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# ON APPROXIMATION OF SOLUTIONS OF SOME QUASILINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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## Abstract

The paper presents some result on the existence and behaviour of solutions of the quasilinear systems of differential equations. The behaviour of integral curves in neighbourhood of the solutions of corresponding linear systems is considered. The obtained results also contain the answer to the question on approximation of solutions, whose existence is established. The errors of the approximation are defined by the functions that can be sufficiently small. The qualitative analysis theory of differential equations and the topological retraction method are used.

## 1. Introduction

Let us consider the system of quasilinear differential equations

$$\dot{x} = C(t)x + G(t) + F(x, t), \qquad \text{(Q)}$$

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and corresponding linear system

$$\dot{x} = C(t)x + G(t), \tag{L}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $n \geq 2$; $t \in I = \langle a, \infty \rangle$, $a \in \mathbb{R}$; $D \subset \mathbb{R}^n$ is open set, $\Omega = D \times I$, $C(t) = (c_{ij}(t))_{n \times n}$ the matrix-function with elements $a_{ij} \in C(I, \mathbb{R})(i, j = 1, \dots, n)$; the $G(t) = (g_1(t), \dots, g_n(t))^T$ is the vector-function with elements $g_i \in C(I, \mathbb{R})$, $F(x, t) = (f_1(x, t), \dots, f_n(x, t))^T$ is the vector-function with elements $f_i \in C(\Omega, \mathbb{R})$, functions $f_i(x, t)$, $i = 1, \dots, n$, satisfy the Lipschitz's condition with respect to the variable $x$.

Above conditions for the functions $c_{ij}$, $g_i$, $f_i(i, j = 1, \dots, n)$, grant the existence and unique solution of every Cauchy's problem for system (Q) in $\Omega$.

Let

$$\Gamma = \{(x, t) \in \Omega : x = \varphi(t), t \in I\}, \tag{1}$$

where $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$, $\varphi_i(t) \in C^1(I, \mathbb{R})$, is an integral curve of system (L).

In this paper, the behaviour of the solutions of system (Q) in the neighbourhood of curve $\Gamma$ is considered. The qualitative analysis theory of differential equations and the topological retraction method of Ważevski [6], are used.

## 2. Notations and Preliminaries

We shall consider the behaviour of integral curves $(x(t), t)$, $t \in I$, of system (Q) with respect to the set

$$\omega = \left\{(x, t) \in \Omega : \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 < r^2(t)\right\}, \tag{2}$$

where $r \in C^1(I, \mathbb{R}^+)$.

The boundary surface of set $\omega$ with respect to the set $\Omega$ is

$$W = \left\{(x, t) \in Cl\omega : B(x, t) := \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 - r^2(t) = 0\right\}.$$

Let us denote the tangent vector field to an integral curve $(x(t), t)$, $t \in I$, of (Q) by $T$. The vector $\nabla B$ is the external normal on surface $W$. We have

$$T = \left(\sum_{j=1}^{n} a_{1j}x_j + f_1, \ldots, \sum_{j=1}^{n} a_{ij}x_j + f_i, \ldots, \sum_{j=1}^{n} a_{nj}x_j + f_n, 1\right),$$

$$\frac{1}{2}\nabla B = \left(x_1 - \varphi_1, \ldots, x_i - \varphi_i, \ldots, x_n - \varphi_n, -\sum_{i=1}^{n} (x_i - \varphi_i)\varphi_i' - rr'\right).$$

By means of scalar product $P(x, t) = (\nabla B, T)$ on $W$, we shall establish the behaviour of integral curve of (Q) with respect to the set $\omega$.

Let us denote by $S^n(I)$ a class of solutions $(x(t), t)$ of the system (Q) defined on $I$, which depends on $n$ parameters. We shall simply say that the class of solutions $S^n(I)$ belongs to the set $\omega$ if graphs of functions in $S^n(I)$ are contained in $\omega$. In that case, we shall write $S^n(I) \subset \omega$. For $p = 0$, we have the notation $S^0(I)$, which means that there exists at least one solution $(x(t), t)$ on $I$ of the system (Q), whose graph belongs to the set $\omega$.

The results of this paper are based on the following lemmas (see [2, 5]):

**Lemma 1.** *If, for the system* (Q), *the scalar product* $P(x, t) < 0$ *on* $W$, *then the system* (Q) *has a class of solutions* $S^n(I)$ *belonging to the set* $\omega$ *for all* $t \in I$, *i.e.,* $S^n(I) \subset \omega$.

**Lemma 2.** *If, for the system* (Q), *the scalar product* $P(x, t) > 0$ *on* $W$, *then the system* (Q) *has at least one solution on* $I$, *whose graph belongs to the set* $\omega$ *for all* $t \in I$, *i.e.,* $S^0(I) \subset \omega$.

According to Lemma 1, the set $W$ is a set of point of strict entrance of integral curves of the system (Q) with respect to the sets $\omega$ and $\Omega$. Hence, all solutions of system (Q) which satisfy condition

$$\sum_{i=1}^{n}(x_i(t_0) - \varphi_i(t_0))^2 < r^2(t_0), \quad (t_0 \in I),$$

also satisfy condition $\sum_{i=1}^{n}(x_i - \varphi_i(t))^2 < r^2(t)$, for every $t > t_0$, i.e., $S^n(I) \subset \omega$.

According to Lemma 2, the set $W$ is a set of points of strict exit of integral curves of the system (Q) with respect to the sets $\omega$ and $\Omega$. Hence, according to Wazewski's retraction method [6], system (Q) has at least one solutions belonging to the set $\omega$ for every $t \in I$, i.e., $S^0(I) \subset \omega$.

## 3. Main Results

**Theorem 1.** *Let* $\Gamma$ *is an integral curve of system* (L), $r \in C^1(I, \mathbb{R}^+)$ *and* $u \in C(I)$.

(a) *If the conditions*

$$\left.\begin{aligned} \sum_{j=1(j\neq i)}^{n} |c_{ij}(t) + c_{ji}(t)| \leqslant -2c_{ii}(t), \quad i = 1, \ldots, n, \\ \sum_{i=1}^{n}(x_i - \varphi_i(t))\, f_i(x, t) < r(t)\, r'(t), \end{aligned}\right\} \tag{3}$$

*are satisfied on* $W$, *then the system* (Q) *has a class of solutions* $S^n(I) \subset \omega$.

(b) *If the conditions*

$$\left.\begin{aligned} \sum_{j=1(j\neq i)}^{n} |c_{ij}(t) + c_{ji}(t)| \leqslant 2c_{ii}(t), \quad i = 1, \ldots, n, \\ \sum_{i=1}^{n} (x_i - \varphi_i(t))\, f_i(x, t) > r(t)\, r'(t), \end{aligned}\right\} \tag{4}$$

*are satisfied on* $W$, *then the system* (Q) *has at least one solution, whose graph belongs to the set* $\omega$, *i.e.,* $S^0(I) \subset \omega$.

**Proof.** For the scalar product $P(x, t) = (\frac{1}{2}\nabla B, T)$, we have

$$P(x, t) = \sum_{i=1}^{n}\left[(x_i - \varphi_i)\left(\sum_{j=1}^{n} c_{ij}x_j + g_i + f_i\right)\right] - \sum_{i=1}^{n}(x_i - \varphi_i)\varphi_i' - rr'$$

$$= \sum_{i=1}^{n} c_{ii}(x_i - \varphi_i)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(c_{ij} + c_{ji})(x_i - \varphi_i)(x_j - \varphi_j)$$

$$+ \sum_{i=1}^{n}\left[(x_i - \varphi_i)\left(\sum_{j=1}^{n} c_{ij}\varphi_j + g_i - \varphi_i' + f_i\right)\right] - rr'$$

$$= \sum_{i=1}^{n} c_{ii}(x_i - \varphi_i)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(c_{ij} + c_{ji})(x_i - \varphi_i)(x_j - \varphi_j)$$

$$+ \sum_{i=1}^{n}(x_i - \varphi_i)\, f_i - rr'.$$

The following estimates for $P(x, t)$ on $W$ are valid.

(a) According to the conditions (3), we have

$$P(x,\,t) \le \sum_{i=1}^{n} c_{ii}(x_i - \varphi_i)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{1}{2}|c_{ij} + c_{ji}|\left[(x_i - \varphi_i)^2 + (x_j - \varphi_j)^2\right]$$

$$+ \sum_{i=1}^{n}(x_i - \varphi_i)\,f_i - rr'$$

$$= \sum_{i=1}^{n}\left[c_{ii} + \sum_{j=1(j\neq i)}^{n} \frac{1}{2}|c_{ij} + c_{ji}|\right](x_i - \varphi_i)^2$$

$$+ \sum_{i=1}^{n}(x_i - \varphi_i)\,f_i - rr' < 0.$$

(b) According to the conditions (4), we have

$$P(x,\,t) \ge \sum_{i=1}^{n}\left[c_{ii} - \sum_{j=1(j\neq i)}^{n} \frac{1}{2}|c_{ij} + c_{ji}|\right](x_i - \varphi_i)^2 + \sum_{i=1}^{n}(x_i - \varphi_i)\,f_i - rr' > 0.$$

According to Lemmas 1 and 2, the above estimates for $P(x,\,t)$ imply the statement of the theorem. □

**Theorem 2.** *Let* $\Gamma$ *is an integral curve of system* (L), $r \in C^1(I,\,\mathbb{R}^+)$ *and* $u \in C(I)$.

(a) *If the conditions*

$$\left.\begin{aligned} \sum_{j=1(j\neq i)}^{n} |c_{ij}(t) + c_{ji}(t)| &\leqslant -2c_{ii}(t) - 2u(t), \quad i = 1,\,\ldots,\,n, \\ \sum_{i=1}^{n}|f_i(x,\,t)| &< u(t)r(t) + r'(t), \end{aligned}\right\} \tag{5}$$

*are satisfied on W, then the system* (Q) *has a class of solutions* $S^n(I) \subset \omega$.

(b) *If the conditions*

$$\left.\begin{aligned} \sum_{j=1(j\neq i)}^{n} |c_{ij}(t) + c_{ji}(t)| \leqslant 2c_{ii}(t) - 2u(t), \quad i = 1, \ldots, n, \\ \sum_{i=1}^{n} |f_i(x, t)| < u(t)r(t) - r'(t), \end{aligned}\right\} \tag{6}$$

*are satisfied on* $W$, *then the system* (Q) *has at least one solution, whose graph belongs to the set* $\omega$, *i.e.,* $S^0(I) \subset \omega$.

**Proof.** For the scalar product $P(x, t) = (\frac{1}{2}\nabla B, T)$, we have the following estimates for $P(x, t)$ on $W$ are valid.

(a) According to the conditions (5), we have

$$P(x, t) \leq \sum_{i=1}^{n}\left[c_{ii} + \sum_{j=1(j\neq i)}^{n}\frac{1}{2}|c_{ij} + c_{ji}|\right](x_i - \varphi_i)^2 + \sum_{i=1}^{n}(x_i - \varphi_i) f_i - rr'$$

$$\leq \sum_{i=1}^{n}(-u)(x_i - \varphi_i)^2 + \sum_{i=1}^{n} r|f_i| - rr'$$

$$= -ur^2 + r\sum_{i=1}^{n}|f_i| - rr' < 0.$$

(b) According to the conditions (6), we have

$$P(x, t) \geq \sum_{i=1}^{n}\left[c_{ii} - \sum_{j=1(j\neq i)}^{n}\frac{1}{2}|c_{ij} + c_{ji}|\right](x_i - \varphi_i)^2 + \sum_{i=1}^{n}(x_i - \varphi_i) f_i - rr'$$

$$\geq \sum_{i=1}^{n} u(x_i - \varphi_i)^2 - \sum_{i=1}^{n} r|f_i| - rr'$$

$$= ur^2 - r\sum_{i=1}^{n}|f_i| - rr' > 0.$$

According to Lemmas 1 and 2, the above estimates for $P(x, t)$ imply the statement of the theorem. □

## 4. Applications

For the quasilinear system, which is known in plasma physics

$$\begin{aligned} \dot{x}_1 &= -\nu_1 x_1 + M_1 x_2 x_3, \\ \dot{x}_2 &= -\nu_2 x_2 + M_2 x_1 x_3, \\ \dot{x}_3 &= -\nu_3 x_3 + M_3 x_1 x_2, \end{aligned} \tag{7}$$

where $\nu_i,\ M_i \in \mathbb{R},\ \nu_i > 0,$ and corresponding linear system

$$\begin{aligned} \dot{x}_1 &= -\nu_1 x_1, \\ \dot{x}_2 &= -\nu_2 x_2, \\ \dot{x}_3 &= -\nu_3 x_3, \end{aligned} \tag{8}$$

we have:

**Theorem 3.** *Let* $r \in C^1(I, \mathbb{R}^+)$ *and* $\nu = \min\{\nu_1, \nu_2, \nu_3\}$. *If*

$$|M_1 + M_2 + M_3| \leqslant \frac{\nu}{r} + \frac{r'}{r^2}, \quad t \in I,$$

*then the system* (7) *has a class of solutions* $S^3(I) \subset \omega$, *where*

$$\omega = \{(x_1, x_2, x_3, t) \in \mathbb{R}^3 \times I : x_1^2 + x_2^2 + x_3^2 < r^2(t)\}.$$

**Proof.** For the scalar product $P(x, t) = (\frac{1}{2}\nabla B, T)$ on $W$, we have

$$\begin{aligned} P(x, t) &= (-\nu_1 x_1 + M_1 x_2 x_3)x_1 + (-\nu_2 x_2 + M_2 x_1 x_3)x_2 \\ &\quad + (-\nu_3 x_3 + M_3 x_1 x_2)x_3 - rr' \\ &= -\nu_1 x_1^2 - \nu_2 x_2^2 - \nu_3 x_3^2 + x_1 x_2 x_3 (M_1 + M_2 + M_3) - rr' \end{aligned}$$

$$\leq -\nu(x_1^2 + x_2^2 + x_3^2) + |x_1x_2x_3|\,|M_1 + M_2 + M_3| - rr'$$

$$< -\nu r^2 + r^3|M_1 + M_2 + M_3| - rr' \leq 0.$$

According to Lemma 1, all solutions of (7) belonging in the neighbourhood of solutions of the system (8), i.e., $S^3(I) \subset \omega$. □

**Example 1.** For the system (7) and the initial condition

$$x_1^2(t_0) + x_2^2(t_0) + x_3^2(t_0) \leqslant \alpha^2 e^{-2\alpha^2 t_0}, \tag{9}$$

we can prove the following:

All solutions $(x_1(t), x_2(t), x_3(t))$ of system (7), which satisfy the initial condition (9) satisfy the condition

$$x_1^2(t) + x_2^2(t) + x_3^2(t) \leqslant \alpha^2 e^{-2\alpha^2 t}, \text{ for } t > t_0 \geqslant \frac{1}{\alpha^2} \ln\left(\frac{|M_1 + M_2 + M_3|}{\nu - \alpha}\right).$$

This result follows from Theorem 3, with $r(t) = \alpha e^{-\alpha^2 t}$, $0 < \alpha < \nu$.

**Remark 1.** The obtained results also contain answer to the question on errors of the approximation of solutions $x(t)$, whose existence is established. Under the conditions of Theorem 1, case (a); or the conditions of Theorem 2, case (a); the integral curve of system (L) is the approximate solution of solutions $x(t)$ of system (Q). The errors of the approximation are defined by the functions $r(t)$ can be arbitrary small, $\forall t \in I$.

**Remark 2.** The obtained results also give the possibility to discuss the stability (instability) of solutions of the system (Q). For example, under the conditions of Theorem 1, case (a); every solution of (Q) with initial value in $\omega$ is $r$-stable (stable with the function of stability $r$), if $r$ tends to zero as $t \to \infty$ and $r'(t) < 0$, $t \in I$. However, if we consider the Theorem 1, case (b), then established solution in $\omega$ is $r$-instable in case where $r'(t) > 0$, $t \in I$. The same conclusion holds for other theorems.

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