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DISTANCE RATIO METRIC ON A HALF-PLANE

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Abstract

We prove Lipschitz continuity of an arbitrary holomorphic mapping $f : \mathbb{H} \to \mathbb{H}$ regarding the distance ratio metric with the Lipschitz constant $C = 2$. This represents a generalization for the upper half-plane domain of Gehring - Palka theorem on Möbius transformations. We also proved that the constant 2 is best possible.

1. Introduction

For a subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}}\right),$$

where $d(x, \partial G)$ denotes the Euclidean distance from x to ∂G. The distance ratio metric was introduced by Gehring and Palka [3] and in the above simplified form by Vuorinen [9]. As the "first approximation" of the

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quasihyperbolic metric, it is frequently used in the study of hyperbolic type metrics ([1, 6, 7, 8, 11]) and geometric theory of functions.

For an open continuous mapping $f : G \to G'$, we consider the following condition: There exists a constant $C \geq 1$ such that for all $x, y \in G$, we have

$$j_{G'}(f(x), f(y)) \leq C j_G(x, y),$$

or, equivalently, that the mapping

$$f : (G, j_G) \to (G', j_{G'}),$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C.

However, unlike other frequently used metrics (hyperbolic metric, absolute ratio metric), the distance ratio metric j_G is not invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for this metric under conformal mappings or Möbius transformations in higher dimension. Gehring et al. [3, 4] proved that j metric is not changed by more than a factor 2 under Möbius transformations.

Theorem A. *If G and G' are proper subdomains of $\mathbb{R}^n$ and if f is a Möbius transformation of G onto G', then for all $x, y \in G$*

$$j_{G'}(f(x), f(y)) \leq 2 j_G(x, y).$$

An interesting problem is to investigate Lipschitz continuity of the distance-ratio metric under some other conformal mappings.

Since proper domains for Möbius transformation are $\mathbb{B}^n$ and $\mathbb{H}^n$, it happens that for lower dimension n much more can be said. For example, in this paper, we obtain an analogous to the above result for $n = 2$ and *arbitrary holomorphic mappings* of a half-plane into itself. Thereby Theorem A is substantially generalized in this case.

2. Results

Because j metric is invariant to the rotation and stretching, we can restrict the proof to the case of upper half-plane $\mathbb{H} := \{z|\Im z > 0\}$.

A generalization of Theorem A for this domain is given by the following:

Theorem 1. *For any holomorphic mapping* f, $f : \mathbb{H} \to \mathbb{H}$, *and all* $z, w \in \mathbb{H}$, *we have*

$$j_{\mathbb{H}}(f(z), f(w)) \le 2j_{\mathbb{H}}(z, w). \tag{1.1}$$

The constant 2 *is best possible.*

Proof. Denote $s = \min\{\Im z, \Im w\}$ and suppose that $\Im f(z) \le \Im f(w)$.

Then

$$j_{\mathbb{H}}(z, w) = \log\left(1 + \frac{|z - w|}{s}\right);\ j_{\mathbb{H}}(f(z), f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{\Im f(z)}\right).$$

Main tool in the proof will be Julia's variant of the famous Schwartz-Pick lemma for the half-plane (cf. [2]).

Theorem B. *For any holomorphic mapping* f, $f : \mathbb{H} \to \mathbb{H}$, *and all* $z, w \in \mathbb{H}$, *we have*

$$\left|\frac{f(z) - f(w)}{f(z) - \overline{f(w)}}\right| \le \left|\frac{z - w}{z - \overline{w}}\right|.$$

Applying this assertion, we get

$$\left|\frac{f(z) - \overline{f(w)}}{f(z) - f(w)}\right|^2 - 1 \ge \left|\frac{z - \overline{w}}{z - w}\right|^2 - 1,$$

that is,

$$\left|\frac{f(z) - f(w)}{z - w}\right|^2 \le \frac{\Im f(z) \Im f(w)}{\Im z \Im w},$$

since for any $x,\ y \in \mathbb{C},$ the identity

$$|x - \overline{y}|^2 - |x - y|^2 = 4\Im x \Im y$$

holds.

Therefore,

$$\frac{|f(z) - f(w)|^2}{\Im^2 f(z)} \leq \frac{|z - w|^2}{s^2} \frac{\Im f(w)}{\Im f(z)} = \frac{|z - w|^2}{s^2} \left(1 + \frac{\Im(f(w) - f(z))}{\Im f(z)}\right),$$

i.e.,

$$\frac{|f(z) - f(w)|}{\Im f(z)} \leq \frac{|z - w|}{s} \sqrt{1 + \frac{|f(z) - f(w)|}{\Im f(z)}}, \qquad (*)$$

since $\Im x \leq |x|,\ x \in \mathbb{C}.$

Denote $\dfrac{|z - w|}{s} := 2X.$ After some elementary calculations, from the relation $(*),$ we get

$$1 + \frac{|f(z) - f(w)|}{\Im f(z)} \leq \left(X + \sqrt{1 + X^2}\right)^2.$$

Hence,

$$j_{\mathbb{H}}(f(z),\ f(w)) = \log\left(1 + \frac{|f(z) - f(w)|}{\Im f(z)}\right) \leq 2 \log\left(X + \sqrt{1 + X^2}\right)$$

$$\leq 2 \log(1 + 2X) = 2 j_{\mathbb{H}}(z,\ w),$$

and the proof is done.

To prove that the constant 2 is best possible, choose

$$f_0(z) = a - \frac{1}{b + z},$$

where a and b are arbitrary real numbers.

Since $\Im f_0(z) = \dfrac{\Im z}{|b+z|^2}$, we conclude that $f_0 : \mathbb{H} \to \mathbb{H}$.

A calculation of j values along the line $\zeta \subset \mathbb{H}$ given by $\zeta := \{z = i - b + t,\ t \in \mathbb{R}\}$ with $w = i - b$, gives

$$j_\zeta(z, w) = \log(1+t);\ j_\zeta(f_0(z), f_0(w)) = \log\left(1 + \frac{\frac{t}{\sqrt{1+t^2}}}{\frac{1}{1+t^2}}\right) = \log\left(1 + t\sqrt{1+t^2}\right).$$

Hence,

$$\frac{j_\zeta(f_0(z), f_0(w))}{j_\zeta(z, w)} = \frac{\log\left(1 + t\sqrt{1+t^2}\right)}{\log(1+t)},$$

and this expression is tending to 2 as $t \to \infty$.

Therefore, the constant 2 is best possible.

Remark. An interesting fact is that for holomorphic mappings $f : \mathbb{D} \to \mathbb{D}$, where $\mathbb{D}$ is the open unit disk, best possible Lipschitz constant C is not known.

In the recent paper [12], the following estimation is given:

$$1 + |a| \le C \le \min\{2(1+|a|), \sqrt{5 + 2|a| + |a|^2}\},$$

where $a := f(0)$.

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