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RATE OF WEAK CONVERGENCE FOR HUBER-DUTTER ESTIMATORS IN A LINEAR MODEL WITH FCA PROCESSES

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Abstract

Consider the following linear regression model $y = x^T\beta + e$, where the error e is a functional coefficient autoregressive (FCA) processes. In this paper, we prove that the Huber-Dutter (HD) estimators for unknown parameters in the above model week converge to the true values with rate $n^{-\frac{1}{2}}$.

1. Introduction

Consider the following regression model:

$$y_t = x_t^T\beta + e_t, \quad t = 1, 2, \cdots, n, \tag{1.1a}$$

where $y_t \in R$, $x_t = \{x_{t1}, \cdots, x_{td}\}^T \in R^d$, β is a d-dimensional unknown parameter, and e_t's are functional coefficient autoregressive processes given as

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$$e_0 = 0, \quad e_t = f_t(\theta)e_{t-1} + \eta_t, \quad t = 1, 2, \cdots, n, \tag{1.1b}$$

where η_t's are non-degenerate iid random errors with zero mean and finite variance σ^2, θ is a one-dimensional unknown parameter and $f_t(\theta)$ is a real valued function defined on a compact set Θ, which contains the true value θ_0 as an inner point.

Model (1.1) includes many special models, such as a ordinary linear regression model (when $f_t(\theta) \equiv 0$), a linear regressive processes (when $f_t(\theta) = \theta$, see Maller [7], Pere [8]), time-dependent autoregressive processes (when $f_t(\theta) = a_t$, see Carsoule and Franses [3], Azrak and Melard [2]), e_t is the same as (1.1b), see Hu [5], Kwoun and Yajima [6].

A number of robust estimators have been proposed and investigated over the last 40 years. One of them is the Huber-Dutter (HD) estimators, see, for example, Silvapulle [10], Douglas [4], Tong et al. [12]. This paper will investigate the week convergence rate of HD estimators for unknown parameters in the model (1.1) under some regularity conditions.

2. Estimation Method

Let $\lambda_0 = \{\beta_0^T, \sigma_0, \theta_0\}$ be the "true" value of unknown parameter. Let $f_t'(\theta_0) = \frac{df_t(\theta)}{d\theta}\Big|_{\theta=\theta_0} \neq 0$ and define $\prod_{i=0}^{-1} f_{t-i}(\theta_0) = 1$, then from (1.1b),

$$e_t = \sum_{j=0}^{t-1}\left(\prod_{i=0}^{j-1} f_{t-i}(\theta_0)\right)\eta_{t-j}, \quad Ee_t = 0, \ \mathrm{Var}e_t = \sigma_0^2\sum_{j=0}^{t-1}\left(\prod_{i=0}^{j-1} f_{t-i}^2(\theta_0)\right), \tag{2.1}$$

$$y_t = x_t^T\beta + f_t(\theta)(y_{t-1} - x_{t-1}^T\beta) + \eta_t, \quad y_0 = x_0 = 0. \tag{2.2}$$

Write the Huber-Dutter loss function by

$$Q(\beta, \sigma, \theta) = \sum_{t=1}^{n}\rho\left(\frac{y_t - x_t^T\beta - f_t(\theta)(y_{t-1} - x_{t-1}^T\beta)}{\sigma}\right)\sigma + A_n\sigma, \tag{2.3}$$

where $\rho \geq 0$ is convex, $\rho(0) = 0$, $\rho(t)/|t| \to k$ as $|t| \to \infty$ for some $k > 0$, and $\{A_n\}$ is a suitably choose sequence of constants. We will obtain the HD estimators for $\lambda = (\beta^T, \sigma, \theta)$ by minimizing $Q(\beta, \sigma, \theta)$ and we denote it by $\hat{\lambda}_n = (\hat{\beta}_n^T, \hat{\sigma}_n, \hat{\theta}_n)$. Then the estimators, if they exist (see the following Proposition 2.1), will satisfy

$$\sum_{t=1}^{n} \psi\left(\frac{\hat{e}_t - f_t(\hat{\theta}_n)\hat{e}_{t-1}}{\hat{\sigma}_n}\right)(x_t - f_t(\hat{\theta}_n)x_{t-1}) = 0, \tag{2.4}$$

$$\sum_{t=1}^{n} \chi\left(\frac{\hat{e}_t - f_t(\hat{\theta}_n)\hat{e}_{t-1}}{\hat{\sigma}_n}\right) = A_n, \tag{2.5}$$

$$\sum_{t=1}^{n} \psi\left(\frac{\hat{e}_t - f_t(\hat{\theta}_n)\hat{e}_{t-1}}{\hat{\sigma}_n}\right) f_t'(\hat{\theta}_n)\hat{e}_{t-1} = 0, \tag{2.6}$$

where $\psi = \rho'$, $\chi(u) = u\psi(u) - \rho(u) = \int_0^u x d\psi(x)$, and $\hat{e}_t = y_t - x_t^T\hat{\beta}_n$.

Same as in Silvapulle [10] and Tong et al. [12], in what follows, it will be assumed that $n \geq d + 2$ and $A_n > 0$.

Without loss of generality, we may assume $k = 1$. Therefore, ψ is bounded and increases from -1 to $+1$. It will also assumed that χ is bounded. From Proposition 1 in [10], we have the following:

Proposition 1. *Suppose that* ρ'' *is continuous and for some* $A > 0$, $v < 1 - AV^{-1}$, *where* v *is the largest jump in the error distribution,* $V = \chi(-\infty) \wedge \chi(\infty)$, *then the equation* $E\{\psi(\frac{e-u}{\sigma}), \chi(\frac{e-u}{\sigma}) - A\} = 0$ *has a solution* $(\mu(A), \sigma(A))$ *with* $\sigma(A) > 0$. *Especially, when* $A = \lim_{n\to\infty} n^{-1}A_n$, *where* A_n *are defined in* (2.3). *We denote it by* (μ_0, σ_0) *with* $\sigma_0 > 0$.

HD estimators include some existed estimators, for example, the least squares estimators ($\rho(u) = u^2$, $A_n = 0$, and $\sigma = 1$), the least absolute deviation estimators ($\rho(u) = |u|$, $A_n = 0$), M-estimators (cf. Tong et al. [12]). If we put $\rho(u) = |u|^q (0 < q \neq 1)$, $A_n = 0$, and σ is a known constant, then (2.4)-(2.6) are the L_q estimate equations for a linear regression model with q-norm distribution errors (Arcones [1], Rouner [9]).

3. Main Result and Preliminary Lemmas

To obtain the result in this paper, the following conditions are sufficient ($\|\cdot\|$ denote the Euclidean norm):

(A_1) $\max_{1\le t\le n}\|x_t\| = o(n^{\frac{1}{3}})$ (as $n \to \infty$).

(A_2) There is a constant $\alpha > 0$ such that for any t and $\theta \in \Theta$,

$$\sum_{j=1}^{t}\left(\prod_{i=0}^{j-1} f_{t-1}^2(\theta)\right) \le \alpha, \quad \sum_{j=t+1}^{n}\left|\prod_{i=0}^{j-t-1} f_{t-1}(\theta)\right| \le \alpha.$$

(A_3) $f_t(\theta)$, $f_t'(\theta)$, $f_t''(\theta)$, $f_t'''(\theta)$ are bounded for any t and $\theta \in \Theta$.

(A_4) $\lim_{n\to\infty} \frac{1}{n} A_n - A = 0$.

(A_5) $E\psi(\frac{\eta_1}{\sigma}) = 0$ for any $\sigma > 0$, $b = E\psi'(\frac{\eta_1}{\sigma_0}) > 0$, $c = E[\psi'(\frac{\eta_1}{\sigma_0})\cdot\frac{\eta_1}{\sigma_0}]$, $r = E[\psi'(\frac{\eta_1}{\sigma_0})\cdot\eta_1^2]$, $br \ge c^2$ and $\mathrm{Var}(\psi'(\frac{\eta_1}{\sigma})) < \infty$, $\mathrm{Var}(\psi''(\frac{\eta_1}{\sigma})) < \infty$, $\mathrm{Var}(\psi'(\frac{\eta_1}{\sigma})\eta_1) < \infty$, $\mathrm{Var}(\psi'(\frac{\eta_1}{\sigma})\eta_1^2) < \infty$, $E\eta_1^4 < \infty$.

Remark. The condition (A_1) is weaker than the corresponding condition in Silvapulle [10], Hu [5]. The condition (A_2) is used by Hu [5], Song et al. [11] and the first boundlessness condition of (A_2) is used by Kwoun and Yajima [6]. The boundlessness of $f_t(\theta)$, $f_t'(\theta)$, $f_t''(\theta)$ in (A_3) are used by [5], [11]. And the conditions (A_4) and (A_5) are used by Silvapulle [10], Tong et al. [12].

Theorem 3.1. *Suppose that conditions* (A_1)-(A_5) *hold. Then*

$$\sqrt{n}(\hat{\lambda}_n - \lambda_0) \text{ is } O_p(1) \text{ as } n \to \infty. \tag{3.1}$$

For the proof of Theorem 3.1, we need the lemma as following:

Lemma 3.1. *Under the conditions* (A_1)-(A_5), *we have*

(i) $\Delta_n(\lambda_0) := \sum_{t=1}^{n} f_t'^2(\theta) E e_{t-1}^2 \to \infty; \quad \Delta_n(\lambda_0) = O(n) \text{ as } n \to \infty.$

(ii) $E e_t^4 \le (E\eta_1^4 + 6\sigma_0^4)\alpha^2.$

(iii) $\mathrm{Var}(\sum_{t=1}^{n} e_t^2) = O(n) \, (n \to \infty).$

Proof. The proof of (i), see Hu [5]. We only prove (ii) and (iii),

$$E e_t^4 = E\left\{\sum_{j=0}^{t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)\eta_{t-j}\right\}^4$$

$$= E\left\{\left[\sum_{j=0}^{t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)\eta_{t-j}\right]^2\left[\sum_{k=0}^{t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)\eta_{t-k}\right]^2\right\}$$

$$= E\left\{\left[\sum_{j=0}^{t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)^2\eta_{t-j}^2\right]\left[\sum_{k=0}^{t-1}\left(\prod_{r=0}^{k-1} f_{t-r}(\theta_0)\right)^2\eta_{t-k}^2\right]\right\}$$

$$+ 4E\left\{\left[\sum_{0\le i<j\le t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)\eta_{t-j}\cdot\left(\prod_{r=0}^{i-1} f_{t-r}(\theta_0)\right)\eta_{t-i}\right]\right.$$

$$\left.\times\left[\sum_{0\le l<m\le t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)\eta_{t-l}\cdot\left(\prod_{r=0}^{m-1} f_{t-r}(\theta_0)\right)\eta_{t-m}\right]\right\}$$

$$:= I_1 + 4I_2. \tag{3.2}$$

$$I_1 = E\left\{\sum_{j=0}^{t-1}\left[\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\eta_{t-j}\right]^4\right\}$$

$$+ 2E\left\{\sum_{m=0}^{t-2}\sum_{k=m+1}^{t-1}\left(\prod_{r=0}^{m-1} f_{t-r}(\theta_0)\eta_{t-m}\right)^2\left(\prod_{r=0}^{k-1} f_{t-r}(\theta_0)\eta_{t-k}\right)^2\right\}$$

$$= E\eta_1^4\left\{\sum_{j=0}^{t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)^4\right\}$$

$$+ 2\sigma_0^4\sum_{m=0}^{t-2}\sum_{k=m+1}^{t-1}\left(\prod_{r=0}^{m-1} f_{t-r}(\theta_0)\right)^2\left(\prod_{r=0}^{k-1} f_{t-r}(\theta_0)\right)^2$$

$$= E\eta_1^4\left\{\left[\sum_{j=0}^{t-1}\prod_{r=0}^{j-1} f_{t-r}^2(\theta_0)\right]\left[\sum_{j=0}^{t-1}\prod_{r=0}^{j-1} f_{t-r}^2(\theta_0)\right]\right\}$$

$$+ 2\sigma_0^4\left\{\sum_{m=0}^{t-2}\prod_{r=0}^{m-1} f_{t-r}^2(\theta_0)\cdot\sum_{k=0}^{t-1}\prod_{r=0}^{k-1} f_{t-r}^2(\theta_0)\right\}$$

$$\le (E\eta_1^4 + 2\sigma_0^4)\alpha^2. \tag{3.3}$$

Similarly, from the condition (A_2), we can obtain that

$$I_2 = E\left[\sum_{0\le i<j\le t-1}\left(\prod_{r=0}^{j-1} f_{t-r}(\theta_0)\right)^2\eta_{t-j}^2\left(\prod_{r=0}^{i-1} f_{t-r}(\theta_0)\right)^2\eta_{t-i}^2\right]$$

$$= \sum_{i=1}^{t-2} \sum_{j=i+1}^{t-1} \left[\prod_{r=0}^{j-1} f_{t-r}(\theta_0) \right]^2 \left[\prod_{r=0}^{j-1} f_{t-r}(\theta_0) \right]^2 E\eta_{t-i}^2 \eta_{t-j}^2$$

$$\leq \sigma_0^4 \alpha^2. \tag{3.4}$$

Hence, we get (ii) from (3.2)-(3.4).

Now we prove (iii)

$$E(\sum_{t=1}^{n} e_t^2)^2 - (\sum_{t=1}^{n} Ee_t^2)^2 = \left[\sum_{t=1}^{n} Ee_t^4 + 2 \sum_{1 \leq i < j \leq n} Ee_i^2 e_j^2 \right]$$

$$- \left[\sum_{t=1}^{n} (Ee_t^2)^2 + 2 \sum_{1 \leq i < j \leq n} Ee_i^2 Ee_j^2 \right]$$

$$= \sum_{t=1}^{n} \left[Ee_t^4 - (Ee_t^2)^2 \right] + 2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \left[Ee_i^2 e_{i+k}^2 - Ee_i^2 \cdot Ee_{i+k}^2 \right]. \tag{3.5}$$

From (ii), we have

$$\sum_{t=1}^{n} [Ee_t^4 - (Ee_t^2)^2] = O(n). \tag{3.6}$$

By recursive computation,

$$Ee_i^2 e_{i+1}^2 = E[e_i^2 (f_{i+1}(\theta_0) e_i + \eta_{i+1})^2] = f_{i+1}^2(\theta_0) Ee_i^4 + \sigma_0^2 Ee_i^2,$$

$$Ee_i^2 e_{i+2}^2 = E[e_i^2 (f_{i+2}(\theta_0) e_{i+1} + \eta_{i+2})^2] = f_{i+2}^2(\theta_0) Ee_i^2 e_{i+1}^2 + \sigma_0^2 Ee_i^2$$

$$= f_{i+2}^2(\theta_0) [f_{i+1}^2(\theta_0) Ee_i^4 + \sigma_0^2 Ee_i^2] + \sigma_0^2 Ee_i^2$$

$$= f_{i+2}^2(\theta_0) f_{i+1}^2(\theta_0) Ee_i^4 + f_{i+2}^2(\theta_0) \sigma_0^2 Ee_i^2 + \sigma_0^2 Ee_i^2,$$

$$
\begin{aligned}
Ee_i^2e_{i+3}^2 &= E[e_i^2(f_{i+3}(\theta_0)e_{i+2}+\eta_{i+3})^2] = f_{i+3}^2(\theta_0)Ee_i^2e_{i+2}^2+\sigma_0^2Ee_i^2\\
&= f_{i+3}^2(\theta_0)[f_{i+2}^2(\theta_0)f_{i+1}^2(\theta_0)Ee_i^4+f_{i+2}^2(\theta_0)\sigma_0^2Ee_i^2+\sigma_0^2Ee_0^2]+\sigma_0^2Ee_0^2\\
&= f_{i+3}^2(\theta_0)f_{i+2}^2(\theta_0)f_{i+1}^2(\theta_0)Ee_i^4+f_{i+3}^2(\theta_0)f_{i+2}^2(\theta_0)\sigma_0^2Ee_i^2\\
&\quad +f_{i+3}^2(\theta_0)\sigma_0^2Ee_i^2+\sigma_0^2Ee_i^2,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
Ee_i^2e_{i+k}^2 &= E[e_i^2(f_{i+k}(\theta_0)e_{i+k-1}+\eta_{i+k})^2]\\
&= f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\cdots f_{i+1}^2(\theta_0)Ee_i^4+f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\\
&\quad \cdots f_{i+2}^2(\theta_0)\sigma_0^2Ee_i^2+f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\cdots f_{i+3}^2(\theta_0)\sigma_0^2Ee_i^2\\
&\quad +\cdots+f_{i+k}^2(\theta_0)\sigma_0^2Ee_i^2+\sigma_0^2Ee_i^2.
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
Ee_{i+1}^2 &= E[f_{i+1}(\theta_0)e_i+\eta_{i+1}]^2 = f_{i+1}^2(\theta_0)Ee_i^2+\sigma_0^2,\\
Ee_{i+2}^2 &= E[f_{i+2}(\theta_0)e_{i+1}+\eta_{i+2}]^2 = f_{i+2}^2(\theta_0)Ee_{i+1}^2+\sigma_0^2\\
&= f_{i+2}^2(\theta_0)[f_{i+1}^2(\theta_0)Ee_i^2+\sigma_0^2]+\sigma_0^2\\
&= f_{i+2}^2(\theta_0)f_{i+1}^2(\theta_0)Ee_i^2+f_{i+2}^2(\theta_0)\sigma_0^2+\sigma_0^2,\\
Ee_{i+3}^2 &= E[f_{i+3}(\theta_0)e_{i+2}+\eta_{i+3}]^2 = f_{i+3}^2(\theta_0)Ee_{i+2}^2+\sigma_0^2\\
&= f_{i+3}^2(\theta_0)[f_{i+2}^2(\theta_0)f_{i+1}^2(\theta_0)Ee_i^2+f_{i+2}^2(\theta_0)\sigma_0^2]+\sigma_0^2\\
&= f_{i+3}^2(\theta_0)f_{i+2}^2(\theta_0)f_{i+1}^2(\theta_0)Ee_i^2+f_{i+3}^2(\theta_0)f_{i+2}^2(\theta_0)\sigma_0^2+f_{i+3}^2(\theta_0)\sigma_0^2+\sigma_0^2,\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\
Ee_{i+k}^2 &= f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\cdots f_{i+1}^2(\theta_0)Ee_i^2+f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\cdots f_{i+2}^2(\theta_0)\sigma_0^2\\
&\quad +\cdots+f_{i+k}^2(\theta_0)\sigma_0^2+\sigma_0^2.
\end{aligned}
\tag{3.8}
$$

Then,

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}[Ee_i^2e_{i+k}^2 - Ee_i^2Ee_{i+k}^2]$$

$$= \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}[f_{i+k}^2(\theta_0)f_{i+k-1}^2(\theta_0)\cdots f_{i+1}^2(\theta_0)](Ee_i^4 - Ee_i^2 \cdot Ee_i^2)$$

$$= \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n-i}\prod_{r=1}^{k} f_{i+r}^2(\theta_0)\right)(Ee_i^4 - Ee_i^2 \cdot Ee_i^2) \le \alpha^2 \cdot \sum_{i=1}^{n-1}(Ee_i^4 - Ee_i^2 \cdot Ee_i^2) = O(n). \tag{3.9}$$

From (3.5), (3.6), and (3.9), we obtain (iii).

4. Proof of Theorem 2.1

Let $F_n(\lambda)$ be the Hessian of $Q(\lambda)$, that is,

$$F_n(\lambda) = \frac{\partial^2 Q}{\partial\lambda^T\partial\lambda} = \begin{bmatrix} \frac{\partial^2 Q}{\partial\beta^T\partial\beta} & \frac{\partial^2 Q}{\partial\beta^T\partial\sigma} & \frac{\partial^2 Q}{\partial\beta^T\partial\theta} \\ * & \frac{\partial^2 Q}{\partial\sigma^2} & \frac{\partial^2 Q}{\partial\sigma\partial\theta} \\ * & * & \frac{\partial^2 Q}{\partial\theta^2} \end{bmatrix}, \tag{4.1}$$

where $*$ indicates that the elements are filled in by symmetry.

Let

$$R_{nl}(\lambda) := \frac{\partial}{\partial\lambda_l} F_n(\lambda) = \begin{bmatrix} \frac{\partial^3 Q}{\partial\lambda_l\partial\beta^T\partial\beta} & \frac{\partial^3 Q}{\partial\lambda_l\partial\beta^T\partial\sigma} & \frac{\partial^3 Q}{\partial\lambda_l\partial\beta^T\partial\theta} \\ * & \frac{\partial^3 Q}{\partial\lambda_l\partial\sigma^2} & \frac{\partial^3 Q}{\partial\lambda_l\partial\sigma\partial\theta} \\ * & * & \frac{\partial^3 Q}{\partial\lambda_l\partial\theta^2} \end{bmatrix}, \tag{4.2}$$

where $\lambda_l = \beta_l (l = 1, 2, \cdots, d)$, $\lambda_{d+l} = \sigma$, $\lambda_{d+2} = \theta$.

First, we prove that

$$R_{nl}(\lambda) = o_p(n^{\frac{3}{2}}), \quad l = 1, 2, \ldots, d+2 \quad (n \to \infty); \tag{4.3}$$

$$\frac{\partial^3 Q}{\partial\beta_l \partial\beta^T \partial\beta} = -\frac{1}{\sigma^2}\sum_{t=1}^{n} \psi''\left(\frac{\eta_t}{\sigma}\right)(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})(x_t - f_t(\theta)x_{t-1})^T$$

$$=: L_n^{(1)};$$

$$|L_n^{(1)}| \le \frac{1}{\sigma^2}\max_{1\le t\le n}\|x_t - f_t(\theta)x_{t-1}\|\sum_{t=1}^{n}|\psi''\left(\frac{\eta_t}{\sigma}\right)| = o(n)\cdot\sum_{t=1}^{n}|\psi''\left(\frac{\eta_t}{\sigma}\right)|.$$

Since $\{\psi''\left(\frac{\eta_t}{\sigma}\right)\}$ are iid variables,

$$E\left[\sum_{t=1}^{n}|\psi''(\frac{\eta_t}{\sigma})|\right] = \sum_{t=1}^{n}E|\psi''(\frac{\eta_t}{\sigma})|^2 = O(n).$$

By Chebyshev inequality, $\sum_{t=1}^{n}\psi''(\frac{\eta_t}{\sigma}) = O_p(n^{\frac{1}{2}})$. Then,

$$L_n^{(1)} = o_p(n^{\frac{3}{2}}). \tag{4.4}$$

Similarly,

$$L_n^{(2)} =: \frac{\partial^3 Q}{\partial\beta_l \partial\beta^T \partial\sigma} = -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left\{\psi'(\frac{\eta_t}{\sigma}) + \psi''(\frac{\eta_t}{\sigma})(\frac{\eta_t}{\sigma})\right\}(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})^T$$

$$= o_p(n^{\frac{3}{2}}), \tag{4.5}$$

$$L_n^{(3)} =: \frac{\partial^3 Q}{\partial\beta_l \partial\sigma^2} = -\frac{1}{\sigma^2}\sum_{t=1}^{n}\left\{\psi''(\frac{\eta_t}{\sigma})(\frac{\eta_t}{\sigma})^2(x_{t,l} - f_t(\theta)x_{t-1,l})\right\}$$

$$-\frac{1}{\sigma^2}\sum_{t=1}^{n}\psi'(\frac{\eta_t}{\sigma})\eta_t(x_{t,l} - f_t(\theta)x_{t-1,l}) = o_p(n^{\frac{3}{2}}), \tag{4.6}$$

$$L_n^{(4)} =: \frac{\partial^3 Q}{\partial\beta_l \partial\beta^T \partial\theta} = -\frac{1}{\sigma^2}\sum_{t=1}^{n} f_t'(\theta)\psi''(\frac{\eta_t}{\sigma})e_{t-1}(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})^T$$

$$-\frac{1}{\sigma}\sum_{t=1}^{n} f_t'(\theta)\psi'(\frac{\eta_t}{\sigma})(x_{t,l} - f_t(\theta)x_{t-1,l})x_{t-1}^T$$

$$-\frac{1}{\sigma^2}\sum_{t=1}^{n} f_t'(\theta)\psi'(\frac{\eta_t}{\sigma})x_{t,l}(x_t - f_t(\theta)x_{t-1})^T$$

$$:= L_{n1}^{(4)} - L_{n2}^{(4)} - L_{n3}^{(4)}. \tag{4.7}$$

Similar to (4.5) and (4.6),

$$L_{n2}^{(4)} = o_p(n^{\frac{3}{2}}), \quad L_{n3}^{(4)} = o_p(n^{\frac{3}{2}}). \tag{4.8}$$

$$L_{n1}^{(4)} = -\frac{1}{\sigma^2}\sum_{t=1}^{n} f_t'(\theta)\psi''(\frac{\eta_t}{\sigma})e_{t-1}(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})^T$$

$$= -\frac{1}{\sigma^2}\sum_{t=1}^{n} f_t'(\theta)\left[\psi''(\frac{\eta_t}{\sigma}) - E\psi''(\frac{\eta_t}{\sigma})\right]$$

$$\times e_{t-1}(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})^T$$

$$-\frac{1}{\sigma^2}\sum_{t=1}^{n} E\left[\psi''(\frac{\eta_t}{\sigma})\right]e_{t-1}(x_{t,l} - f_t(\theta)x_{t-1,l})(x_t - f_t(\theta)x_{t-1})^T$$

$$:= J_{n1}^{(4)} - J_{n2}^{(4)}. \tag{4.9}$$

Note that $\{f_t'(\theta)[\psi''(\frac{\eta_t}{\sigma}) - E\psi''(\frac{\eta_t}{\sigma})]e_{t-1}\}$ is a $F_{t,n} := \sigma(\eta_1, \eta_2, \cdots, \eta_t)$, $(t \le n)$ martingale-difference array. Then,

$$E(J_{n1}^{(4)})^2 = -\frac{1}{\sigma^4}\sum_{t=1}^{n}[f_t'(\theta)]^2 \cdot E\left\{\left[\psi''(\frac{\eta_t}{\sigma}) - E\psi''(\frac{\eta_t}{\sigma})\right]^2 e_{t-1}^2\right\}$$

$$\cdot (x_{t,l} - f_t(\theta)x_{t-1,l})^2(x_t - f_t(\theta)x_{t-1})^T \cdot (x_t - f_t(\theta)x_{t-1})$$

$$\leq \frac{1}{\sigma^4}\max_{1\leq t\leq n}|f_t'(\theta)|^2\|x_t - f_t(\theta)x_{t-1}\|^4 \cdot \sum_{t=1}^{n} E\left[\psi''(\frac{\eta_t}{\sigma})\right]^2 Ee_{t-1}^2$$

$$= \frac{1}{\sigma^4}\max_{1\leq t\leq n}|f_t'(\theta)|^2\|x_t - f_t(\theta)x_{t-1}\|^4 E[\psi''(\frac{\eta_1}{\sigma})]^2 \cdot \sum_{t=1}^{n} Ee_{t-1}^2$$

$$= o_p(n^{\frac{4}{3}}) \cdot O(n) = o_p(n^{\frac{7}{3}}).$$

By Chebyshev inequality, $J_{n1}^{(4)} = o_p(n^{\frac{7}{6}})$, then

$$J_{n1}^{(4)} = o(n^{\frac{3}{2}}). \tag{4.10}$$

$$J_{n2}^{(4)} = \frac{2}{\sigma^4}\operatorname{Var}\left\{\sum_{t=1}^{n} e_{t-1}f_t'(\theta)(x_{t,l} - f_t(\theta)x_{t-1,l})^2(x_t - f_t(\theta)x_{t-1})^T\right\}$$

$$= \frac{b^2}{\sigma^4}\operatorname{Var}\left\{\sum_{t=2}^{n}\left[\sum_{k=0}^{t-2}(\prod_{i=0}^{k-1} f_{t-i}(\theta_0))\eta_{t-k}\right] f_t'(\theta)[x_{t,l} - f_t(\theta)x_{t-1,l}][x_t - f_t(\theta)x_{t-1}]^T\right\}.$$

Exchange the order of summing,

$$\operatorname{Var}J_{n2}^{(4)} = \frac{b^2}{\sigma^4}\operatorname{Var}$$

$$\left\{\sum_{k=2}^{n}\left[\sum_{t=0}^{n-k}\left(\prod_{i=0}^{t-1} f_{t+k-i}(\theta)\right)(x_{t+k-i} - f_{t+k}(\theta)x_{t+k-1,l})(x_{t+k} - f_{t+k}(\theta)x_{t+k-1})^T f_{t+k}'(\theta)\right]\eta_k\right\}$$

$$= b^2\sum_{k=2}^{n}\left[\sum_{t=0}^{n-k}(\prod_{i=0}^{t-1} f_{t+k-i}(\theta_0))f_{t+k}'(\theta)\right.$$

$$\cdot (x_{t+k,l} - f_{t+k}(\theta)x_{t+k-1,l})(x_{t+k} - f_{t+k}(\theta)x_{t+k-1})^T(x_{t+k} - f_{t+k}(\theta)x_{t+k-1}) \Bigg]$$

$$\leq b^2 \max_j |f_j'(\theta)|^2 \|x_j - f_j(\theta)x_{j-1}\|^4 \sum_{k=2}^{n} \left[\sum_{t=0}^{n-k} \left(\prod_{i=0}^{t-1} f_{t+k-i}(\theta) \right) \right]^2$$

$$\leq b^2 \max_j |f_j'(\theta)|^2 \|x_j - f_j(\theta)x_{j-1}\|^4 \sum_{k=2}^{n} \left[1 + \sum_{m=k+1}^{n} \left| \prod_{i=0}^{m-k-1} f_{m-i}(\theta) \right| \right]^2$$

$$\leq b^2 \max_j |f_j'(\theta)|^2 \|x_j - f_j(\theta)x_{j-1}\|^4 \sum_{k=2}^{n} [1 + \alpha]^2 = o(n^{\frac{4}{3}}) \cdot O(n) = o(n^{\frac{7}{3}}),$$

$$J_{n2}^{(4)} = o_p(n^{\frac{7}{6}}). \tag{4.11}$$

From (4.7), (4.8), (4.9), (4.10), and (4.11), we have

$$L_n^{(4)} = o_p(n^{\frac{3}{2}}). \tag{4.12}$$

For the estimation of $\frac{\partial^3 Q}{\partial \beta_l \partial \sigma \partial \theta}, \frac{\partial^3 Q}{\partial \beta_l \partial \theta^2}, \frac{\partial^3 Q}{\partial \sigma \partial \beta^T \partial \beta_l}, \frac{\partial^3 Q}{\partial \sigma \partial \beta^T \partial \sigma}, \cdots, \frac{\partial^3 Q}{\partial \theta^3}$, we can use the methods seminary to that in (4.4) or (4.12) to obtain (4.3).

Let $T_n(\lambda) = n^{-\frac{1}{2}} \frac{\partial Q}{\partial \lambda}$, $B_n(\lambda) = n^{-1} \frac{\partial^2 Q}{\partial \lambda^T \partial \lambda}$, we will prove that

$$T_n(\lambda_0 + n^{-\frac{1}{2}} r) = T_n(\lambda_0) + B_n(\lambda_0) r + p_n(r), \tag{4.13}$$

where $\sup_{\|r\| \leq K} \|p_n(r)\| \to 0$ as $n \to \infty$, for any $K > 0$.

In fact, by Taylor's theorem,

$$T_n(\lambda_0 + n^{-\frac{1}{2}} r) = T_n(\lambda_0) + B_n(\lambda_0) r + n^{-\frac{3}{2}} R_n(r, s), \tag{4.14}$$

where $R_n(r, s)$ has l-th element

$$R_{n,l}(r, s_l) = r^T[R_{nl}(\lambda_0 + n^{-\frac{1}{2}}s_l r)]r, \quad S_l \in [0, 1].$$

For $\|r\| \le K$, we have

$$|R_{n,l}(r, s_l)| \le K^2 \sum_{a,b=1}^{d+2} \left| \frac{\partial^2}{\partial\lambda_a \partial\lambda_b} \left(\frac{\partial}{\partial\lambda_l} (\bar{\lambda}) \right) \right|_{\bar{\lambda}=\lambda_0+n^{-\frac{1}{2}}s_l}.$$

Then by (4.3), $n^{-\frac{3}{2}}R_n(r, s) = o_p(1)$. So (4.13) holds.

From Lemma 2 and Lemma 3(a) in Douglas [4], we can obtain the result of Theorem 2.1 basing on the expansion (4.13).

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