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A CHARACTERIZATION OF ADDITIVE DERIVATIONS ON VON NEUMANN ALGEBRAS

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Abstract

Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Suppose that $\varphi : \mathcal{M} \to \mathcal{M}$ is an additive map and ξ is a scalar with $\xi \neq \pm 1$. Then φ satisfies $\varphi(AB - \xi BA) = \varphi(A)B - \xi B\varphi(A) + A\varphi(B) - \xi\varphi(B)A$ for any $A, B \in \mathcal{M}$ with $AB + BA = 0$, if and only if $\varphi(I)$ is in the center of $\mathcal{M}$, $\varphi(\xi I) = 2\xi\varphi(I) + \varphi(I)$ and $\varphi(AB) = \varphi(A)B + A\varphi(B) - AB\varphi(I)$ for all $A, B \in \mathcal{M}$. Particularly, if φ is linear, then φ is a derivation.

1. Introduction

Let $\mathcal{R}$ be an associative ring (or an algebra over a field $\mathbb{F}$). Recall that an additive map δ on $\mathcal{R}$ is called an additive derivation, if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{R}$; is called an additive Jordan derivation, if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{R}$

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(equivalently, $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathcal{R}$, if the characteristic of $\mathcal{R}$ is not 2); is called a Lie derivation, if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in \mathcal{R}$, where $[A, B] = AB - BA$ is the Lie product of A and B. The structures of derivations, Jordan derivations, and Lie derivations had been studied intensively for many researchers (for example, see [1, 4, 6, 11] and the references therein).

For a scalar $\xi \in \mathbb{F}$ and for $A, B \in \mathcal{R}$, we say that A commutes with B up to a factor ξ, if $AB = \xi BA$. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups ([3, 8]). Motivated by this, a binary operation $[A, B]_\xi = AB - \xi BA$, called ξ-Lie product of A and B, was introduced in [13]. An additive map $L : \mathcal{R} \to \mathcal{R}$ is called an additive ξ-Lie derivation, if $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for all $A, B \in \mathcal{R}$. This conception unifies the above three notions. It is clear that a ξ-Lie derivation is a derivation, Lie derivation and Jordan derivation if $\xi = 0, 1, -1$, respectively. The structure of ξ-Lie derivations on various operator algebras was also discussed by several authors (for example, see [9, 13, 17, 18]).

Recently, the question of under what conditions that an additive map becomes a derivation attracted much attention of many researchers. For example, Qi and Hou [14] characterized the linear maps L between $\mathcal{J}$-subspace lattice algebras that satisfies $L([A, B]) = [L(A), B] + [A, L(B)]$ for any A, B with $[A, B] = 0$. Let X be a Banach space with $\dim X \geq 3$ and $\mathcal{B}(X)$ be the algebra of all bounded linear operators acting on X. Lu and Ji in [10] showed that, if $\delta : \mathcal{B}(X) \to \mathcal{B}(X)$ is a linear map satisfying $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for any $A, B \in \mathcal{B}(X)$ with $AB = 0$ (resp., $AB = P$, where P is a fixed nontrivial idempotent), then $\delta = \tau + \nu$, where τ is a derivation of $\mathcal{B}(X)$ and $\nu : \mathcal{B}(X) \to \mathbb{C}I$ is a

linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp., $AB = P$). Let $\mathcal{A}$ be a unital prime algebra over a field $\mathbb{F}$ and $\xi \in \mathbb{F}$. In [16], Qi et al. considered the additive maps L on $\mathcal{A}$ satisfying the condition $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for all $A, B \in \mathcal{A}$ with $[A, B]_\xi = 0$. For other related results, we refer to [2, 5, 7, 10, 14, 15] and the references therein.

The purpose of the present paper is to give a new characterization of additive derivations on von Neumann algebras without central summands of type I_1. Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1 and ξ is a scalar with $\xi \neq \pm 1$. Denote by $\mathcal{Z}(\mathcal{M})$ the center of $\mathcal{M}$. We prove that an additive map $\varphi : \mathcal{M} \to \mathcal{M}$ satisfies $\varphi(AB - \xi BA) = \varphi(A)B - \xi B\varphi(A) + A\varphi(B) - \xi\varphi(B)A$ for any $A, B \in \mathcal{M}$ with $AB + BA = 0$, if and only if $\varphi(I) \in \mathcal{Z}(\mathcal{M})$, $\varphi(\xi I) = (2\xi + 1)\varphi(I)$ and $\varphi(AB) = \varphi(A)B + A\varphi(B) - AB\varphi(I)$ for all $A, B \in \mathcal{M}$ (Main Theorem). Furthermore, it is shown that φ is a derivation, if φ is linear (Corollary).

2. Main Result and its Proof

In this section, we will give our main result and its proof.

Main Theorem. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Suppose that $\varphi : \mathcal{M} \to \mathcal{M}$ is an additive map and ξ is a scalar with $\xi \neq \pm 1$. Then φ satisfies $\varphi([A, B]_\xi) = [\varphi(A), B]_\xi + [A, \varphi(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $AB + BA = 0$, if and only if $\varphi(I) \in \mathcal{Z}(\mathcal{M})$, $\varphi(\xi I) = (2\xi + 1)\varphi(I)$ and $\varphi(AB) = \varphi(A)B + A\varphi(B) - AB\varphi(I)$ for all $A, B \in \mathcal{M}$, where $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$.*

Particularly, if φ is linear, then we have the following Corollary:

Corollary. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Suppose that $\varphi : \mathcal{M} \to \mathcal{M}$ is a linear map and ξ is a scalar with $\xi \neq \pm 1$. Then φ satisfies $\varphi([A, B]_\xi) = [\varphi(A), B]_\xi + [A, \varphi(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $AB + BA = 0$, if and only if φ is a derivation.*

Proof. The "if" part is clear. For the "only if" part, by Main Theorem, we have $\varphi(\xi I) = 2\xi\varphi(I) + \varphi(I)$ and $\varphi(AB) = \varphi(A)B + A\varphi(B) - AB\varphi(I)$ for all $A, B \in \mathcal{M}$. To complete the proof, we only need to check $\varphi(I) = 0$. Indeed, if φ is linear, then $\xi\varphi(I) = \varphi(\xi I) = 2\xi\varphi(I) + \varphi(I)$, and so $\varphi(I) = 0$ as $\xi \neq -1$. □

Before proving the Main Theorem, we need some notations. Let $\mathcal{M}$ be any von Neumann algebra and $A \in \mathcal{M}$. Recall that the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = 0$. If A is self-adjoint, then the core of A, denoted by $\underline{A}$, is $\sup\{S \in \mathcal{Z}(\mathcal{M}) : S = S^*, S \leq A\}$. Particularly, if $A = P$ is a projection, it is clear that $\underline{P}$ is the largest central projection $\leq P$. A projection P is called core-free if $\underline{P} = 0$. It is easy to see that $\underline{P} = 0$, if and only if $\overline{I - P} = I$. For more properties of core-free projections, see [12].

We first give two useful lemmas, which are needed to prove Main Theorem.

Lemma 1 ([12])**.** *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. Then each nonzero central projection $C \in \mathcal{M}$ is the carrier of a core-free projection in $\mathcal{M}$. Particularly, there exists a nonzero core-free projection $P \in \mathcal{M}$ with $\overline{P} = I$.*

Lemma 2 ([12]). *Let $\mathcal{M}$ be a von Neumann algebra. For projections $P, Q \in \mathcal{M}$, if $\overline{P} = \overline{Q} \neq 0$ and $P + Q = I$, then $T \in \mathcal{M}$ commutes with PXQ and QXP for all $X \in \mathcal{M}$ implies $T \in \mathcal{Z}(\mathcal{M})$.*

Proof of Main Theorem. By a simple calculation, it is easy to check that the "if" part holds.

For the "only if" part, assume that $\xi \neq \pm 1$ and $\varphi : \mathcal{M} \to \mathcal{M}$ is an additive map satisfying $\varphi([A, B]_\xi) = [\varphi(A), B]_\xi + [A, \varphi(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $AB + BA = 0$. We will prove the "only if" part by several claims.

By Lemma 1, we can find a core-free projection $P \in \mathcal{M}$ with $\overline{P} = I$. In the sequel fix such a projection P. By the definitions of core and central carrier, we have $\underline{P} = \underline{I - P} = 0$ and $\overline{I - P} = I$. For the convenience, write $P_1 = P$, $P_2 = I - P$ and $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$, $i, j \in \{1, 2\}$. Then $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$.

Claim 1. $P_1\varphi(I)P_2 = P_2\varphi(I)P_1 = 0$ and $P_2\varphi(P_1)P_2 = P_1\varphi(P_2)P_1 = 0$.

Since $P_1P_2 + P_2P_1 = P_2P_1 + P_1P_2 = 0$, we have $[\varphi(P_1), P_2]_\xi + [P_1, \varphi(P_2)]_\xi = 0$ and $[\varphi(P_2), P_1]_\xi + [P_2, \varphi(P_1)]_\xi = 0$, that is,

$$\varphi(P_1)P_2 - \xi P_2\varphi(P_1) + P_1\varphi(P_2) - \xi\varphi(P_2)P_1 = 0, \tag{1}$$

and

$$\varphi(P_2)P_1 - \xi P_1\varphi(P_2) + P_2\varphi(P_1) - \xi\varphi(P_1)P_2 = 0. \tag{2}$$

Multiplying by P_1 and P_2 from left and the right in Equation (1), respectively, and multiplying by P_2 and P_1 from left and the right in Equation (2), respectively, one gets $P_1\varphi(P_1)P_2 + P_1\varphi(P_2)P_2 = 0$ and $P_2\varphi(P_2)P_1 + P_2\varphi(P_1)P_1 = 0$, that is,

$$P_1\varphi(I)P_2 = 0 \text{ and } P_2\varphi(I)P_1 = 0.$$

Multiplying by P_2 from both sides in Equation (1) and multiplying by P_1 from both sides in Equation (2), one obtains $(1-\xi)P_1\varphi(P_2)P_1 = 0$ and $(1-\xi)P_2\varphi(P_1)P_2 = 0$. Note that $\xi \neq 1$. So

$$P_1\varphi(P_2)P_1 = 0 \text{ and } P_2\varphi(P_1)P_2 = 0.$$

Now define a map $\delta : \mathcal{M} \to \mathcal{M}$ by

$$\delta(A) = \varphi(A) + TA - AT \quad \text{for all } A \in \mathcal{M},$$

where $T = P_1\varphi(P_1)P_2 - P_2\varphi(P_1)P_1$. Clearly, δ is also additive and satisfies

$$AB + BA = 0 \Rightarrow \delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi.$$

It follows from Claim 1 that

$$P_1\delta(I)P_2 = P_2\delta(I)P_1 = P_2\delta(P_1)P_2 = P_1\delta(P_2)P_1 = 0.$$

In addition, we have

$$\begin{aligned}\delta(P_1) &= \varphi(P_1) + TP_1 - P_1T = P_1\varphi(P_1)P_1 \\ &= P_1\delta(P_1)P_1 - P_1(TP_1 - P_1T)P_1 = P_1\delta(P_1)P_1 \in \mathcal{M}_{11}, \end{aligned} \tag{3}$$

and

$$\begin{aligned}\delta(P_2) &= \varphi(P_2) + TP_2 - P_2T = P_2\varphi(P_2)P_2 \\ &= P_2\delta(P_2)P_2 - P_2(TP_2 - P_2T)P_2 = P_2\delta(P_2)P_2 \in \mathcal{M}_{22}. \end{aligned} \tag{4}$$

In the following, we will prove that δ is a generalized derivation by Claims 2-5.

Claim 2. $\delta(\mathcal{M}_{ii}) \subseteq \mathcal{M}_{ii}$, $i = 1, 2$.

We only give the proof of $\delta(\mathcal{M}_{11}) \subseteq \mathcal{M}_{11}$, and the proof of another inclusion is similar.

In fact, for any $A_{11} \in \mathcal{M}_{11}$, by the equation $A_{11}P_2 + P_2A_{11} = 0$, we have $[\delta(A_{11}), P_2]_\xi + [A_{11}, \delta(P_2)]_\xi = 0$, which and Equation (4) yield

$$\delta(A_{11})P_2 - \xi P_2\delta(A_{11}) = 0. \tag{5}$$

Multiplying by P_1 from the left in Equation (5), one gets $P_1\delta(A_{11})P_2 = 0$; multiplying by P_2 from both sides in Equation (5) and noting that $\xi \neq 1$, one has $P_2\delta(A_{11})P_2 = 0$.

Similarly, by the relation $P_2A_{11} + A_{11}P_2 = 0$, one can check $P_2\delta(A_{11})P_1 = 0$. Hence $\delta(\mathcal{M}_{11}) \subseteq \mathcal{M}_{11}$. The claim holds.

Claim 3. $\delta(\mathcal{M}_{ij}) \subseteq \mathcal{M}_{ij}, 1 \leq i \neq j \leq 2$.

For any $A_{ij} \in \mathcal{M}_{ij}(i \neq j)$, since $A_{ij}A_{ij} + A_{ij}A_{ij} = 0$, we have $[\delta(A_{ij}), A_{ij}]_\xi + [A_{ij}, \delta(A_{ij})]_\xi = 0$, that is,

$$\begin{aligned}&\delta(A_{ij})A_{ij} - \xi A_{ij}\delta(A_{ij}) + A_{ij}\delta(A_{ij}) - \xi\delta(A_{ij})A_{ij}\\ &= (1-\xi)(\delta(A_{ij})A_{ij} + A_{ij}\delta(A_{ij})) = 0.\end{aligned}$$

As $\xi \neq 1$, the above equation can be reduced to

$$\delta(A_{ij})A_{ij} + A_{ij}\delta(A_{ij}) = 0. \tag{6}$$

Since $(P_i + A_{ij})(A_{ij} - P_j) + (A_{ij} - P_j)(P_i + A_{ij}) = 0$. By Equations (3)-(4) and Equation (6), one gets

$$\begin{aligned}0 &= [\delta(P_i) + \delta(A_{ij}), A_{ij} - P_j]_\xi + [P_i + A_{ij}, \delta(A_{ij}) - \delta(P_j)]_\xi\\ &= \delta(P_i)A_{ij} - \delta(A_{ij})P_j + \xi P_j\delta(A_{ij}) + P_i\delta(A_{ij}) - A_{ij}\delta(P_j) - \xi\delta(A_{ij})P_i. \quad (7)\end{aligned}$$

Multiplying by P_i and P_j from both sides in Equation (7), respectively, by Equations (3) and (4), we obtain $(1-\xi)P_i\delta(A_{ij})P_i = 0$ and $(1-\xi)P_j\delta(A_{ij})P_j = 0$, which imply

$$P_i\delta(A_{ij})P_i = 0 \text{ and } P_j\delta(A_{ij})P_j = 0. \tag{8}$$

Finally, we still need to prove $P_j\delta(A_{ij})P_i = 0$. To do this, note that $A_{ij}(P_i - P_j) + (P_i - P_j)A_{ij} = 0$. So

$$\begin{aligned} -(1+\xi)\delta(A_{ij}) &= [\delta(A_{ij}),\ P_i - P_j]_\xi + [A_{ij},\ \delta(P_i) - \delta(P_j)]_\xi \\ &= \delta(A_{ij})P_i - \delta(A_{ij})P_j - \xi P_i\delta(A_{ij}) \\ &\quad + \xi P_j\delta(A_{ij}) - A_{ij}\delta(P_j) - \xi\delta(P_i)A_{ij}. \end{aligned}$$

Multiplying by P_j and P_i from the left and the right in the above equation, respectively, gets $-(1+\xi)P_j\delta(A_{ij})P_i = (1+\xi)P_j\delta(A_{ij})P_i$. This implies $P_j\delta(A_{ij})P_i = 0$. By Equation (8), one gets $\delta(A_{ij}) \in \mathcal{M}_{ij}$, completing the proof of the claim.

Claim 4. $\delta(I) \in \mathcal{Z}(\mathcal{M})$.

For any $A_{ij} \in \mathcal{M}_{ij}\ (1 \le i \ne j \le 2)$, by Equation (7) and Claim 3, we have proved that

$$\delta(P_i)A_{ij} - A_{ij}\delta(P_j) = 0.$$

Combining the equation and Equations (3)-(4), one obtains

$$\delta(I)A_{ij} = \delta(P_i)A_{ij} = A_{ij}\delta(P_j) = A_{ij}\delta(I).$$

It follows from Lemma 2 that $\delta(I) \in \mathcal{Z}(\mathcal{M})$.

Claim 5. $\delta(AB) = \delta(A)B + A\delta(B) - AB\delta(I)$ for all $A, B \in \mathcal{M}$, that is, δ is a generalized derivation.

We will prove the claim by four steps.

Step 1. The following statements hold:

(1) For any $A_{ii} \in \mathcal{M}_{ii}$ and $B_{ij} \in \mathcal{M}_{ij}$, we have $\delta(A_{ii}B_{ij}) = \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(I)$, $1 \le i \ne j \le 2$.

(2) For any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{jj} \in \mathcal{M}_{jj}$, we have $\delta(A_{ij}B_{jj}) = \delta(A_{ij})B_{jj} + A_{ij}\delta(B_{jj}) - A_{ij}B_{jj}\delta(I)$, $1 \le i \ne j \le 2$.

Take any $A_{ii} \in \mathcal{M}_{ii}$ and $B_{ij} \in \mathcal{M}_{ij}$. Since $(A_{ii} + A_{ii}B_{ij})(B_{ij} - P_j) + (B_{ij} - P_j)(A_{ii} + A_{ii}B_{ij}) = 0$, by Claims 2-3, one gets

$$\begin{aligned} 0 &= [\delta(A_{ii}) + \delta(A_{ii}B_{ij}),\ B_{ij} - P_j]_\xi + [A_{ii} + A_{ii}B_{ij},\ \delta(B_{ij}) - \delta(P_j)]_\xi \\ &= \delta(A_{ii})B_{ij} - \delta(A_{ii}B_{ij}) + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(P_j) \\ &= \delta(A_{ii})B_{ij} - \delta(A_{ii}B_{ij}) + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(I). \end{aligned}$$

(1) holds.

For any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{jj} \in \mathcal{M}_{jj}$, note that $(A_{ij}B_{jj} + B_{jj})(A_{ij} - P_i) + (A_{ij} - P_i)(A_{ij}B_{jj} + B_{jj}) = 0$. By a similar argument to that of (1), it is easy to check that (2) holds.

Step 2. For any $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$, we have $\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I)$, $i = 1, 2$.

For any A_{ii}, $B_{ii} \in \mathcal{M}_{ii}$ and any $S_{ij} \in \mathcal{M}_{ij}$, by Step 1, on the one hand, we have

$$\delta(A_{ii}B_{ii}S_{ij}) = \delta(A_{ii}B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I).$$

On the other hand,

$$\begin{aligned} \delta(A_{ii}B_{ii}S_{ij}) &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii}S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I) \\ &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - 2A_{ii}B_{ii}S_{ij}\delta(I). \end{aligned}$$

Comparing the above two equations obtains

$$(\delta(A_{ii}B_{ii}) - \delta(A_{ii})B_{ii} - A_{ii}\delta(B_{ii}) + A_{ii}B_{ii}\delta(I))S_{ij} = 0,$$

for all $S_{ij} \in \mathcal{M}_{ij}$ as $\delta(I) \in \mathcal{Z}(\mathcal{M})$, that is,

$$(\delta(A_{ii}B_{ii}) - \delta(A_{ii})B_{ii} - A_{ii}\delta(B_{ii}) + A_{ii}B_{ii}\delta(I))SP_j = 0,$$

holds for all $S \in \mathcal{M}$. Note that $\overline{P_j} = I$. It follows from the definition of the central carrier that $\operatorname{span}\{TP_j(x) : T \in \mathcal{M}, x \in H\}$ is dense in H. So $\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I)$.

Step 3. For any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{ji} \in \mathcal{M}_{ji}$, we have $\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}) - A_{ij}B_{ji}\delta(I)$, $1 \le i \ne j \le 2$.

For any $A_{ij} \in \mathcal{M}_{ij}$ and $B_{ji} \in \mathcal{M}_{ji}$, since

$$[(-A_{12}B_{21} + A_{12} + B_{21} - P_2), (P_1 + A_{12} + B_{21} + B_{21}A_{12})]_{-1} = 0,$$

by Claims 2-4, one can get

$$\begin{aligned}
0 &= [-\delta(A_{12}B_{21}) + \delta(A_{12}) + \delta(B_{21}) - \delta(P_2),\ P_1 + A_{12} + B_{21} + B_{21}A_{12}]_{\xi} \\
&\quad + [-A_{12}B_{21} + A_{12} + B_{21} - P_2,\ \delta(P_1) + \delta(A_{12}) + \delta(B_{21}) + \delta(B_{21}A_{12})]_{\xi} \\
&= (\xi - 1)\delta(A_{12}B_{21}) + (1 - \xi)\delta(A_{12})B_{21} + (1 - \xi)A_{12}\delta(B_{21}) \\
&\quad + (\xi - 1)A_{12}B_{21})\delta(I) + (\xi - 1)\delta(B_{21}A_{12}) + (1 - \xi)\delta(B_{21})A_{12} \\
&\quad + (1 - \xi)B_{21}\delta(A_{12}) + (\xi - 1)B_{21}A_{12}\delta(I).
\end{aligned}$$

As $\xi \ne 1$, the above equation implies

$$\delta(A_{12}B_{21}) = \delta(A_{12})B_{21} + A_{12}\delta(B_{21}) - A_{12}B_{21}\delta(I),$$

and

$$\delta(B_{21}A_{12}) = \delta(B_{21})A_{12} + B_{21}\delta(A_{12}) - B_{21}A_{12}\delta(I).$$

Step 4. For any $A, B \in \mathcal{M}$, we have $\delta(AB) = \delta(A)B + A\delta(B) - AB\delta(I)$.

Take any $A = A_{11} + A_{12} + A_{21} + A_{22}, B = B_{11} + B_{12} + B_{21} + B_{22} \in \mathcal{M}$. Combining Steps 1-3 and Claims 2-3, it is easily checked that $\delta(AB) = \delta(A)B + A\delta(B) - AB\delta(I)$. So δ is a generalized derivation.

Claim 6. $\delta(\xi I) = 2\xi\delta(I) + \delta(I)$. Therefore, the theorem holds.

For any $A_{12} \in \mathcal{M}_{12}$, as $A_{12}(I - 2P_1) + (I - 2P_1)A_{12} = 0$, we have

$$[\delta(A_{12}),\ I - 2P_1]_\xi + [A_{12},\ \delta(I) - 2\delta(P_1)]_\xi = \delta([A_{12},\ I - 2P_1]_\xi).$$

It follows from Claim 3 and Equations (3)-(4) that

$$\delta(\xi A_{12}) = \xi\delta(A_{12}) + (1 + \xi)\delta(I)A_{12}. \tag{9}$$

On the other hand, by Claim 5, we have

$$\delta(\xi A_{12}) = \delta(\xi I)A_{12} + \xi\delta(A_{12}) - \xi A_{12}\delta(I). \tag{10}$$

Comparing Equations (9) and (10), one achieves $(\delta(\xi I) - (1 + 2\xi)\delta(I))A_{12} = 0$, that is,

$$(\delta(\xi I) - (1 + 2\xi)\delta(I))\mathcal{M}P_2 = 0.$$

Note that $\overline{P_2} = I$. It follows from the definition of the central carrier that $\text{span}\{TP_2(x) : T \in \mathcal{M},\ x \in H\}$ is dense in H. It follows that $\delta(\xi I) = (2\xi + 1)\delta(I)$.

Claim 7. $\varphi(AB) = \varphi(A)B + A\varphi(B) - AB\varphi(I)$ holds for all $A, B \in \mathcal{M}$ and $\varphi(\xi I) = (2\xi + 1)\varphi(I)$.

Note that $\varphi(A) = \delta(A) - TA + AT$ for each $A \in \mathcal{M}$. By Claims 5-6, it is easily check that the claim holds.

The proof of the theorem is finished. □

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