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INVESTIGATIONS OF THE ESSENTIAL SPECTRUM OF A MODEL OPERATOR ASSOCIATED TO A SYSTEM OF THREE PARTICLES ON A LATTICE

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Abstract

We consider a model operator H associated with the system of three particles on a d-dimensional lattice that interact via nonlocal pair potentials. We describe the essential spectrum of H by the spectrum of two channel operators.

The location of the essential spectrum of the three-particle discrete Schrödinger operators has been extensively studied in many works, see, for example, [1, 2]. The main result of [1] is that the essential spectrum of the three-particle discrete Schrödinger operator consists of only finitely many bounded closed intervals, although the corresponding two-particle operators might posses infinitely many eigenvalues for some value of the two-particle quasi-momentum.

In the present paper, we consider a model operator H associated to a system of three particles on a d-dimensional lattice. The spectrum of channel operators are described by the spectrum of corresponding family of Friedrichs models. We establish that the essential spectrum of H is the

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union of the spectra of two channel operators. The upper bound for the number of closed intervals of the essential spectrum is found. We note that the operator H can be considered as a non-compact perturbation of the operators investigated in [3, 4].

Denote by $\mathbb{T}^d$ the d-dimensional torus and by $L_2((\mathbb{T}^d)^2)$ the Hilbert space of square integrable (complex) functions defined on $(\mathbb{T}^d)^2$.

Let us consider an operator H acting on the Hilbert space $L_2((\mathbb{T}^d)^2)$:

$$H := H_0 - V_1 - V_2,$$

where the operators H_0 and V_α, $\alpha = 1, 2$ are defined by

$$(H_0 f)(p, q) := w(p, q) f(p, q);$$

$$(V_1 f)(p, q) := \sum_{i=1}^{n} v_{1i}(q) \int_{\mathbb{T}^d} v_{1i}(s) f(p, s) ds,$$

$$(V_2 f)(p, q) := \sum_{i=1}^{n} v_{2i}(p) \int_{\mathbb{T}^d} v_{2i}(s) f(s, q) ds.$$

Here $f \in L_2((\mathbb{T}^d)^2)$, the number n is a positive integer with $n \geq 3$, the functions $v_{\alpha i}(\cdot)$, $\alpha = 1, 2$, $i = 1, \ldots, n$ and $w(\cdot, \cdot)$ are real-valued continuous functions on $\mathbb{T}^d$ and $(\mathbb{T}^d)^2$, respectively. Under these assumptions, the operator H is bounded and self-adjoint. We recall that the operators V_α, $\alpha = 1, 2$ are called nonlocal interaction operators.

Denote $w_1(p, q) := w(p, q)$ and $w_2(p, q) := w(q, p)$. To formulate main results of the paper, we introduce the channel operators acting on $L_2((\mathbb{T}^d)^2)$ as $H_\alpha := H_0 - V_\alpha$ and two families of bounded self-adjoint operators (Friedrichs models) $h_\alpha(p)$, $p \in \mathbb{T}^d$, which acts in $L_2(\mathbb{T}^d)$ as

$$(h_\alpha(p)f)(q) := w_\alpha(p, q)f(q) - \sum_{i=1}^{n} v_{\alpha i}(q) \int_{\mathbb{T}^d} v_{\alpha i}(s)f(s)ds.$$

For any fixed $p \in \mathbb{T}^d$, we define the analytic functions in $\mathbb{C} \setminus [m_\alpha(p);\ M_\alpha(p)]$ by

$$I_{ij}^{(\alpha)}(p; z) := \int_{\mathbb{T}^d} \frac{v_{\alpha i}(s)v_{\alpha j}(s)ds}{w_\alpha(p, s) - z}, \quad i, j = 1, \ldots, n;$$

$$\Delta_\alpha(p; z) := \det\left(\delta_{ij} - I_{ij}^{(\alpha)}(p; z)\right)_{i,j=1}^{n}, \quad \delta_{ij} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and the function $\Delta_\alpha(p; \cdot)$ is the Fredholm determinant associated with the operator $h_\alpha(p)$, where the numbers $m_\alpha(p)$ and $M_\alpha(p)$ are defined by

$$m_\alpha(p) := \min_{q \in \mathbb{T}^d} w_\alpha(p, q), \quad M_\alpha(p) := \max_{q \in \mathbb{T}^d} w_\alpha(p, q).$$

The following lemma describes the relation between the eigenvalues of the operator $h_\alpha(p)$ and zeroes of the function $\Delta_\alpha(p; \cdot)$.

Lemma 1. *The number* $z \in \mathbb{C} \setminus [m_\alpha(p); M_\alpha(p)]$ *is an eigenvalue of the operator* $h_\alpha(p)$ *if and only if* $\Delta_\alpha(p; z) = 0$.

We denote by $\sigma(\cdot)$, $\sigma_{\text{ess}}(\cdot)$, and $\sigma_{\text{disc}}(\cdot)$ the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator, respectively, and set

$$m := \min_{p, q \in \mathbb{T}^d} w(p, q), \quad M := \max_{p, q \in \mathbb{T}^d} w(p, q).$$

From Lemma 1, it follows that

$$\sigma_{\text{disc}}(h_\alpha(p)) = \{z \in \mathbb{C} \setminus [m_\alpha(p);\ M_\alpha(p)] : \Delta_\alpha(p; z) = 0\}.$$

Now we give the main results of the paper.

The spectrum of the channel operator H_α is described in the following theorem:

Theorem 1. *For the spectrum of the operator* H_α, *the equality holds*

$$\sigma(H_\alpha) = \bigcup_{p\in\mathbb{T}^d} \sigma_{\text{disc}}(h_\alpha(p)) \cup [m;\ M].$$

The following theorem describes the essential spectrum of the operator H.

Theorem 2. *The essential spectrum of* H *coincides with the union of the spectrum of* H_1 *and* H_2, *that is, the equality* $\sigma_{\text{ess}}(H) = \sigma(H_1) \cup \sigma(H_2)$ *holds. Moreover, the set* $\sigma_{\text{ess}}(H)$ *consists of no more than* $2n+1$ *bounded closed intervals.*

Comparing Theorems 1 and 2, we conclude that

$$\sigma_{\text{ess}}(H) = \bigcup_{\alpha=1}^{2} \bigcup_{p\in\mathbb{T}^d} \sigma_{\text{disc}}(h_\alpha(p)) \cup [m;\ M].$$

The subsets $[m;\ M]$ and

$$\bigcup_{\alpha=1}^{2} \bigcup_{p\in\mathbb{T}^d} \sigma_{\text{disc}}(h_\alpha(p))$$

of $\sigma_{\text{ess}}(H)$ are called three-particle and two-particle branches of the essential spectrum of H.

It is obvious that for a given H, the operators H_1 and H_2 are uniquely chooses by the property of decomposability into a direct integral. Since the channel operators H_1 and H_2 have a structure simpler than that of H, and Theorems 1 and 2 play an important role in further investigating the spectrum of the operator H.

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