligned}$$

because $A(t) = J^{-1}H(t)$ and $A(t)^T = -H(t)J^{-1}$. This implies that

$$\frac{d}{dt}(U(t)^T JU(t)) = 0.$$

Then there exists a constant matrix C such that $U(t)^T JU(t) = C, \forall t \in \mathbb{R}$; as $U(\frac{T}{2})^T JU(\frac{T}{2}) = (1/2)J$, we obtain $C = (1/2)J$, hence equality (3.5). We show in the same manner equality (3.6). □

Putting $X(T) = U(T)(V(0))^{-1}$, the method gives us the monodromy matrix $W = X(T)$. Let $(t_k^1)_{1\leq k\leq n}$ and $(t_k^2)_{1\leq k\leq n}$ the sequences defined by $t_k^1 = \frac{T}{2}(1 + \frac{k-1}{n})$ and $t_k^2 = \frac{T}{2}(1 - \frac{k-1}{n})$. The resolution of (3.4) leads us to the resolution of the system

$$\forall k = 1, 2, 3, \ldots, n, \quad \begin{cases} \dfrac{dU_k(t)}{dt} = A(t_k^1 + t)U_k(t), \\ \dfrac{dV_k(t)}{dt} = -A(t_k^2 - t)V_k(t), \end{cases} \tag{3.7}$$

on the interval $]0, \frac{T}{2n}]$ (by the one of the method of Runge Kutta by example). The initial condition $\begin{pmatrix} U_{k+1}(0) \\ V_{k+1}(0) \end{pmatrix}$ of Equation (3.7) at iteration $k+1$ is obtained from the normalization of solution of (3.7) at iteration k at the time $t = T/2n$. We can have this normalization with the QR-algorithm (or singular value decomposition (SDV) algorithm) [11, 13]. At iteration $k = n$, we have the matrix $W = X(T) \equiv U_n(T/2n)(V_n(T/2n))^{-1}$. Then we give the following proposition:

Proposition 3.1. *For all* $n > 1$, *the matrix* $W = U_n(T/2n)(V_n(T/2n))^{-1}$ *is J-symplectic and gives a good approximation of the monodromy matrix* $X(T)$ *of system* (1.2).

Proof. From (3.6), we get $\forall n > 1, (V_n(T/2n)^{-1})^T J(V_n(T/2n))^{-1} = 2J$. Thus, we have

$$\begin{aligned} W^T JW &= \left[U_n(T/2n)(V(T/2n))^{-1}\right]^T J\left[U_n(T/2n)(V(T/2n))^{-1}\right] \\ &= (V_n(T/2n)^{-1})^T (U_n(T/2n)^T JU_n(T/2n))(V_n(T/2n))^{-1} \\ &= (1/2)(V_n(T/2n))^{-T} J(V_n(T/2n))^{-1} \\ &= J. \end{aligned}$$

Hence the symplecticity of W. Moreover, $\forall k = 1, 2, \ldots, n$, $W_k(t) = U_k(t)$ $(V_k(t))^{-1}$, $0 \le t \le T/2n$, where $U_k(t)$ and $V_k(t)$ verify system (3.7).

Remark that

$$\frac{dV_k^{-1}(t)}{dt} = -V_k^{-1}(t)\frac{dV_k(t)}{dt}V_k^{-1}(t) = V_k^{-1}(t)A(t_k^2 - t).$$

Then we obtain $\forall k = 1, 2, \ldots, n$, $\forall t \in [0, T/2n]$,

$$\begin{aligned} \frac{dW_k(t)}{dt} &= \frac{dU_k(t)}{dt}(V_k(t))^{-1} + U_k(t)\frac{d(V_k(t))^{-1}}{dt} \\ &= A(t_k^1 + t)W_k(t) + W_k(t)A(t_k^2 - t). \end{aligned}$$

Specifically, for $k = n$, we have

$$\frac{dW_k(t)}{dt} = A\left(T - \frac{T}{2n} + t\right)W_k(t) + W_k(t)A\left(\frac{T}{2n} - t\right), \quad \forall t \in [0, T/2n], \tag{3.8}$$

with $A\left(\frac{T}{2n} - t\right) \to 0$ when $t \to T/2n$.

This implies that $W_n(t)A\left(\frac{T}{2n} - t\right) \to 0$. Thus, the solution $W_n(t)$ of Equation (3.8) converges to the monodromy matrix $X(T)$ of Equation (1.2), i.e., $W_n(T/2n) \equiv X(T)$. □

The following algorithm shows the steps of the computation of an approximation of the monodromy matrix $X(T)$ of Hamiltonian system (1.2).

Algorithm 3.1.

(1) Put $U_1(0) = V_1(0) = \frac{I_{2N}}{\sqrt{2}}$.

(2) For all $k = 1, 2, 3, \ldots, n$.

(a) Resolve on $[0, \frac{T}{2n}]$

$$\frac{d}{dt}\begin{bmatrix} U_k(t) \\ V_k(t) \end{bmatrix} = \begin{bmatrix} A(t_k^1 + t) & 0 \\ 0 & -A(t_k^2 - t) \end{bmatrix}\begin{bmatrix} U_k(t) \\ V_k(t) \end{bmatrix},$$

on initial conditions $U_k(0)$ and $V_k(0)$, where $t_k^1 = \frac{T}{2}(1 + \frac{k-1}{n})$ and $t_k^2 = \frac{T}{2}(1 - \frac{k-1}{n})$.

(b) Determine the orthogonal matrix Q_k with the QR-decomposition such that

$$\begin{pmatrix} U_k(\frac{T}{2n}) \\ V_k(\frac{T}{2n}) \end{pmatrix} = Q_k R_k.$$

(c) Put

$$U_{k+1}(0) = Q_k(1 : 2N, 1 : 2N) \text{ and } V_{k+1}(0) = Q_k(2N + 1 : 4N, 1 : 2N).$$

(3) Compute

$$W = U_{n+1}(\frac{T}{2n})\left(V_{n+1}(\frac{T}{2n})\right)^{-1}.$$

3.2. Numerical examples

Example 3.1. Consider the system of differential equations (see [8] and [14, Vol. 2, p. 412])

$$\begin{cases} q_1 \dfrac{d^2\eta_1}{dt^2} + p_1\eta_1 + [a\eta_1 \cos 2\gamma t + (b\cos 2\gamma t + c\sin 2\gamma t)\eta_3] = 0, \\ q_2 \dfrac{d^2\eta_2}{dt^2} + p_2\eta_2 + g\eta_3 \sin 5\gamma t = 0, \\ q_3 \dfrac{d^2\eta_3}{dt^2} + p_3\eta_3 + [(b\cos 2\gamma t + c\sin 2\gamma t)\eta_1 + g\eta_2 \sin 5\gamma t] = 0, \end{cases} \tag{3.9}$$

which can be reduced to the canonical Hamiltonian as below:

Equation (3.9) gives

$$\frac{d^2\eta}{dt^2} + P(t)\eta = 0, \tag{3.10}$$

with

$$\eta = \begin{pmatrix} \dfrac{\eta_1}{\sqrt{q_1}} \\ \dfrac{\eta_2}{\sqrt{q_2}} \\ \dfrac{\eta_3}{\sqrt{q_3}} \end{pmatrix} \text{ and } P(t) = \begin{pmatrix} \dfrac{p_1 + a\cos 2\gamma t}{q_1} & 0 & \dfrac{b\cos 2\gamma t + c\sin 2\gamma t}{\sqrt{q_1q_3}} \\ 0 & \dfrac{p_2}{q_2} & \dfrac{g\sin 5\gamma t}{\sqrt{q_2q_3}} \\ \dfrac{b\cos 2\gamma t + c\sin 2\gamma t}{\sqrt{q_1q_3}} & \dfrac{g\sin 5\gamma t}{\sqrt{q_2q_3}} & \dfrac{p_3}{q_3} \end{pmatrix}. \tag{3.11}$$

Putting

$$x = \begin{pmatrix} \eta \\ \dfrac{\eta}{dt} \end{pmatrix}, \quad J = \begin{pmatrix} 0_3 & -I_3 \\ I_3 & 0_3 \end{pmatrix}, \quad H(t) = \begin{pmatrix} P(t) & 0_3 \\ 0_3 & I_3 \end{pmatrix},$$

we obtain the following canonical Hamiltonian equation (3.12):

$$J\frac{dX(t)}{dt} = H(t)X(t), \quad X(0) = I_6. \tag{3.12}$$

The matrix $H(t)$ is Hamiltonian and $(2\pi/\gamma)$-periodic. We look at the symplectic matrix $\mathrm{W} = \mathrm{X}(2\pi/\gamma)$ with $\gamma = \sqrt{7}$, $q_1 = q_2 = q_3 = 1$, $p_1 = 4$, $p_2 = 3$, and $p_3 = 2$.

Taking $\varepsilon = a = g$, $\delta = b$, and $c = 0$. Put $W(\varepsilon, \delta) = X(2\pi/\gamma)$, we analyze the strong stability of system (3.12) for some values of ε and δ.

• For $\varepsilon = 15.5$ and $\delta = 1$, Algorithm 3.1 gives the monodromy matrix

$$W(\varepsilon, \delta) = X(T) \equiv$$

$$\begin{pmatrix} -0.0604 & -0.0077 & -0.0449 & -0.0075 & 0.0002 & 0.0072 \\ 0.0272 & -0.1065 & 0.6474 & -0.0002 & -0.5193 & -0.0090 \\ -2.7419 & 0.4722 & -0.6373 & 0.0072 & 0.0090 & -0.3678 \\ 114.3965 & 3.4182 & -2.8568 & -0.0604 & -0.0272 & -2.7419 \\ -3.4182 & 2.5044 & -0.6295 & 0.0077 & -0.1065 & -0.4722 \\ -2.8568 & 0.6295 & 2.0711 & -0.0449 & -0.6474 & -0.6373 \end{pmatrix}$$

verifying the equality $\|W(\varepsilon, \delta)^T JW(\varepsilon, \delta) - J\| \approx 9.4396 \times 10^{-14}$, i.e., $W(\varepsilon, \delta)$ is J-symplectic. The computation of projectors with Algorithm 2.2 gives

$$\mathbb{P}_0 = 10^{-10} \times$$

$$\begin{pmatrix} 0.0009 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0.0010 & 0.0001 & -0.0016 & 0.0000 & 0.0013 & -0.0000 \\ -0.0022 & 0.0016 & 0.0004 & -0.0000 & 0.0000 & -0.0013 \\ 0.2837 & 0.0030 & 0.0005 & 0.0009 & -0.0010 & -0.0022 \\ -0.0030 & -0.0028 & 0.0004 & -0.0000 & 0.0001 & -0.0016 \\ 0.0005 & -0.0004 & 0.0025 & -0.0000 & 0.0016 & 0.0004 \end{pmatrix} \approx 0,$$

$\mathbb{P}_1 = I_6, \quad \mathbb{P}_\infty = 10^{-10} \times$

$$\begin{pmatrix} 0.0009 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0010 & 0.0001 & 0.0016 & 0.0000 & -0.0013 & -0.0000 \\ -0.0022 & -0.0016 & 0.0004 & 0.0000 & 0.0000 & 0.0013 \\ -0.2837 & 0.0030 & -0.0005 & 0.0009 & 0.0010 & -0.0022 \\ -0.0030 & 0.0028 & 0.0004 & 0.0000 & 0.0001 & 0.0016 \\ -0.0005 & -0.0004 & -0.0025 & -0.0000 & -0.0016 & 0.0004 \end{pmatrix} \approx 0,$$

$$\mathbb{P}_r = 0, \quad \mathbb{P}_g = I_6.$$

We get $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I_6$, which show that the monodromy matrix W is strongly stable. So system (3.12) is strongly stable.

• For $\varepsilon = 15$ and $\delta = 1.5$, Algorithm 3.1 gives the monodromy matrix

$W(\varepsilon, \delta) = X(T) \equiv$

$$\begin{pmatrix} -0.4306 & -0.0281 & -0.0536 & -0.0049 & 0.0004 & 0.0247 \\ 0.0361 & -0.1361 & 0.6264 & -0.0004 & -0.5202 & -0.0089 \\ -3.7989 & 0.3584 & -0.5711 & 0.0247 & 0.0089 & -0.2866 \\ 105.5918 & 4.5828 & -3.9892 & -0.4306 & -0.0361 & -3.7989 \\ -4.5828 & 2.3282 & -0.4948 & 0.0281 & -0.1361 & -0.3584 \\ -3.9892 & 0.4948 & 2.0946 & -0.0536 & -0.6264 & -0.5711 \end{pmatrix}$$

verifying the equality $\|W(\varepsilon, \delta)^T JW(\varepsilon, \delta) - J\| \approx 1.0999 \times 10^{-13}$, i.e., $W(\varepsilon, \delta)$ is J-symplectic. The computation of projectors with Algorithm 2.2 gives

$\mathbb{P}_0 = 10^{-14} \times$

$$\begin{pmatrix} -0.0018 & 0.0001 & 0.0002 & -0.0000 & 0.0000 & -0.0001 \\ 0.0016 & 0.0004 & -0.0020 & -0.0000 & 0.0016 & 0.0001 \\ 0.0121 & 0.0006 & -0.0014 & -0.0001 & -0.0001 & -0.0004 \\ 0.2301 & -0.0171 & 0.0472 & -0.0018 & -0.0016 & 0.0121 \\ 0.0171 & -0.0052 & -0.0025 & -0.0001 & 0.0004 & -0.0006 \\ 0.0472 & 0.0025 & 0.0024 & 0.0002 & 0.0020 & -0.0014 \end{pmatrix} \approx 0,$$

$\mathbb{P}_1 = I_6, \quad \mathbb{P}_\infty = 10^{-14} \times$

$$\begin{pmatrix} 0 & -0.0001 & 0.0002 & 0.0000 & 0.0000 & 0.0001 \\ -0.0016 & 0 & 0.0020 & -0.0000 & -0.0016 & 0.0001 \\ 0.0121 & -0.0006 & 0 & 0.0001 & -0.0001 & 0.0004 \\ -0.2301 & -0.0171 & -0.0472 & 0 & 0.0016 & 0.0121 \\ 0.0171 & 0.0052 & -0.0025 & 0.0001 & 0 & 0.0006 \\ -0.0472 & 0.0025 & -0.0024 & 0.0002 & -0.0020 & 0 \end{pmatrix} \approx 0,$$

$$\mathbb{P}_r = 0, \quad \mathbb{P}_g = I_6.$$

We get $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I_6$, which show that the monodromy matrix W is strongly stable. So system (3.12) is strongly stable.

- For $\varepsilon = 15$ and $\delta = 2$, Algorithm 3.1 gives the monodromy matrix

$W(\varepsilon, \delta) = X(T) \equiv$

$$\begin{pmatrix} -0.3558 & -0.0317 & -0.0769 & -0.0031 & 0.0005 & 0.0281 \\ 0.0482 & -0.1355 & 0.6259 & -0.0005 & -0.5202 & -0.0093 \\ -5.0845 & 0.2314 & -0.4383 & 0.0281 & 0.0093 & -0.1819 \\ 106.5830 & 6.1329 & -5.3619 & -0.3558 & -0.0482 & -5.0845 \\ -6.1329 & 2.1743 & -0.3358 & 0.0317 & -0.1355 & -0.2314 \\ -5.3619 & 0.3358 & 2.2768 & -0.0769 & -0.6259 & -0.4383 \end{pmatrix}$$

verifying the equality $\|W(\varepsilon, \delta)^T JW - J\| \approx 2.0674 \times 10^{-13}$, i.e., $W(\varepsilon, \delta)$ is J-symplectic. The computation of projectors with Algorithm 2.2 gives

$\mathbb{P}_0 =$

$$\begin{pmatrix} 0.2357 & -0.0072 & 0.0311 & 0.0007 & -0.0005 & 0.0060 \\ 0.1558 & -0.0047 & 0.0206 & 0.0005 & -0.0003 & 0.0040 \\ 2.0370 & -0.0618 & 0.2691 & 0.0060 & -0.0040 & 0.0520 \\ 79.7441 & -2.4199 & 10.5334 & 0.2357 & -0.1558 & 2.0370 \\ 2.4199 & -0.0734 & 0.3197 & 0.0072 & -0.0047 & 0.0618 \\ 10.5334 & -0.3197 & 1.3914 & 0.0311 & -0.0206 & 0.2691 \end{pmatrix} \neq 0,$$

$\mathbb{P}_1 =$

$$\begin{pmatrix} 0.5287 & -0.0000 & -0.0623 & -0.0000 & 0.0009 & 0.0000 \\ 0.0000 & 1.0095 & 0.0000 & -0.0009 & -0.0000 & -0.0080 \\ -4.0741 & -0.0000 & 0.4619 & -0.0000 & 0.0080 & 0.0000 \\ -0.0000 & 4.8399 & -0.0000 & 0.5287 & 0.0000 & -4.0741 \\ -4.8399 & -0.0000 & -0.6393 & -0.0000 & 1.0095 & 0.0000 \\ 0.0000 & 0.6393 & 0.0000 & -0.0623 & -0.0000 & 0.4619 \end{pmatrix} \neq I_6,$$

$\mathbb{P}_\infty =$

$$\begin{pmatrix} 0.2357 & 0.0072 & 0.0311 & -0.0007 & -0.0005 & -0.0060 \\ -0.1558 & -0.0047 & -0.0206 & 0.0005 & 0.0003 & 0.0040 \\ 2.0370 & 0.0618 & 0.2691 & -0.0060 & -0.0040 & -0.0520 \\ -79.7441 & -2.4199 & -10.5334 & 0.2357 & 0.1558 & 2.0370 \\ 2.4199 & 0.0734 & 0.3197 & -0.0072 & -0.0047 & -0.0618 \\ -10.5334 & -0.3197 & -1.3914 & 0.0311 & 0.0206 & 0.2691 \end{pmatrix} \neq 0,$$

$$\mathbb{P}_r = [\ ] \quad \text{and} \quad \mathbb{P}_g = [\ ],$$

with the projectors $\mathbb{P}_r$ and $\mathbb{P}_g$ are not computed. $\mathbb{P}_0 \neq 0$, $\mathbb{P}_\infty \neq 0$, and $\mathbb{P}_1 \neq I_6$ imply that the monodromy matrix W is not stable (so not strongly stable). This implies that system (3.12) is not strongly stable.

Example 3.2. Consider the equation of Mathieu

$$\frac{d^2y}{dt^2} + (a + b\sin 2t)y = 0, \tag{3.13}$$

where $a, b \in \mathbb{R}$ [14, Vol.1, p. 109]. Putting

$$x = \begin{pmatrix} y \\ \frac{dy}{dt} \end{pmatrix}, \; J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } H(t) = \begin{pmatrix} b\sin 2t + a & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain the following canonical Hamiltonian equation (3.14):

$$J\frac{dX}{dt} = H(t)X, \quad X(0) = I_2, \tag{3.14}$$

where the matrix $H(t)$ is Hamiltonian and π-periodic.

- For $a = 7$ and $b = 4$, Algorithm 3.1 gives the monodromy matrix

$$W(a, b) = X(T) \equiv \begin{pmatrix} 0.0606 & 0.3804 \\ -2.7112 & -0.5184 \end{pmatrix},$$

verifying the equality $\|W(a, b)^T JW(a, b) - J\| \approx 2.3315 \times 10^{-15}$, i.e., $W(\varepsilon, \delta)$ is J-symplectic. The computation of projectors with Algorithm 2.2 gives

$$\mathbb{P}_0 = 10^{-34} \times \begin{pmatrix} -0.7128 & -0.0210 \\ 0.1499 & -0.6806 \end{pmatrix} \approx 0, \quad \mathbb{P}_1 = I_2, \quad \mathbb{P}_\infty = 0_2,$$

$$\mathbb{P}_r = I_2, \quad \mathbb{P}_g = 10^{-34} \begin{pmatrix} 0 & 0.0210 \\ -0.1499 & 0 \end{pmatrix} \approx 0_2.$$

We get $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = \mathbb{P}_1 = I_2$, which show that the monodromy matrix W is strongly stable. So system (3.14) is strongly stable.

- For $a = 16.1916618724166685\ldots$ and $b = 5$, Algorithm 3.1 gives the monodromy matrix $W(a, b)$ verifying the equality $\|W(a, b)^T JW(a, b) - J\|$ $\approx 3.3307 \times 10^{-16}$, i.e., $W(a, b)$ is J-symplectic. Moreover, this matrix has the mixed eigenvalue 1. Note that any strongly stable matrix does not have mixed eigenvalue. The computation of projectors with Algorithm 2.2 gives

$$\mathbb{P}_0 = 10^{-97} \times \begin{pmatrix} -0.0144 & -0.0012 \\ 0.1702 & 0.0143 \end{pmatrix} \approx 0_2, \quad \mathbb{P}_1 = I_2,$$

$$\mathbb{P}_\infty = 10^{-97} \times \begin{pmatrix} 0 & 0.0012 \\ -0.1702 & 0 \end{pmatrix} \approx 0_2,$$

with

$$\mathbb{P}_r = 0_2 \quad \text{and} \quad \mathbb{P}_g = 0_2.$$

We have $\mathbb{P}_0 = \mathbb{P}_\infty = 0$ and $\mathbb{P}_r + \mathbb{P}_g = 0_2 \neq I_2$, which show that the monodromy matrix is not strongly stable. So system (3.14) is not strongly stable.

• For $a = 10$ and $b = 7$, Algorithm 3.1 gives the monodromy matrix

$$W(a, b) = X(T) \equiv \begin{pmatrix} -0.9513 & 0.0549 \\ 1.1393 & -1.1169 \end{pmatrix},$$

verifying the equality $\|W(a, b)^T JW(a, b) - J\| \approx 2.1094 \times 10^{-15}$, i.e., $W(a, b)$ is J-symplectic. The computation of projectors with Algorithm 2.2 gives

$$\mathbb{P}_0 = \begin{pmatrix} 0.6572 & 0.1042 \\ 2.1629 & 0.3428 \end{pmatrix} \neq 0_2, \quad \mathbb{P}_1 = 10^{-15} \times \begin{pmatrix} 0.2220 & -0.0278 \\ 0 & -0.2220 \end{pmatrix} \approx 0_2,$$

$$\mathbb{P}_\infty = \begin{pmatrix} 0.3428 & -0.1042 \\ -2.1629 & 0.6572 \end{pmatrix} \neq 0_2, \quad \mathbb{P}_r = [\ ] \text{ and } \mathbb{P}_g = [\ ].$$

This shows that the monodromy matrix is not strongly stable. So system (3.14) is not strongly stable.

4. Conclusion

We explained how to analyze the strong stability of Hamiltonian systems with periodic coefficients stability. In this study, we propose a method to extract the monodromic matrix $X(T)$ of the system. This method, called the method of double orthogonal sweeps, is based on orthogonal sweeps method presented by Godunov [6]. We showed that the matrix $W = U(T)(V(0))^{-1} \equiv U_n(T/2n)(V_n(T/2n))^{-1}$, for an integer $n > 1$, is J-symplectic and gives a good approximation of the monodromy matrix $X(T)$ of (1.2). Thereafter, we deduce the strong stability of the matrix W using Algorithm 2.2. Such system (1.2) is stable (respectively, strongly stable) if and only if $X(T)$ is stable (strongly stable).

We illustrate our theory by two numerical examples. In each example, the monodromy matrix of system is computed using Algorithm 3.1 for some values of the parameters of the system; after that we use Algorithm 2.2 to know if $X(T)$ is strong stable or not.

In a future work, we will search the strong stability zone of two Hamiltonian systems with periodic coefficients given in numerical example.

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