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NORMAL VARIANCE-MEAN MIXTURES (III) OPTION PRICING THROUGH STATE-PRICE DEFLATORS

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Abstract

An alternative to the Black-Scholes-Vasicek deflator is proposed. It is based on a simple multivariate exponential normal variance-mean mixture Lévy process. Closed-form analytical integral formulas for pricing the European geometric basket option with a deflated normal variance-mean price process are obtained. Applications to the variance-gamma, the normal inverse Gaussian and the normal tempered stable processes are included. An extended Black-Scholes formula that takes into account the correlation structure of the market is also derived.

1. Introduction

The present contribution follows up the author's investigations around the normal variance-mean (NVM) mixture model introduced in Barndorff-Nielsen et al. [6]. It is motivated by the search for non-Gaussian multivariate option pricing models as an alternative to the

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multivariate Black-Scholes model. Our emphasis is on results that can be formulated for a possibly large class of NVM mixture models in the spirit of Korsholm [32]; Bingham and Kiesel [7]; and Tjetjep and Seneta [50] among others.

The concept of state-price deflator or stochastic discount factor, which has been introduced by Duffie [17], pp. 23 and 97, is a convenient ingredient of general financial pricing rules. It contains information about the valuation of payments in different states at different points in time. The state-price deflator is a natural extension of the notion of state prices that were introduced earlier and studied by Arrow [2, 3, 4, 5]; Debreu [14]; Negishi [44]; and Ross [48], a milestone in the history of asset pricing (see Dimson and Mussavian [16]). Though general frameworks for deriving state-price deflators exist (e.g., Milterssen and Persson [42]; Jeanblanc et al. [31]; Munk [43]), there are not many papers, which propose explicit expressions for them and their corresponding distribution functions. A short account of the content follows.

Section 2 recalls the multivariate NVM mixture model in its simplest Lévy process form. It generalizes the construction of the multivariate exponential variance-gamma process considered in Cont and Tankov [12]; Luciano and Schoutens [35]; and first studied in Hürlimann [24, 25]. Theoretical and statistical justifications for its use are briefly mentioned. In Section 3, we derive the multivariate NVM deflator as alternative to the multivariate Black-Scholes-Vasicek (BSV) deflator introduced in Hürlimann [21] (see also Hürlimann [23, 28]). Analytical integral formulas for pricing the European geometric basket call option within the multivariate NVM market with NVM deflator are derived and discussed in Section 4. The final Section 5 introduces a multivariate subordinated asset price model, whose univariate version coincides with the model by Hurst et al. [20]. As a special case, it includes an extended Black-Scholes formula that takes into account the correlation structure of the market. The present novel approach is general and simple. From a statistical perspective, parameter estimation can be done with a multivariate moment method (see Hürlimann [27]). Moreover, the derivation of the extended Black-Scholes formula has an elementary probabilistic flavour.

2. A Selection of Multivariate Normal Variance-Mean Mixture Processes

The univariate *normal variance-mean* (NVM) *mixture* process is defined as a drifted Brownian motion time changed by an independent *mixing process*. Viewed from the initial time 0, it is defined by

$$X_t = \theta \cdot G_t + \tau \cdot W_{G_t}, \quad t > 0, \quad \tau > 0, \quad -\infty < \theta < \infty, \qquad (2.1)$$

where W_t is a standard Wiener process and G_t is an independent *subordinator*, that is an increasing, positive Lévy process. Since X_t is a Lévy process, the dynamics of the process is determined by its distribution at unit time. The random variable $X = X_1$ follows a distribution, whose cumulant generating function (cgf) $C_X(u) = \ln E[\exp(uX)]$ is assumed to exist over some open interval. By considering in parallel different subordinators, a unified approach to several important NVM mixture models is possible. Our focus is restricted to three of them (for further possible choices, consult Hürlimann [26], Section 4).

(a) Variance-gamma (VG) process

The subordinator $G_t \sim \Gamma(\nu^{-1}t, \nu^{-1})$ is a *gamma* process with unit mean rate and variance rate ν. The random variable $X = X_1 \sim VG(\theta, \tau^2, \nu)$ follows a three parameter VG distribution with cgf:

$$C_X(u) = -\nu^{-1} \cdot \ln\{1 - \nu \cdot (\theta u + \frac{1}{2}\tau^2 u^2)\}, \quad \tau, \nu > 0, \quad -\infty < \theta < \infty. \qquad (2.2)$$

This formula is obtained from the cgf $C_G(u) = -\nu^{-1} \cdot \ln(1 - \nu \cdot u)$ of the gamma random variable $G = G_1$ by conditioning using that $X|G \sim N(\theta G, \tau^2 G)$ is normally distributed. The increments of the process follow a VG distribution, namely, $X_{t+s} - X_s \sim VG(\theta t, \tau^2 t, \nu / t)$, $0 \le s < t$.

(b) Normal inverse Gaussian (NIG) process

The cgf of the *inverse Gaussian* subordinator is $C_{G_t}(u) = \nu^{-1}t \cdot \{1 - \sqrt{1 - 2\nu \cdot u}\}$, $\nu > 0$. The random variable $X \sim NIG(\theta, \tau^2, \nu)$ follows a three parameter NIG distribution with cgf:

$$C_X(u) = \nu^{-1} \cdot \{1 - \sqrt{1 - 2\nu \cdot (\theta u + \frac{1}{2}\tau^2 u^2)}\}, \quad \tau, \nu > 0, \quad -\infty < \theta < \infty. \tag{2.3}$$

The increments of the process are $NIG(\theta t, \tau^2 t, \nu / t)$ distributed.

(c) Normal tempered stable (NTS) process

The subordinator follows a *classical tempered stable* process with cgf $C_{G_t}(u) = \nu^{-\alpha} t / \alpha \cdot \{1 - (1 - 2\nu^{\alpha} u)^{\alpha/2}\}$, $\nu > 0$, $\alpha \in (0, 2)$. The random variable $X \sim NTS(\theta, \tau^2, \nu, \alpha)$ follows a four parameter NTS distribution with cgf:

$$C_X(u) = \nu^{-\alpha} t / \alpha \cdot \{1 - (1 - 2\nu^{\alpha}(\theta u + \frac{1}{2}\tau^2 u^2))^{\alpha/2}\}, \quad \tau, \nu > 0, \quad \alpha \in (0, 2),$$
$$-\infty < \theta < \infty. \tag{2.4}$$

The increments of the process are $NTS(\theta t, \tau^2 t, \nu t^{-\alpha^{-1}}, \alpha)$ distributed. The special case $\alpha = 1$ is the NIG process.

Similarly to those multivariate versions mentioned in Hürlimann [24] for the VG process, different multivariate models for each NVM mixture process can be defined. For simplicity, we consider here only multivariate Lévy processes with NVM components of the type

$$X_t^{(k)} = \theta_k \cdot G_t + \tau_k \cdot W_{G_t}^{(k)}, \quad k = 1, \ldots, n, \tag{2.5}$$

where the $W_t^{(k)}$'s are *correlated* standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} dt$. Despite all its shortcomings, the use of the model

(2.5) is justified theoretically by looking at the variance of its NVM margins $X^{(k)} = X_1^{(k)}$ at unit time, namely,

$$\mathrm{Var}[X^{(k)}] = \tau_k^2 + \mathrm{Var}[G] \cdot \theta_k^2. \tag{2.6}$$

Each variance decomposes into an *idiosyncratic* component τ_k^2, that is attributed to the Brownian motion, and an *exogeneous* component $\mathrm{Var}[G] \cdot \theta_k^2$, that is due to the time change of the Brownian motion. The parameters θ_k govern the exposures of the margins to the global *market uncertainty* measured by the common variance $\mathrm{Var}[G]$. Similarly, one notes that the skewness and kurtosis are also affected by the single marginal settings and the common subordinated parameter(s) (see Hürlimann [27], Theorem 2.1). On the other hand, the statistical moment method developed in Hürlimann [27] and its successful application to real-world data justifies its use in practical work. For these reasons, it is legitimate to focus first on the model (2.5). One knows that the cgf of the NVM mixture model is given by

$$C_{X_t}(u) = C_{G_t}(\theta^T u + \frac{1}{2} u^T \Sigma u), \quad \theta = (\theta_1, \dots, \theta_n), \quad \Sigma = (\rho_{ij}\tau_i\tau_j),$$

$$u = (u_1, \dots, u_n). \tag{2.7}$$

The joint cgf of the considered multivariate NVM processes can be expressed in closed-form.

Proposition 2.1. *The joint cgf of the multivariate NVM mixture process* $X_t = (X_t^{(1)}, \dots, X_t^{(n)})$ *with parameters* $\theta = (\theta_1, \dots, \theta_n), \Sigma = (\rho_{ij}\tau_i\tau_j)$, *and subordinator* G_t, *is determined as follows*:

Multivariate VG process

The multivariate VG random vector $X_t \sim VG(\theta t, \Sigma t, \nu/t)$ *has joint cgf*

$$C_{X_t}(u) = -\nu^{-1} t \cdot \ln\{1 - \nu \cdot (\theta^T u + \frac{1}{2} u^T \Sigma u)\}, \quad u = (u_1, \dots, u_n). \tag{2.8}$$

Multivariate NIG process

The multivariate NIG random vector $X_t \sim NIG(\theta t, \sum t, \nu/t)$ *has joint cgf*

$$C_{X_t}(u) = \nu^{-1}t \cdot \{1 - \sqrt{1 - 2\nu \cdot (\theta^T u + \frac{1}{2}u^T \sum u)}\}, \quad u = (u_1, \dots, u_n). \tag{2.9}$$

Multivariate NTS process

The multivariate NTS random vector $X_t \sim NTS(\theta t, \sum t, \nu t^{-\alpha^{-1}}, \alpha)$ *has joint cgf*

$$C_{X_t}(u) = \nu^{-\alpha}t / \alpha \cdot \{1 - (1 - 2\nu^{\alpha} \cdot (\theta^T u + \frac{1}{2}u^T \sum u))^{\alpha/2}\}, \quad u = (u_1, \dots, u_n). \tag{2.10}$$

Proof. This follows from (2.7) and the explicit forms for the cgf of the subordinators.

3. Unified State-Price Deflator Representation for the Exponential NVM Mixture Process

In the following, we simplify and generalize the procedure in Hürlimann [24], Section 4. Consider the following class of asset pricing models. Given the current prices of $n \geq 1$ risky assets at initial time 0 their future prices at time $t > 0$ are described by exponential NVM mixture processes

$$S_t^{(k)} = S_0^{(k)} \exp((\mu_k - \omega_k)t + X_t^{(k)}), \quad k = 1, \dots, n, \tag{3.1}$$

where μ_k represents the mean logarithmic rate of return of the k-th risky asset per time unit, and the random vector $X_t = (X_t^{(1)}, \dots, X_t^{(n)})$ follows a multivariate NVM mixture process. Using the defining relationship $E[S_t^{(k)}] = S_0^{(k)} \exp(\mu_k t)$ at unit time, one sees that $\omega_k = C_{X^{(k)}}(1) < \infty$, $k = 1, \dots, n$, where one assumes that the cgf of

$X^{(k)} = X_1^{(k)}$ exists over some open interval, which contains one. Suppose that the *multivariate NVM deflator of dimension n* has the same form as the price processes in (3.1). For some parameter α and vector $\beta = (\beta_1, \dots, \beta_n)$ (both to be determined), one sets for it (an Esscher transformed measure)

$$D_t = \exp(-\alpha t - \beta^T X_t), \quad t > 0. \tag{3.2}$$

A simple cgf calculation shows that the state-price deflator martingale conditions

$$E[D_t] = e^{-rt}, \quad E[D_t S_t^{(k)}] = S_0^{(k)}, \quad t > 0, \tag{3.3}$$

are equivalent with the system of $n + 1$ equations in the $2n + 1$ unknowns $\alpha, \beta_k, \omega_k$ (use that X_t is a Lévy process, hence $C_{X_t}(u) = t \cdot C_X(u)$):

$$r - \alpha + C_X(-\beta) = 0, \quad \mu_k - \omega_k - \alpha + C_X(\beta^{(k)}) = 0,$$

$$\beta^{(k)} = (\beta_1^{(k)}, \dots, \beta_n^{(k)}), \quad \beta_j^{(k)} = \delta_j^k - \beta_j, \quad j, k = 1, \dots, n. \tag{3.4}$$

Inserting the first equation into the second ones yields the necessary relationships

$$\mu_k - r - \omega_k + C_X(\beta^{(k)}) - C_X(-\beta) = 0, \quad k = 1, \dots, n. \tag{3.5}$$

Since the system (3.4) has n degrees of freedom, the unknown ω_k can be chosen arbitrarily, say

$$\omega_k = \mu_k - r = C_{X^{(k)}}(1), \quad k = 1, \dots, n, \tag{3.6}$$

which is interpreted as the (time-independent) *NVM market price of the k-th risky asset*. With the made restriction on the cgf, this value is always finite. Inserted into (3.5) shows that the parameter vector β is determined by the equations

$$C_X(\beta^{(k)}) = C_X(-\beta), \quad k = 1, \dots, n. \tag{3.7}$$

We are ready to show the following unified NVM deflator representation:

Theorem 3.1 (Multivariate NVM deflator)**.** *Given is a risk-free asset with constant return r and* $n \geq 1$ *risky assets with real-world prices* (3.1), *where one assumes that the cgf of* $X^{(k)} = X_1^{(k)}$ *exists over some open interval, which contains one. Then, the multivariate NVM deflator of the exponential NVM mixture process is determined by*

$$D_t = \exp(-\alpha t - \sum_{k=1}^{n} \beta_k X_t^{(k)}), \quad \alpha = r + C_X(-\beta),$$

$$\beta_k \tau_k^2 = \theta_k + (\frac{1}{2} + \gamma_k)\tau_k^2, \quad \gamma_k = \sum_{j \neq k} \rho_{kj} \frac{\tau_j}{\tau_k}, \quad k = 1, \ldots, n. \tag{3.8}$$

Moreover, in the univariate case $n = 1$, *one has* $\gamma_1 = 0$.

Proof. The first equation in (3.4) yields α. Since X_t, G_t are Lévy processes, one has $C_{X_t}(u) = t \cdot C_X(u)$, $C_{G_t} = t \cdot C_G(u)$, hence (2.7) is equivalent with the equation $C_X(u) = C_G(\theta^T u + \frac{1}{2} u^T \Sigma u)$. It follows that the conditions (3.7) are equivalent with the equations:

$$\theta^T \beta^{(k)} + \frac{1}{2} \beta^{(k)^T} \Sigma \beta^{(k)} + \theta^T \beta - \frac{1}{2} \beta^T \Sigma \beta = 0, \quad k = 1, \ldots, n.$$

A straightforward calculation shows that the latter is equivalent with the stated conditions for β_k, $k = 1, \ldots, n$, where in case $n = 1$, one has $\gamma_1 = 0$ (empty sum). □

4. Pricing Geometric Basket Options for the Exponential NVM Mixture Process

In the literature, one distinguishes between two types of basket options. The *arithmetic basket option* is defined on the weighted arithmetic average of asset prices such that

$$S_t = \sum_{k=1}^{n} c_k S_t^{(k)}, \tag{4.1}$$

where the weights (c_k) can be negative, and in this situation, it includes *spread options*. The *geometric basket option* is defined on the weighted geometric average of asset prices

$$S_t = \prod_{k=1}^{n} [S_t^{(k)}]^{c_k}, \quad c_k > 0, \quad \sum_{k=1}^{n} c_k = 1. \tag{4.2}$$

Since distribution functions of weighted sums of correlated asset prices can usually not be written in explicit closed form, the pricing of arithmetic basket options is rather challenging. Different and mostly approximate methods to price them have been developed so far by many authors including Turnbull and Wakeman [51]; Milevsky and Posner [41]; Krekel et al. [34]; Carmona and Durrleman [10]; Borovka et al. [8]; Wu et al. [53]; Venkatramanan and Alexander [52]; Alexander and Venkatramanan [1]; and Brigo et al. [9]. The pricing of the geometric basket option is more straightforward.

To illustrate the usefulness in option pricing of the multivariate NVM deflator, we restrict the attention to the pricing of geometric basket options. We obtain a general analytical NVM pricing formula, which can be viewed as a generalization of the Black-Scholes formula. Moreover, explicit analytical formulas are displayed for the exponential VG and NIG price processes. In particular, a simpler alternative to the NIG closed-form formula by Wu et al. [53] is derived.

Consider a European geometric basket call option with maturity date T and exercise price K in the *multivariate NVM market* with $n \geq 1$ risky assets that follow the price process (3.1) and is subject to the NVM deflator (3.8). Its price at initial time 0 is given by

$$C = E[D_T (S_T - K)_+]. \tag{4.3}$$

A straightforward calculation, which takes into account the normalizing condition $\omega_k = \mu_k - r,\ k = 1, \dots, n,$ shows that

$$D_T S_T = S_0 \cdot \exp\{-C_X(-\beta)T + \sum_{k=1}^{n}(c_k - \beta_k)X_T^{(k)}\},$$

$$D_T K = Ke^{-rT} \cdot \exp\{-C_X(-\beta)T - \sum_{k=1}^{n}\beta_k X_T^{(k)}\}. \tag{4.4}$$

In the following, we suppose that the density function $f_{G_T}(x)$ of the mixing random variable G_T exists. Through conditioning one can rewrite (4.3) as $C = e^{-C_X(-\beta)T} \cdot \int_0^\infty C(w) f_{G_T}(w)dw$ with

$$C(w) = E[(S_0 \cdot \exp\{\sum_{k=1}^{n}(c_k - \beta_k)(\theta_k G_T + \tau_k W_{G_T}^{(k)})\}$$

$$- Ke^{-rT} \cdot \exp\{-\sum_{k=1}^{n}\beta_k(\theta_k G_T + \tau_k W_{G_T}^{(k)})\})_+ | G_T = w]. \tag{4.5}$$

Each of the two conditional correlated normally distributed sums in (4.5) is normally distributed, and their joint distribution is bivariate normal. Therefore, the distribution of the conditional random couple $(\sum_{k=1}^{n}(c_k - \beta_k)X_T^{(k)}, -\sum_{k=1}^{n}\beta_k X_T^{(k)} | G_T = w)$ is determined by the conditional means

$$E[\sum_{k=1}^{n}(c_k - \beta_k)X_T^{(k)} | G_T = w] = m_1 w, \quad m_1 = \sum_{k=1}^{n}(c_k - \beta_k)\theta_k,$$

$$E[-\sum_{k=1}^{n}\beta_k X_T^{(k)} | G_T = w] = m_2 w, \quad m_2 = -\sum_{k=1}^{n}\beta_k\theta_k, \tag{4.6}$$

the conditional variances

$$\operatorname{Var}[\sum_{k=1}^{n}(c_k-\beta_k)X_T^{(k)}|G_T=w]=s_1^2w,\quad s_1^2=\sum_{i,j=1}^{n}\rho_{ij}(c_i-\beta_i)\tau_i(c_j-\beta_j)\tau_j,$$

$$\operatorname{Var}[-\sum_{k=1}^{n}\beta_kX_T^{(k)}|G_T=w]=s_2^2w,\quad s_2^2=\sum_{i,j=1}^{n}\rho_{ij}\beta_i\tau_i\beta_j\tau_j, \tag{4.7}$$

and the conditional covariance

$$\operatorname{Cov}[\sum_{k=1}^{n}(c_k-\beta_k)X_T^{(k)},-\sum_{k=1}^{n}\beta_kX_T^{(k)}|G_T=w]=\rho s_1s_2w,$$

$$\rho s_1s_2=\sum_{i,j=1}^{n}\rho_{ij}(\beta_i-c_i)\tau_i\beta_j\tau_j. \tag{4.8}$$

Now, let $\Phi(x)$ denotes the standard normal distribution, $\overline{\Phi}(x)=1-\Phi(x)$ its survival function, and $\varphi(x)=\Phi'(x)$ its density. The bivariate standard normal density is defined and denoted by

$$\varphi_2(x,\ y;\ \rho)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2-2\rho xy+y^2\right)\right\}.$$

From (4.5) and the definitions (4.6)-(4.8), one obtains

$$C(w)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(S_0e^{m_1w+s_1\sqrt{w}x}-Ke^{-rT+m_2w+s_2\sqrt{w}y})_+\varphi_2(x,\ y;\ \rho)dxdy. \tag{4.9}$$

The expression in the bracket of (4.9) is non-negative provided $x\geq x(y)$ with

$$x(y)=\frac{\ln(K/S_0)-rT}{s_1\sqrt{w}}+\frac{(m_2-m_1)}{s_1}\sqrt{w}+\frac{s_2}{s_1}y.$$

Since $\varphi_2(x,\ y;\ \rho)=\varphi(y)\varphi((x-\rho y)/\sqrt{1-\rho^2})/\sqrt{1-\rho^2}$ a separation of the double integral yields $C(w)=\int_{-\infty}^{\infty}J(y,\ w)\varphi(y)dy$ with the inner integral

$$J(y,w) = (1/\sqrt{1-\rho^2}) \cdot \int_{x(y)}^{\infty} \{S_0 e^{m_1 w + s_1 \sqrt{w} x} - K e^{-rT + m_2 w + s_2 \sqrt{w} y}\}$$

$$\times\ \varphi((x-\rho y)/\sqrt{1-\rho^2})dx. \tag{4.10}$$

A straightforward application of Lemma A1 in the Appendix yields

$$J(y,w) = S_0 e^{m_1 w + \rho s_1 \sqrt{w} y + \frac{1}{2}(1-\rho^2)s_1^2 w} \Phi\left(\frac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}\, s_1 \sqrt{w}\right)$$

$$- K e^{-rT + m_2 w + s_2 \sqrt{w} y} \Phi\left(\frac{\rho y - x(y)}{\sqrt{1-\rho^2}}\right).$$

To simplify notation, rewrite the arguments within the normal distribution functions as

$$\frac{\rho y - x(y)}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2}\, s_1 \sqrt{w} = a + cy, \quad \frac{\rho y - x(y)}{\sqrt{1-\rho^2}} = b + cy, \text{ with}$$

$$a = \frac{rT - \ln(K/S_0) + (m_1 - m_2 + (1-\rho^2)s_1^2)w}{s_1\sqrt{w}\sqrt{1-\rho^2}},$$

$$b = \frac{rT - \ln(K/S_0) + (m_1 - m_2)w}{s_1\sqrt{w}\sqrt{1-\rho^2}}, \quad c = \frac{\rho s_1 - s_2}{s_1\sqrt{1-\rho^2}}.$$

Furthermore, one has

$$e^{\rho s_1 \sqrt{w} y}\varphi(y) = e^{\frac{1}{2}\rho^2 s_1^2 w}\varphi(y - \rho s_1 \sqrt{w}), \quad e^{s_2\sqrt{w} y}\varphi(y) = e^{\frac{1}{2}s_2^2 w}\varphi(y - s_2\sqrt{w}).$$

Now, using twice the Lemma A2 of the Appendix, one obtains

$$C(w) = S_0 e^{(m_1 + \frac{1}{2}s_1^2)w}\Phi\left(\frac{a + c\rho s_1\sqrt{w}}{\sqrt{1+c^2}}\right) - K e^{-rT + (m_2 + \frac{1}{2}s_2^2)w}\Phi\left(\frac{b + cs_2\sqrt{w}}{\sqrt{1+c^2}}\right).$$

Based on the above expressions for the coefficients a, b, and c, one obtains further

$$C(w) = S_0 e^{(m_1 + \frac{1}{2}s_1^2)w} \Phi\left(\frac{rT - \ln(K / S_0)}{\Omega\sqrt{w}} + \frac{m_1 - m_2 + \Omega_1^2}{\Omega} \sqrt{w} \right)$$

$$- Ke^{-rT + (m_2 + \frac{1}{2}s_2^2)w} \Phi\left(\frac{rT - \ln(K / S_0)}{\Omega\sqrt{w}} + \frac{m_1 - m_2 - \Omega_2^2}{\Omega} \sqrt{w} \right),$$

$$\Omega_k^2 = s_k^2 - \rho s_1 s_2, \quad k = 1, 2, \quad \Omega = \sqrt{\Omega_1^2 + \Omega_2^2}. \tag{4.11}$$

Summarizing, we have shown the following main result:

Theorem 4.1 (Geometric basket call option formula in the multivariate NVM market)**.** *Given is the multivariate exponential NVM mixture process* (3.1) *subject to the NVM deflator* (3.8). *Then*, *in the above notations*, *one has*

$$C = E[D_T(S_T - K)_+] = e^{-C_X(-\beta)T} \cdot \{S_0 \cdot \Psi_{G_T}(a_1, b_1, d) - Ke^{-rT} \cdot \Psi_{G_T}(a_2, b_2, d)\}, \tag{4.12}$$

$$\Psi_{G_T}(a, b, d) = \int_0^\infty e^{aw} \Phi(b\sqrt{w} + \tfrac{d}{\sqrt{w}}) f_{G_T}(w) dw,$$

$$a_k = m_k + \frac{1}{2} s_k^2, \quad b_k = \frac{m_1 - m_2 + (-1)^{k-1}\Omega_k^2}{\Omega},$$

$$k = 1, 2, \quad d = \frac{rT - \ln(K / S_0)}{\Omega}. \tag{4.13}$$

A particular instance, for which the formula (4.12) simplifies considerably, is the special case $d = 0$, that is the exercise price is set equal to $K = S_0 e^{rT}$. In this situation, (4.12) rewrites as

$$C = E[D_T(S_T - S_0 e^{-rT})_+] = e^{-C_X(-\beta)T} S_0 \cdot \int_0^\infty \{e^{a_1 w} \Phi(b_1 \sqrt{w})$$

$$- e^{a_2 w} \Phi(b_2 \sqrt{w})\} f_{G_T}(w) dw. \tag{4.14}$$

At this point, an important connection with the standard no-arbitrage framework of mathematical finance must be mentioned (e.g., Wüthrich et al. [54], Subsection 2.5; and Wüthrich and Merz [55], Chapter 2). By the fundamental theorem of asset pricing, the assumption of no-arbitrage (weak form of efficient market hypothesis) is equivalent with the existence of an equivalent martingale measure for deflated price processes. In complete markets, the equivalent martingale measure is unique, perfect replication of contingent claims holds, and straightforward pricing applies. In incomplete markets, an economic model is required to decide upon which equivalent martingale measure is appropriate. Now, let P denotes the real-world measure and P^* denotes an equivalent martingale measure. Then, one can either work under P, where the price processes are deflated with a state-price deflator. Alternatively, one can work under P^* by discounting the price processes with the bank account numeraire. Working with financial instruments only, one often works under P^*. But, if additionally insurance liabilities are considered, one works under P (see Wüthrich et al. [54], Remark 2.13). A recent non-trivial example is pricing of the "guaranteed maximum inflation death benefit (GMIDB) option" (Hürlimann [22]; Equation (5.4) in Hürlimann [23]). Theorem 4.1 demonstrates the practicability of the state-price deflator approach for exponential NVM price processes as applied to the European geometric basket call option. In cases, where the density function of the mixing random variable is known, the formulas (4.12)-(4.14) yield a closed-form analytical pricing system for the geometric basket option. The conditions under which the multivariate NVM market is complete and arbitrage-free, that is there exists a unique equivalent martingale measure and prices are uniquely defined (whether under P^* or under P with state-price deflator), remain to be found. This is a non-trivial problem that has been tackled so far solely for the multivariate Black-Scholes model (see Dhaene et al. [15]).

Example 4.1. Multivariate exponential VG asset pricing model.

The subordinator $G_T \sim \Gamma(\gamma T, \gamma)$, $\gamma = \nu^{-1}$ is gamma distributed with density

$$f_{G_T}(x) = \gamma \cdot (\gamma x)^{\gamma T - 1} \cdot e^{-\gamma x} / \Gamma(\gamma). \tag{4.15}$$

Making the change of variable $z = (1 - a)w$, the Ψ-function in (4.13) is of the form stated in Kotz et al. [33], p. 296 (European risk-neutral call option price for an exponential VG price process), which is the original closed form formula by Madan et al. [37], Theorem 2 and Appendix (see also Madan [36], Subsection 6.3.1, Equation (14)). The advantage of the alternative pricing formula (4.12) is the simple parameter structure (4.13). Also, the special case (4.14) simplifies considerably. A similar analytical pricing system for the Margrabe option has been considered previously in Hürlimann [24, 25]. In particular, for an Erlang subordinator with integer parameter $\gamma = \nu^{-1}$ formulas of the type (4.14) reduce to finite sum expressions (see Hürlimann [24], Appendix 2). As a new result, we derive a closed-form expression for the Ψ-function as a sum of an incomplete beta function and an integrated Macdonald function. It is much simpler than the original formula in terms of the Macdonald function (modified Bessel function of the second kind) and the degenerate hyper-geometric function of two variables.

In the following, let $Ga(\gamma) \sim \Gamma(\gamma, 1)$ be a standardized gamma random variable with density $f_{Ga(\gamma)}(x) = x^{\gamma - 1} \cdot e^{-x} / \Gamma(\gamma)$, $Be(\frac{1}{2}, \gamma)$ be a beta with distribution function

$$F_{Be(\frac{1}{2}, \gamma)}(x) = \frac{1}{B(\frac{1}{2}, \gamma)} \cdot \int_0^x t^{-\frac{1}{2}} (1 - t)^{\gamma - 1} dt, \tag{4.16}$$

and $VG(\gamma, \alpha, \beta)$ a variance-gamma with density (e.g., Hürlimann [29], Equation (A4.21))

$$f_{VG(\gamma,\alpha,\beta)}(x) = \frac{(\alpha\beta)^{\gamma}}{\sqrt{\pi}\Gamma(\gamma)}\left(\frac{|x|}{\alpha+\beta}\right)^{\gamma-\frac{1}{2}} \cdot \exp\left(-\frac{1}{2}(\alpha-\beta)x\right) \cdot K_{\gamma-\frac{1}{2}}\left(\frac{1}{2}(\alpha+\beta)|x|\right),$$

$$x \neq 0. \tag{4.17}$$

Theorem 4.2 (Closed-form VG Ψ-function representation)**.** *The normal gamma mixed integral* $\Psi(a, b, \gamma) = \int_0^\infty \Phi(a\sqrt{x^{-1}} + b\sqrt{x}) f_{Ga(\lambda)}(x)dx$ *satisfies the following probabilistic representation*:

$$\Psi(a, b, \gamma) = \frac{1}{2}\{1 + \operatorname{sgn}(b) \cdot F_{Be(\frac{1}{2},\gamma)}\left(\frac{\frac{1}{2}b^2}{1+\frac{1}{2}b^2}\right)\}$$

$$+ \operatorname{sgn}(a) \cdot \int_0^{|a|} f_{VG(\gamma, \sqrt{2+b^2}+\operatorname{sgn}(a)b, \sqrt{2+b^2}-\operatorname{sgn}(a)b)}(x)dx. \tag{4.18}$$

Proof. This is shown in the Appendix. □

In particular, the VG formula (4.14) can be rewritten in terms of the incomplete beta function.

Example 4.2. Multivariate exponential NIG asset pricing model.

The subordinator $G_T \sim IG(\sqrt{\frac{T}{\nu}}, \sqrt{\frac{1}{T\nu}})$ is inverse Gaussian distributed with density

$$f_{G_T}(x) = \sqrt{\frac{T}{2\pi\nu}} x^{-\frac{3}{2}} \cdot \exp\{-\frac{(x-T)^2}{2\nu T x}\}. \tag{4.19}$$

Pricing of the geometric basket option under an equivalent martingale measure has been previously considered in Wu et al. [53]. However, their fast Fourier transform pricing formula is more complex than the closed-form expression (4.22) below.

In the following, let $SNRIG(\alpha, \delta)$ be a symmetric NVM mixture random variable with reciprocal inverse Gaussian mixing density

$$f_{SNRIG(\alpha,\delta)}(x) = \frac{1}{\pi} \cdot \alpha \cdot e^{\alpha\delta} \cdot K_0(\alpha\sqrt{\delta^2 + x^2}). \tag{4.20}$$

It is obtained from the generalized hyperbolic distribution as $SNRIG(\alpha, \delta) = GH(\frac{1}{2}, \alpha, 0, \delta)$. The density of the normal inverse Gaussian $NIG(\alpha, \beta, \delta) = GH(-\frac{1}{2}, \alpha, \beta, \delta)$ is denoted by

$$f_{NIG(\alpha,\beta,\delta)}(x) = \frac{1}{\pi} \cdot \alpha\delta \cdot e^{\delta\sqrt{\alpha^2-\beta^2}+\beta x} \cdot \frac{K_1(\alpha\sqrt{\delta^2 + x^2})}{\sqrt{\delta^2 + x^2}}. \tag{4.21}$$

Theorem 4.3 (Closed-form NIG Ψ-function representation)**.** *The normal gamma mixed integral* $\Psi(a, b, \gamma) = \int_0^\infty \Phi(a\sqrt{x^{-1}} + b\sqrt{x})f_{G_T}(x)dx$ *satisfies the following probabilistic representation*:

$$\Psi(a, b, \gamma) = \frac{1}{2} + \operatorname{sgn}(b) \cdot F_{SNRIG(\sqrt{\frac{T}{\nu}}, \sqrt{\frac{1}{\nu T}})}(|b|)$$

$$+ \operatorname{sgn}(a) \cdot F_{NIG(\sqrt{\frac{1}{\nu T}+b^2}, -\operatorname{sgn}(a)b, \sqrt{\frac{T}{\nu}})}(|a|). \tag{4.22}$$

An extension of the probabilistic representations (4.18) and (4.22) to arbitrary generalized inverse Gaussian mixing densities is found in Hürlimann [30].

Example 4.3. Multivariate exponential NTS asset pricing model.

The classical tempered stable subordinator has not a tractable density function but a relatively simple characteristic function. In this situation, one approximates the integrals (4.13)-(4.14) by use of the fast Fourier transform (FFT) of f_{G_T} (e.g., Hürlimann [27], Appendix 1).

5. A Simple Multivariate Subordinated Asset Price Model

The special choice $\beta_k = 0,\ k = 1, \ldots, n,$ of the multivariate NVM deflator (3.8) implies that $C_X(-\beta) = 0,\ \alpha = r,\ \theta_k = -(\frac{1}{2} + \gamma_k)\tau_k^2 = -\frac{1}{2}\tau_k \cdot \sum_{j=1}^{n} \rho_{kj}\tau_j,\ k = 1, \ldots, n.$ It follows that the NVM deflator degenerates to the risk-free discount factor $D_t = e^{-rt}$. In this *multivariate subordinated market*, the risky assets follow the price process (insert (3.6) into (3.1))

$$S_t^{(k)} = S_0^{(k)} \exp\left(rt - (\frac{1}{2} + \gamma_k) \cdot \tau_k^2 \cdot G_t + \tau_k \cdot W_{G_t}^{(k)}\right), \quad k = 1, \ldots, n, \tag{5.1}$$

where the $W_t^{(k)}$'s are *correlated* standard Wiener processes such that $E[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij}$ and G_t is an independent subordinator. The pricing of the European geometric basket call option with maturity date T and exercise price K simplifies somewhat.

Theorem 5.1 (Geometric basket call option formula in the multivariate subordinated market)**.** *Given is the multivariate subordinated asset price model* (5.1) *subject to the risk-free discount factor* $D_t = e^{-rt}$. *Then one has*

$$C = E[D_T(S_T - K)_+] = S_0 \cdot \Psi_{G_T}^{-}(m, s, d) - Ke^{-rT} \cdot \Psi_{G_T}^{+}(m, s, d), \tag{5.2}$$

$$\Psi_{G_T}^{-}(m, s, d) = \int_0^\infty e^{(m+\frac{1}{2}s^2)w} \Phi\left(\frac{d + (m + s^2)w}{s\sqrt{w}}\right) f_{G_T}(w)dw,$$

$$\Psi_{G_T}^{+}(m, s, d) = \int_0^\infty \Phi\left(\frac{d + mw}{s\sqrt{w}}\right) f_{G_T}(w)dw, \tag{5.3}$$

$$m = -\frac{1}{2}\sum_{k=1}^{n} c_k \tau_k \sum_{j=1}^{n} \rho_{kj}\tau_j, \quad s^2 = \sum_{i,j=1}^{n} \rho_{ij} c_i \tau_i c_j \tau_j, \quad d = \ln(S_0 / K) + rT.$$

Proof. This result is a special instance of Theorem 4.1 and can be similarly derived in a simpler way (one-dimensional integration only). □

The specialization to a geometric basket with a single risky asset is instructive.

Corollary 5.1 (European state-price deflated call option price for a single subordinated asset)**.** *Given is the multivariate subordinated asset price model* (5.1) *subject to the risk-free state-price deflator* $D_t = e^{-rt}$. *Then one has*

$$C = E[D_T(S_T^{(1)} - K)_+] = S_0^{(1)} \cdot \Psi_{G_T}^{-}(\tau_1, \{\rho_{1j}\tau_j\}, d) - Ke^{-rT} \cdot \Psi_{G_T}^{+}(\tau_1, \{\rho_{1j}\tau_j\}, d), \tag{5.4}$$

$$\Psi_{G_T}^{-}(\tau_1, \{\rho_{1j}\tau_j\}, d) = \int_0^\infty e^{-\frac{1}{2}\tau_1 \cdot (\sum_{j\neq 1}\rho_{1j}\tau_j)w}\, \Phi\left(\frac{d - \frac{1}{2}\tau_1 \cdot (\sum_{j\neq 1}\rho_{1j}\tau_j)w + \frac{1}{2}\tau_1^2 w}{\tau_1\sqrt{w}}\right) f_{G_T}(w)dw,$$

$$\Psi_{G_T}^{+}(\tau_1, \{\rho_{1j}\tau_j\}, d) = \int_0^\infty \Phi\left(\frac{d - \frac{1}{2}\tau_1 \cdot (\sum_{j\neq 1}\rho_{1j}\tau_j)w - \frac{1}{2}\tau_1^2 w}{\tau_1\sqrt{w}}\right) f_{G_T}(w)dw,$$

$$d = \ln(S_0 / K) + rT. \tag{5.5}$$

Proof. Set $c_1 = 1, c_2 = \ldots = c_n = 0$ to see that $m = -\frac{1}{2}\tau_1 \cdot (\sum_{j\neq 1}\rho_{1j}\tau_j)$ $-\frac{1}{2}\tau_1^2$, $s^2 = \tau_1^2$, and the result follows immediately through insertion into the formula (5.2). □

Corollary 5.1 is related to some remarkable previous results. Suppose $\rho_{1j} = 0$, $j = 2, \ldots, n$, that is the single asset is uncorrelated with the other risky assets, then $\sum_{j \neq 1} \rho_{1j} \tau_j = 0.$ Then, one recovers the option formula by Hurst et al. [20] for the univariate subordinated asset price model (see also Rachev and Mittnik [47]; and Rachev et al. [46], Subsection 7.6 with correction of the misprint in the stock price model, however). If $G_T = T$ with probability one (Black-Scholes-Merton model), then (5.4) yields a new *extended Black-Scholes formula* that takes into account the correlation structure of the market. Besides the multivariate exponential VG, NIG, and NTS processes, the proposed multivariate subordinator market contains a rich class of feasible models. If G_t is the stable process proposed by Mandelbrot [39, 40] and Fama [18], one has a logstable model. If G_t is the CIR process by Cox et al. [13], one obtains a multivariate version of the model by Heston [19] (see Rachev et al. [46], Subsection 7.6). The CGMY and Meixner specifications by Carr et al. [11], resp., Schoutens and Teugels [49]; and Pitman and Yor [45], are also subordinated models, as shown by Madan and Yor [38]. Some differences with the approach by Hurst et al. [20] can be noted. While these authors work under an equivalent martingale measure P^* in the univariate case $n = 1$ only, we work in the multivariate case under P using a state-price deflator. Hurst et al. argue in Section 5 that an equivalent martingale measure is not expected to exist unless one assumes that $\mu_1 = r.$ In this case, their (equivalent martingale measure) market price of risk reads (Equation (5.11))

$$\omega_1^* = -(\theta_1 + \frac{1}{2}\tau_1^2)/\tau_1. \tag{5.6}$$

The assumption $\mu_1 = r$ corresponds to the choice $\mu_1 - \omega_1 = r$ in the deflator approach. By virtue of (3.6), the (state-price deflator) market price of risk equals

$$\omega_1 = \mu_1 - r = C_{X^{(1)}}(1) = C_G(\theta_1 + \frac{1}{2}\tau_1^2) = C_G(-\frac{1}{2}\tau_1^2 \cdot (\sum_{j \neq 1} \rho_{1j}\tau_j)). \tag{5.7}$$

Under the made assumptions, both market prices of risk are equal and vanish, if and only if one has $\sum_{j \neq 1} \rho_{1j} \tau_j = 0$, and in this situation, the formula (5.4) coincides with the Hurst-Platen-Rachev formula. Our multivariate subordinated asset price model is a genuine extension because it covers cases with non-vanishing market prices of risk for which $\mu_1 \neq r$. For this, the condition $\sum_{j \neq 1} \rho_{1j} \tau_j \neq 0$ in financial markets is rather the rule than the exception. Therefore, the new approach is of primordial importance. Since the equivalent martingale approach and the state-price deflator approach are equivalent (comments after Theorem 4.1), there is no need to display the measure P^* in the multivariate case.

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■

Appendix: Integral Identities of Normal Type and Probabilistic Ψ-Function Representations

The crucial identities used in the derivation of Theorem 4.1 are stated and proved separately.

Lemma A1. *For any real numbers* $b, c, \mu,$ *and* $\sigma > 0$, *one has the identity*

$$\sigma^{-1} \cdot \int_c^\infty e^{bx} \cdot \varphi((x-\mu)/\sigma)dx = e^{b\mu+\frac{1}{2}b^2\sigma^2} \cdot \Phi\left(\frac{\mu-c}{\sigma} + b\sigma\right). \tag{A1.1}$$

Proof. Consider first the case $\mu = 0, \sigma = 1$. From the relation $e^{bx}\varphi(x) = e^{\frac{1}{2}b^2}\varphi(x-b)$, one gets $\int_c^\infty e^{bx} \cdot \varphi(x)dx = e^{\frac{1}{2}b^2} \cdot \int_{c-b}^\infty \varphi(t)dt = e^{\frac{1}{2}b^2} \cdot \Phi(b-c)$. Using this, one obtains by a change of variables $\sigma^{-1} \cdot \int_c^\infty e^{bx} \cdot \varphi((x-\mu)/\sigma)dx = \int_{(c-\mu)/\sigma}^\infty e^{b\mu+b\sigma t} \cdot \varphi(t)dt = e^{b\mu+\frac{1}{2}b^2\sigma^2} \cdot \Phi\left(\frac{\mu-c}{\sigma} + b\sigma\right)$. □

Lemma A2. *For any real numbers* $a, b, \mu,$ *and* $\sigma > 0$, *one has the identity*

$$\sigma^{-1} \cdot \int_{-\infty}^\infty \Phi(a+bx)\varphi((x-\mu)/\sigma)dx = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right). \tag{A1.2}$$

Proof. Consider the functions $F(z) = \int_{-\infty}^\infty \Phi(z+x)\varphi(x)dx$, $G_a(z) = \int_{-\infty}^\infty \Phi(a+zx)\varphi(x)dx$. One notes that $F(0) = \int_{-\infty}^\infty \Phi(x)\varphi(x)dx = \frac{1}{2}$ and $F'(z) = \int_{-\infty}^\infty \varphi(z+x)\varphi(x)dx = \frac{\sqrt{2}}{2}\varphi(\frac{z}{\sqrt{2}})$, from which it follows that $F(a) = F(0) + \int_0^a F'(z)dz = \Phi(\frac{a}{\sqrt{2}})$. On the other hand, one has $G_a(1) = F(a) = \Phi(\frac{a}{\sqrt{2}})$ and $G_a'(z) = z \cdot \int_{-\infty}^\infty \varphi(a+zx)\varphi(x)dx = \frac{z}{(1+z^2)^{3/2}}$

$\varphi(\frac{a}{\sqrt{1+z^2}})$, hence $G_a(b) = G_a(1) + \int_1^b G'(z)dz = \Phi(\frac{a}{\sqrt{1+b^2}})$. It follows that $\sigma^{-1} \cdot \int_{-\infty}^{\infty} \Phi(a+bx)\varphi((x-\mu)/\sigma)dx = G_{a+b\mu}(b\sigma) = \Phi\left(\frac{a+b\mu}{\sqrt{1+b^2\sigma^2}}\right)$. □

Proof of Theorem 4.2. A repeated application of the method used in Lemma A2 will do. In the special case $a = 0,\ b \geq 0,$ set $J(z) = \Psi(0, z, \gamma)$ $= \int_0^\infty \Phi(z\sqrt{x}) f_{Ga(\lambda)}(x)dx,\ z \geq 0.$ One has $J(0) = \frac{1}{2}$, $J(b) = J(0) + \int_0^b J'(z)dz.$ A calculation shows that

$$J'(z) = \int_0^\infty \sqrt{x}\varphi(z\sqrt{x}) f_{Ga(\lambda)}(x)dx = \frac{1}{\sqrt{2\pi}\Gamma(\gamma)} \cdot \int_0^\infty x^{(\gamma+\frac{1}{2})-1} e^{-(1+\frac{1}{2}z^2)x} dx$$

$$= \frac{1}{\sqrt{2}B(\frac{1}{2},\gamma)} \cdot \frac{1}{(1+\frac{1}{2}z^2)^{\gamma+\frac{1}{2}}},$$

where the last expression follows by noting that the integrand is related to a gamma density. With the change of variable $z = \sqrt{2}u,$ one sees that

$$J(b) = \frac{1}{2} + \frac{1}{B(\frac{1}{2},\gamma)} \cdot \int_0^{b/\sqrt{2}} \frac{du}{(1+u^2)^{\gamma+\frac{1}{2}}}.$$

A further change of variables $u = \sqrt{t/(1-t)}$ shows that

$$\int_0^x \frac{du}{(1+u^2)^{\gamma+\frac{1}{2}}} = \frac{1}{2} \cdot \int_0^{x^2/(1+x^2)} t^{-\frac{1}{2}}(1-t)^{\gamma-1} dt.$$

With (4.16), this implies (4.18) for $a = 0,\ b \geq 0.$ The formula for $a = 0,$ $b < 0$ follows by noting that $\Phi(b\sqrt{x}) = \Phi(-|b|\sqrt{x}) = 1 - \Phi(|b|\sqrt{x}),$ hence

$J(b) = 1 - J(|b|)$. In general, if $a \geq 0$, set $I(z) = \int_0^\infty \Phi(z\sqrt{x^{-1}} + b\sqrt{x}) f_{Ga(\lambda)}(x)dx$, $z \geq 0$. One has $I(0) = J(b)$, $I(a) = J(b) + \int_0^a I'(z)dz$, and

$$I'(z) = \int_0^\infty \sqrt{x^{-1}}\varphi(z\sqrt{x^{-1}} + b\sqrt{x}) f_{Ga(\lambda)}(x)dx$$

$$= \frac{e^{-bz}}{\sqrt{2\pi}\Gamma(\gamma)} \cdot \int_0^\infty x^{(\gamma - \frac{1}{2}) - 1} e^{-(1 + \frac{1}{2}b^2)x - \frac{1}{2}z^2 x^{-1}} dx.$$

The integrand is related to a generalized inverse Gaussian density of the form

$$f_{GIG(\alpha,\, \beta,\, p)}(x) = \frac{(\alpha / \beta)^{\frac{1}{2}p}}{2K_p(\sqrt{\alpha\beta})} x^{p-1} e^{-\frac{1}{2}(\alpha x + \beta x^{-1})},$$

$$p = \gamma - \frac{1}{2}, \quad \alpha = 2(1 + \frac{1}{2}b^2), \quad \beta = z^2.$$

Set $c = \sqrt{2(1 + \frac{1}{2}b^2)}$ to see that

$$I'(z) = \sqrt{\frac{2}{\pi}} \cdot \frac{z^{\gamma - \frac{1}{2}} e^{-bz} K_{\gamma - \frac{1}{2}}(cz)}{\Gamma(\gamma) c^{\gamma - \frac{1}{2}}}.$$

A comparison with (4.17) shows that $I'(z) = f_{VG(\gamma,\, \alpha,\, \beta)}(z)$, $\alpha + \beta = 2c$, $\alpha - \beta = 2b$. The formula (4.18) for $a \geq 0$ follows. If $a < 0$, one notes that

$$\Phi(a\sqrt{x^{-1}} + b\sqrt{x}) = \Phi(-|a|\sqrt{x^{-1}} + b\sqrt{x}) = 1 - \Phi(|a|\sqrt{x^{-1}} - b\sqrt{x}),$$

hence $\Psi(a, b, \gamma) = 1 - \Psi(|a|, -b, \gamma)$. $\square$

Proof of Theorem 4.3. If $a = 0,\ b \geq 0,$ set $J(z) = \Psi(0,\ z,\ \gamma) = \int_0^\infty \Phi(z\sqrt{x})$ $f_{G_T}(x)dx,\ z \geq 0.$ One has $J(b) = \frac{1}{2} + \int_0^b J'(z)dz.$ A calculation shows that

$$J'(z) = \int_0^\infty \sqrt{x}\varphi(z\sqrt{x})f_{G_T}(x)dx = \frac{1}{2\pi}\cdot\sqrt{\frac{T}{\nu}e^{\frac{1}{\nu}}}\cdot\int_0^\infty \frac{1}{x}\exp(-\frac{1}{2}\{(\frac{1}{\nu T}+z^2)x+\frac{T}{\nu}x^{-1}\})dx.$$

The integrand is related to a harmonic law $H(\alpha,\ \beta) = GIG(\alpha,\ \beta,\ 0)$ with density

$$f_{H(\alpha,\beta)}(x) = \frac{1}{2K_0(\sqrt{\alpha\beta})}\frac{1}{x}e^{-\frac{1}{2}(\alpha x+\beta x^{-1})},\qquad \alpha = \frac{1}{\nu T}+z^2,\qquad \beta = \frac{T}{\nu},$$

such that

$$J'(z) = \frac{1}{\pi}\cdot\sqrt{\frac{T}{\nu}e^{\frac{1}{\nu}}}\cdot K_0(\sqrt{\frac{1}{\nu^2}+\frac{T}{\nu}z^2}).$$

Comparing with (4.20) shows (4.22) for $a = 0,\ b \geq 0.$ The formula for $a = 0,\ b < 0$ follows from the relation $J(b) = 1 - J(|b|).$ If $a \geq 0,$ then one has $I(a) = J(b) + \int_0^a I'(z)dz$ with $I(z) = \int_0^\infty \Phi(z\sqrt{x^{-1}} + b\sqrt{x})f_{G_T}(x)dx,\ z \geq 0.$ Through calculation, one obtains

$$\begin{aligned} I'(z) &= \int_0^\infty \sqrt{x^{-1}}\varphi(z\sqrt{x^{-1}}+b\sqrt{x})f_{G_T}(x)dx \\ &= \frac{1}{2\pi}\cdot\sqrt{\frac{T}{\nu}e^{\frac{1}{\nu}-bz}}\cdot\int_0^\infty \frac{1}{x^2}\exp(-\frac{1}{2}\{(\frac{1}{\nu T}+b^2)x+(\frac{T}{\nu}+z^2)x^{-1}\})dx, \end{aligned}$$

whose integrand is related to the density of a generalized inverse Gaussian $GIG(\alpha, \beta, p)$ with parameters $p = -1$, $\alpha = \frac{1}{\nu T} + b^2$, $\beta = \frac{T}{\nu} + z^2$. It follows that

$$I'(z) = \frac{1}{\pi} \cdot \sqrt{\frac{T}{\nu}} \sqrt{\frac{1}{\nu T} + b^2} \cdot e^{\frac{1}{\nu} - bz} \cdot \frac{K_1\left(\sqrt{\frac{1}{\nu T} + b^2} \sqrt{\frac{T}{\nu} + z^2}\right)}{\sqrt{\frac{T}{\nu} + z^2}}.$$

Comparing with (4.21) shows that $I'(z) = f_{NIG\left(\sqrt{\frac{1}{\nu T} + b^2}, -b\sqrt{\frac{T}{\nu}}\right)}(z)$ and (4.22) for $a \geq 0$ is shown. The formula for $a < 0$ is obtained from the relationship $\Psi(a, b, \gamma) = 1 - \Psi(|a|, -b, \gamma)$. $\square$