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ON THE DYNAMICS OF THE RECURSIVE SEQUENCE $x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{2p} x_{n-4} + \gamma x_{n-2} x_{n-4}^{2q}}$

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Abstract

In this paper, we investigate the global behaviour of the difference equation $x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{2p} x_{n-4} + \gamma x_{n-2} x_{n-4}^{2q}}$, $n = 0, 1, \ldots$ with positive parameters and non-negative initial conditions.

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1. Introduction

Consider the fifth-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{2p} x_{n-4} + \gamma x_{n-2} x_{n-4}^{2q}}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where the parameters $\alpha, \beta,$ and γ are positive real numbers, p and q are positive integers, and the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}$ and x_0 are non-negative real numbers such that

$$\beta + \gamma x_{n-2}^{2p} x_{n-4} + \gamma x_{n-2} x_{n-4}^{2q} > 0, \quad n = 0, 1, \ldots.$$

We investigate the global asymptotic behaviour and the periodic character of the solutions of the difference equation (1.1), by generalizing the results due to M. Emre Erdogan et al. [1], corresponding to the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{2} x_{n-4} + \gamma x_{n-2} x_{n-4}^{2}}, \quad n = 0, 1, \ldots,$$

where the parameters $\alpha, \beta,$ and γ are positive real numbers and the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1},$ and x_0 are arbitrary non-negative real numbers. Similar recursive sequences were studied previously; for example, see [1-12].

The study of these equations is quite challenging and rewarding and is still in its infancy. We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore, that results about such equations offer prototypes for the development of the basic theory of the global behaviour of nonlinear difference equations.

We need the following definitions and theorem [13]:

Definition 1. Let I be an interval of real numbers and $f : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \tag{1.2}$$

with $x_{-k}, \dots, x_0 \in I$. Let $\bar{x}$ be the equilibrium point of Equation (1.2). The linearized equation of Equation (1.2) about the equilibrium point $\bar{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + \cdots + c_{k+1} y_{n-k}, \quad n = 0, 1, \dots, \tag{1.3}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\bar{x}, \dots, \bar{x}),\ c_2 = \frac{\partial f}{\partial x_{n-1}}(\bar{x}, \dots, \bar{x}), \dots, c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\bar{x}, \dots, \bar{x}).$$

The characteristic equation of Equation (1.3) is

$$\lambda^{k+1} - c_1\lambda^k - c_2\lambda^{k-1} - \cdots - c_k\lambda - c_{k+1} = 0. \tag{1.4}$$

Definition 2. Let $\bar{x}$ be the equilibrium point of Equation (1.2).

(1) The equilibrium $\bar{x}$ is called *locally stable* if for every $\epsilon > 0$, there exists $\delta > 0$, such that if $x_{-k}, \dots, x_0 \in I$, and $|x_0 - \bar{x}| + \cdots + |x_{-k} - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$, for all $n \geq -k$.

(2) The equilibrium $\bar{x}$ is called *locally asymptotically stable* if it is locally stable and if there exists $\gamma > 0$, such that if $x_0, \dots, x_{-k} \in I$ and $|x_0 - \bar{x}| + \cdots + |x_{-k} - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.

(3) The equilibrium $\bar{x}$ is called a *global attractor* if for every $x_0, \dots, x_{-k} \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(4) The equilibrium $\bar{x}$ is called *globally asymptotically stable* if it is locally stable and is a global attractor.

Definition 3. A solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1.2) is called *nonoscillatory*, if there exists $N \geq -k$ such that either $x_n > \bar{x}, \forall n \geq N$ or $x_n < \bar{x}, \forall n \geq N,$ and it is called *oscillatory* if it is not nonoscillatory.

Definition 4. A positive semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all greater than or equal to the equilibrium $\bar{x}$ with $l \geq -k$ and $m \leq \infty$ such that either $l = -k$ or $l > -k$ and $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} < \bar{x}$. A negative semicycle of a solution $\{x_n\}_{n=-k}^{\infty}$ of Equation (1.2) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$, all less than $\bar{x}$ with $l \geq -k$ and $m \leq \infty$ such that either $l = -k$ or $l > -k$ and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \bar{x}$.

Theorem 1. (i) *If all roots of Equation* (1.4) *have absolute value less than one*, *then the equilibrium point* $\bar{x}$ *of Equation* (1.2) *is locally asymptotically stable*.

(ii) *If at least one of the roots of Equation* (1.4) *has absolute value greater than one*, *then the equilibrium point* $\bar{x}$ *of Equation* (1.2) *is unstable*.

2. Dynamics of Equation (1.1)

In this section, we investigate the dynamics of Equation (1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\beta / \gamma) y_n$ reduces Equation (1.1) to the difference equation:

$$y_{n+1} = \frac{r y_{n-1}}{1 + A y_{n-2}^{2p} y_{n-4} + C y_{n-2} y_{n-4}^{2q}}, \quad n = 0, 1, \ldots, \tag{2.1}$$

where $r = \alpha / \beta > 0$, $A = (\beta / \gamma)^{2p} > 0$, and $C = (\beta / \gamma)^{2q} > 0$.

Note that $\bar{y}_1 = 0$ is always an equilibrium point of Equation (2.1). When $r > 1$, Equation (2.1) also possesses equilibrium point satisfy this equation:

$$A\bar{y}_2^{2p+1} + C\bar{y}_2^{2q+1} = r - 1, \tag{2.2}$$

since the powers $2p + 1$ and $2q + 1$ are odd, then Equation (2.2) has at least one positive equilibrium point $\bar{y}_2 > 0$.

Theorem 2. *The following statements are true*:

(i) *If* $r < 1$, *then the equilibrium point* $\bar{y}_1 = 0$ *of Equation* (2.1) *is locally asymptotically stable.*

(ii) *If* $r > 1$, *then the equilibrium point* $\bar{y}_1 = 0$ *of Equation* (2.1) *is an unstable* (*saddle point*).

(iii) *When* $r > 1$, *then the positive equilibrium point* $\bar{y}_2$ *of Equation* (2.1) *is unstable.*

Proof. The linearized equation of Equation (2.1) about the equilibrium point $\bar{y}_1 = 0$ is

$$z_{n+1} = rz_{n-1}, \quad n = 0, 1, \dots,$$

so the characteristic equation of Equation (2.1) about the equilibrium point $\bar{y}_1 = 0$ is $\lambda^5 - r\lambda^3 = 0$, and hence the proof of (i) and (ii) follows from Theorem 1.

For (iii), we assume that $r > 1$; then the linearized equation of Equation (2.1) about the equilibrium point $\bar{y}_2$ has the form

$$z_{n+1} = z_{n-1} - \left[\frac{(r-1) + (2p-1)A\bar{y}^{2p+1}}{r}\right]z_{n-2}$$

$$- \left[\frac{(r-1) + (2q-1)C\bar{y}^{2q+1}}{r}\right]z_{n-4}, \quad n = 0, 1, \dots.$$

So, the characteristic equation of Equation (2.1) about the equilibrium point $\overline{y}_2$ is

$$\lambda^5 - \lambda^3 + \left[\frac{(r-1)+(2p-1)A\overline{y}^{2p+1}}{r}\right]\lambda^2 + \left[\frac{(r-1)+(2q-1)C\overline{y}^{2q+1}}{r}\right] = 0. \tag{2.3}$$

It is clear that Equation (2.3) has a root in the interval $(-\infty, -1)$ and so equilibrium point $\overline{y}_2$ is an unstable equilibrium point. This completes the proof. □

Theorem 3. *Assume that* $r > 1$, *and let* $\{y_n\}_{n=-4}^{\infty}$ *be a solution of Equation* (2.1) *such that*

$$y_{-4},\ y_{-2},\ y_0 \geq \overline{y}_2 \text{ and } y_{-3},\ y_{-1} < \overline{y}_2, \tag{2.4}$$

or

$$y_{-4},\ y_{-2},\ y_0 < \overline{y}_2 \text{ and } y_{-3},\ y_{-1} \geq \overline{y}_2. \tag{2.5}$$

Then $\{y_n\}_{n=-4}^{\infty}$ *oscillates about* $\overline{y}_2$ *with a semicycle of length one.*

Proof. Assume that (2.4) holds. (The case where (2.5) holds is similar and will be omitted.) Then,

$$y_1 = \frac{ry_{-1}}{1 + Ay_{-2}^{2p}y_{-4} + Cy_{-2}y_{-4}^{2q}} < \frac{r\overline{y_2}}{1 + A\overline{y}_2^{2p+1} + C\overline{y}_2^{2q+1}} = \frac{r\overline{y_2}}{1 + r - 1} = \overline{y}_2,$$

and

$$y_2 = \frac{ry_0}{1 + Ay_{-1}^{2p}y_{-3} + Cy_{-1}y_{-3}^{2q}} \geq \frac{r\overline{y_2}}{1 + A\overline{y}_2^{2p+1} + C\overline{y}_2^{2q+1}} = \frac{r\overline{y_2}}{1 + r - 1} = \overline{y}_2,$$

and then the proof follows by induction. □

Theorem 4. *Assume that* $r < 1$; *then the equilibrium point* $\bar{y}_1 = 0$ *of Equation* (2.1) *is globally asymptotically stable.*

Proof. We know by Theorem 2 that the equilibrium point $\bar{y}_1 = 0$ of Equation (2.1) is locally asymptotically stable. So, it suffices to show that $\lim_{n\to\infty} y_n = 0$. Let $\{y_n\}_{n=-4}^{\infty}$ be a solution of Equation (2.1), then

$$0 \leq y_{n+1} = \frac{ry_{n-1}}{1 + Ay_{n-2}^{2p}y_{n-4} + Cy_{n-2}y_{n-4}^{2q}} < ry_{n-1} < y_{n-1},$$

which implies that $\lim_{n\to\infty} y_n = 0$. This completes the proof. □

Theorem 5. *Assume that* $r = 1$; *then Equation* (2.1) *possesses the prime period two solutions*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots,$$

with $\Phi > 0$. *Furthermore, every solution of Equation* (2.1) *converges to period two solutions* (2.5) *with* $\Phi \geq 0$.

Proof. Let

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots,$$

be period two solutions of Equation (2.1). Then

$$\Phi = \frac{r\Phi}{1 + A\Psi^{2p+1} + C\Psi^{2q+1}} \quad \text{and} \quad \Psi = \frac{r\Psi}{1 + A\Phi^{2p+1} + C\Phi^{2q+1}}.$$

So

$$\Phi\Psi = \frac{(r-1)(\Phi - \Psi)}{A(\Psi^{2p} - \Phi^{2p}) + C(\Psi^{2p} - \Phi^{2p})} \geq 0,$$

which implies that $r - 1 \leq 0$. If $r < 1$, then this implies that $\Phi < 0$ or $\Psi < 0$, which is impossible, so $r = 1$. To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-4}^{\infty}$ be a solution of Equation (2.1); then

$$y_{n+1} - y_{n-1} = \frac{-Ay_{n-1}y_{n-2}^{2p}y_{n-4} - Cy_{n-1}y_{n-2}y_{n-4}^{2q}}{1 + Ay_{n-2}^{2p}y_{n-4} + Cy_{n-2}y_{n-4}^{2q}} \leq 0.$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$) and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus,

$$\Phi = \frac{\Phi}{1 + A\Psi^{2p+1} + C\Psi^{2q+1}} \quad \text{and} \quad \Psi = \frac{\Psi}{1 + A\Phi^{2p+1} + C\Phi^{2q+1}},$$

which implies that

$$\Phi\Psi = 0.$$

This completes the proof. □

Theorem 6. *Assume that* $r > 1$; *then Equation* (2.1) *possesses an unbounded solution.*

Proof. From Theorem 3, we can assume without loss of generality that the solution $\{y_n\}_{n=-4}^{\infty}$ of Equation (2.1) is such that

$$y_{2n-1} < \overline{y}_2 \text{ and } y_{2n} > \overline{y}_2, \text{ for } n \geq 0.$$

Then

$$y_{2n+2} = \frac{ry_{2n}}{1 + Ay_{2n-1}^{2p}y_{2n-3} + Cy_{2n-1}y_{2n-3}^{2q}} > \frac{ry_{2n}}{1 + (r-1)} = y_{2n},$$

and

$$y_{2n+3} = \frac{ry_{2n+1}}{1 + Ay_{2n}^{2p}y_{2n-2} + Cy_{2n}y_{2n-2}^{2q}} < \frac{ry_{2n+1}}{1 + (r-1)} = y_{2n+1},$$

from which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \text{ and } \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete. □

Remark. If $p = 1 = q$, the results in [1] follow directly.

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