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EXACT SOLUTIONS FOR A NEW INTEGRABLE SYSTEM (KDV6)

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Abstract

We obtain exact solutions for the new integrable system

$$\begin{cases} u_t - 6uu_x - u_{xxx} + w_x = 0 \\ w_{xxx} + 4uw_x + 2u_x w = 0, \end{cases}$$

which has been derived from the new integrable sixth-order KdV equation $(\text{KdV6})(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0$. Using the generalized tanh- coth method, periodic and soliton solutions are formally derived for it.

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1. Introduction

The sixth-order wave equation (KdV6):

$$(\partial_x^3 + 8u_x\partial_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0, \tag{1.1}$$

has been derived recently by the five authors of [4] as a integrable particular case of the general sixth-order nonlinear wave equation

$$\begin{cases} u_{xxxxxx} + \alpha u_x u_{xxxx} + \beta u_{xx} u_{xxx} + \gamma u_x^2 u_{xx} \\ + \delta u_{tt} + \rho u_{xxxt} + \omega u_x u_{xt} + \sigma u_t u_{xx} = 0, \end{cases} \tag{1.2}$$

where α, β, γ, δ, ρ, ω, σ are arbitrary parameters, and $u = u(x, t)$, is a differentiable function. There are some reasons for which the study of the model given by Eq. (1.2) is important. For instance, the two authors of [2], have derived two integrable nonlinear integro-differential equations: the first equation

$$\begin{cases} 5\partial_x^{-1} v_{tt} + 5v_{xxt} - 15vv_t - 15v_x\partial_x^{-1} v_t - 45v^2 v_x \\ + 15v_x v_{xx} + 15vv_{xxx} - v_{xxxxx} = 0, \end{cases} \tag{1.3}$$

describes the propagation of waves in two opposite directions and represent bi-directional version of Caudrey et al. and Sawada and Kotera equation [1], [8], and after the potential transformation

$$v(x, t) = u_x(x, t), \tag{1.4}$$

reduces to sixth-order KdV equation

$$\begin{cases} 5u_{tt} + 5u_{xxxt} - 15u_x u_{xt} - 15u_{xx} u_t \\ - 45u_x^2 u_{xx} + 15u_{xx} u_{xxx} + 15u_x u_{xxxx} - u_{xxxxxx} = 0. \end{cases} \tag{1.5}$$

The second equation

$$\begin{cases} 5\partial_x^{-1} v_{tt} + 5v_{xxt} - 15vv_t - 15v_x\partial_x^{-1} v_t - 45v^2 v_x \\ + \dfrac{75}{2} v_x v_{xx} + 15vv_{xxx} - v_{xxxxx} = 0, \end{cases} \tag{1.6}$$

represent bidirectional version of the Kaup-Kupershmidt equation [6], and using Eq. (1.4) reduces to

$$\begin{cases} 5u_{tt} - 5u_{xxxt} - 15u_x u_{xt} - 15u_{xx} u_t \\ - 45u_x^2 u_{xx} + \dfrac{75}{2} u_{xx} u_{xxx} + 15u_x u_{xxxx} - u_{xxxxxx} = 0. \end{cases} \tag{1.7}$$

Other important particular case of the Eq. (1.2) which has been studied by several authors [5], [9] is given by

$$\begin{cases} u_{tt} - u_{xxxt} - 2u_{xxxxxx} + 18u_x u_{xxxx} \\ + 36u_{xx} u_{xxx} - 36u_x^2 u_{xx} = 0. \end{cases} \tag{1.8}$$

The five authors of [4], have found four cases in which Eq. (1.2) pass the Painlevé test. One of those cases is the Eq. (1.1), which is equivalent to the Korteweg-de Vries equation with a source satisfying a third-order ordinary differential equation (see [4]):

$$\begin{cases} v_t + v_{xxx} + 12vv_x - w_x = 0, \\ w_{xxx} + 8vw_x + 4wv_x = 0, \end{cases} \tag{1.9}$$

where

$$\begin{cases} v = u_x, \\ w = u_t + u_{xxx} + 6u_x^2. \end{cases} \tag{1.10}$$

Recently, Kupershmidt [7] has showed that the Eq. (1.9) is integrable in the usual sense, which is a truly remarkable fact, since $w = 0$ leaves only the unperturbed KdV itself. It is easy to see that with the change of variable

$$\begin{cases} v(x, t) = \dfrac{1}{2} u(x, -t), \\ w(x, t) = \dfrac{1}{2} w(x, -t), \end{cases} \tag{1.11}$$

the system given by Eq. (1.9) reduces to new integrable system (see [7])

$$\begin{cases} u_t - 6uu_x - u_{xxx} + w_x = 0, \\ w_{xxx} + 4uw_x + 2u_x w = 0. \end{cases} \tag{1.12}$$

The objective of this work, is to present exact solutions to the system given by Eq.(1.12), therefore solutions for the (KdV6) (Eq. (1.1)), and in

this way, to further complement other studies related to this new KdV6 equation (Eq. (1.1)). For this, we will use the generalized tanh-coth method [3]. This paper is organized as follows: In Section 2, we will review briefly the generalized tanh-coth method [3]. In Section 3, we give the mathematical framework to search exact solutions for the system given by Eq. (1.12). Finally, some conclusions are given.

2. The Generalized tanh-coth Method

The wave transformation

$$\xi = x + \lambda t + \xi_0, \tag{2.1}$$

convert a PDE that does not explicitly involve independent variables to an ODE

$$P(v, v', v'', ...) = 0. \tag{2.2}$$

Using the idea of the tanh-coth method introduced by Wazwaz [10], the generalized tanh-coth method admits the use of a finite expansion

$$\sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{M+1}^{2M} a_i \phi(\xi)^{M-i}, \tag{2.3}$$

where $\phi(\xi)$ satisfies the Riccati equation

$$\phi'(\xi) = \mu(\phi(\xi)^2 + k), \tag{2.4}$$

k, μ are constants, whose solutions are given by

$$\phi(\xi) = \begin{cases} -\dfrac{1}{\mu\xi}, & k = 0 \\ \sqrt{k}\tan(\mu\sqrt{k}\xi) & k > 0 \\ -\sqrt{k}\cot(\mu\sqrt{k}\xi) & k > 0 \\ -\sqrt{-k}\tanh(\mu\sqrt{-k}\xi) & k < 0 \\ -\sqrt{-k}\coth(\mu\sqrt{-k}\xi) & k < 0, \end{cases} \tag{2.5}$$

and where M is a positive integer that will be determined. Substituting Eq. (2.3) into Eq. (2.2) and using Eq. (2.4), results in an algebraic equation in powers of $\phi(\xi)$. Balancing the linear terms of highest order in

the resulting equation with the highest order nonlinear term to obtain M, will yield a set of algebraic equations for k, λ, μ, a_1, ..., a_{2M} because all coefficients of $\phi(\xi)^i (i = 1, 2, ..., 2M)$ have to vanish.

3. Exact Solutions to New System (Eq. (1.12))

Using the traveling wave transformation

$$\begin{cases} u(x, t) = v(\xi), \\ w(x, t) = w(\xi), \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{3.1}$$

the system given by Eq. (1.12) reduces to

$$\begin{cases} (\lambda v(\xi) - 3v^2(\xi) - v''(\xi) + w(\xi))' = 0, \\ w'''(\xi) + 4v(\xi)w'(\xi) + 2v'(\xi)w(\xi) = 0. \end{cases} \tag{3.2}$$

Integrating the first equation in Eq. (3.2) and setting the constant of integration to zero we obtain

$$w(\xi) = -\lambda v(\xi) + 3v^2(\xi) + v''(\xi). \tag{3.3}$$

Substituting Eq. (3.3) into second equation in Eq. (3.2), and after other integration and setting the constant of integration to zero we obtain

$$-\lambda v''(\xi) + 5(v'(\xi))^2 + 10v(\xi)v''(\xi) - 3\lambda v^2(\xi) + 10v^3(\xi) + v'''(\xi) = 0. \tag{3.4}$$

Balancing $v(\xi)v''(\xi)$ with $v^3(\xi)$ we find

$$2M + 2 = 3M \tag{3.5}$$

so that

$$M = 2. \tag{3.6}$$

According with the generalized tanh-coth method, we seek solutions to Eq. (3.4) using the finite expansion

$$\sum_{i=0}^{2} a_i \phi(\xi)^i + \sum_{3}^{4} a_i \phi(\xi)^{2-i}. \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.4), using Eq. (2.4) and solving the resulting system respect to unknown variables a_0, a_1, a_2, a_3, a_4, λ, μ and k, we obtain the following sets of distinct solutions of parameters: (in all cases, $a_1 = a_3 = 0$)

$$\begin{cases} a_0 = \dfrac{\lambda}{2}, \\ a_2 = 0, \\ a_4 = -\dfrac{\lambda^2}{8\mu^2}, \\ k = -\dfrac{\lambda}{4\mu^2}, \end{cases} \tag{3.8}$$

$$\begin{cases} a_0 = \dfrac{(\pm 25 - 7\sqrt{10})\lambda}{10(\mp 10 + \sqrt{10})}, \\ a_0 = 0, \\ a_4 = \dfrac{(-11 \pm 2\sqrt{10})\lambda^2}{80\mu^2}, \\ k = \dfrac{(10 \mp \sqrt{10})\lambda}{40\mu^2}, \end{cases} \tag{3.9}$$

$$\begin{cases} a_0 = \dfrac{\lambda}{2}, \\ a_2 = -2\mu^2, \\ a_4 = 0, \\ k = -\dfrac{\lambda}{4\mu^2}, \end{cases} \tag{3.10}$$

$$\begin{cases} a_0 = \dfrac{(-4 \pm \sqrt{10})\lambda}{20}, \\ a_2 = -2\mu^2, \\ a_4 = 0, \\ k = \dfrac{(10 \mp \sqrt{10})\lambda}{40\mu^2}, \end{cases} \tag{3.11}$$

$$\begin{cases} a_0 = \dfrac{\lambda}{4}, \\ a_2 = -2\mu^2, \\ a_4 = -\dfrac{\lambda}{128\mu^2}, \\ k = -\dfrac{\lambda}{16\mu^2}, \end{cases} \tag{3.12}$$

and

$$\begin{cases} a_0 = \dfrac{(2 \pm \sqrt{10})\lambda}{40}, \\ a_2 = -2\mu^2, \\ a_4 = \dfrac{(-11 \pm 2\sqrt{10})\lambda}{1280\mu^2}, \\ k = \dfrac{(10 \mp \sqrt{10})\lambda}{160\mu^2}, \end{cases} \tag{3.13}$$

where λ and μ are selected as free parameters.

Consequently, using Eqs. (3.7), (3.3), (3.1) and (2.5) we consider only the following sets of solutions to new system (Eq. (1.12)):

with respect to Eq. (3.8)

$$\begin{cases} \begin{cases} u_1(x,\,t) = \dfrac{\lambda}{2} + \dfrac{\lambda}{2}\tan^2\left[\dfrac{1}{2}\sqrt{-\lambda}(x + \lambda t + \xi_0)\right], \\ w_1(x,\,t) = 0, \end{cases} \\ \begin{cases} u_2(x,\,t) = \dfrac{\lambda}{2} + \dfrac{\lambda}{2}\cot^2\left[\dfrac{1}{2}\sqrt{-\lambda}(x + \lambda t + \xi_0)\right], \\ w_2(x,\,t) = 0, \end{cases} \\ \begin{cases} u_3(x,\,t) = \dfrac{\lambda}{2} - \dfrac{\lambda}{2}\tanh^2\left[\dfrac{1}{2}\sqrt{\lambda}(x + \lambda t + \xi_0)\right], \\ w_3(x,\,t) = 0, \end{cases} \\ \begin{cases} u_4(x,\,t) = \dfrac{\lambda}{2} - \dfrac{\lambda}{2}\coth^2\left[\dfrac{1}{2}\sqrt{\lambda}(x + \lambda t + \xi_0)\right], \\ w_4(x,\,t) = 0, \end{cases} \end{cases} \tag{3.14}$$

with respect to Eq. (3.9)

$$
\begin{cases}
\begin{cases}
u_5(x,t)=\dfrac{(25-7\sqrt{10})\lambda}{10(-10+\sqrt{10})}+\dfrac{(-11+2\sqrt{10})\lambda}{2(10-\sqrt{10})}\cot^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right],\\
w_5(x,t)=-\dfrac{3\lambda^2}{20(-10+\sqrt{10})^2}\\
\left\{(-179+62\sqrt{10}+(-11+2\sqrt{10})\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right.\\
\left.\csc^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases}\\
\begin{cases}
u_6(x,t)=\dfrac{(25-7\sqrt{10})\lambda}{10(-10+\sqrt{10})}+\dfrac{(-11+2\sqrt{10})\lambda}{2(10-\sqrt{10})}\tan^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right],\\
w_6(x,t)=\dfrac{3\lambda^2}{20(-10+\sqrt{10})^2},\\
\left\{\left(179-62\sqrt{10}+(-11+2\sqrt{10})\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right)\right.\\
\left.\sec^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases}\\
\begin{cases}
u_7(x,t)=\dfrac{(-25-7\sqrt{10})\lambda}{10(10+\sqrt{10})}-\dfrac{(-112\sqrt{10})\lambda}{2(10+\sqrt{10})}\coth^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right],\\
w_7(x,t)=-\dfrac{3\lambda^2}{20(-10+\sqrt{10})^2}\\
\left\{\left(179+62\sqrt{10}+(11+2\sqrt{10})\cosh\left[\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right)\right.\\
\left.\operatorname{csch}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases}\\
\begin{cases}
u_8(x,t)=\dfrac{(-25-7\sqrt{10})\lambda}{10(10+\sqrt{10})}-\dfrac{(-112\sqrt{10})\lambda}{2(10+\sqrt{10})}\tanh^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right],\\
w_8(x,t)=-\dfrac{3\lambda^2}{20(10+\sqrt{10})^2}\\
\left\{\left(-179-62\sqrt{10}+(11+2\sqrt{10})\cosh\left[\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right)\right.\\
\left.\operatorname{sech}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases}
\end{cases}
\tag{3.15}
$$

with respect to Eq. (3.11)

$$
\begin{cases}
\begin{cases}
u_9(x,\,t) = \dfrac{(-4+\sqrt{10})\lambda}{20} - \dfrac{(10-\sqrt{10})\lambda}{20}\tan^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right], \\
w_9(x,\,t) = -\dfrac{3\lambda^2}{200}\left(-9+4\sqrt{10}+\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right) \\
\sec^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right],
\end{cases} \\
\begin{cases}
u_{10}(x,\,t) = \dfrac{(-4+\sqrt{10})\lambda}{20} - \dfrac{(10-\sqrt{10})\lambda}{20}\cot^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right], \\
w_{10}(x,\,t) = \dfrac{3\lambda^2}{200}\left(9-4\sqrt{10}+\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right) \\
\csc^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right],
\end{cases} \\
\begin{cases}
u_{11}(x,\,t) = \dfrac{(-4+\sqrt{10})\lambda}{20} + \dfrac{(10-\sqrt{10})\lambda}{20}\tanh^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right], \\
w_{11}(x,\,t) = -\dfrac{3\lambda^2}{200}\left(-9+4\sqrt{10}+\cosh\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right) \\
\operatorname{sech}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right],
\end{cases} \\
\begin{cases}
u_{12}(x,\,t) = \dfrac{(-4+\sqrt{10})\lambda}{20} + \dfrac{(10-\sqrt{10})\lambda}{20}\coth^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right], \\
w_{12}(x,\,t) = -\dfrac{3\lambda^2}{200}\left(9-4\sqrt{10}+\cosh\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right) \\
\operatorname{csch}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right],
\end{cases}
\end{cases}
\tag{3.16}
$$

with respect to Eq. (3.12)

$$\begin{cases} \begin{cases} u_{13}(x,\,t) = \dfrac{\lambda}{4} + \dfrac{\lambda}{8}\tan^2\left[\dfrac{1}{4}\sqrt{-\lambda}(x + \lambda t + \xi_0)\right] \\ + \dfrac{\lambda}{8}\cot^2\left[\dfrac{1}{4}\sqrt{-\lambda}(x + \lambda t + \xi_0)\right], \\ w_{13}(x,\,t) = 0, \end{cases} \\ \begin{cases} u_{14}(x,\,t) = \dfrac{\lambda}{4} - \dfrac{\lambda}{8}\tanh^2\left[\dfrac{1}{4}\sqrt{\lambda}(x + \lambda t + \xi_0)\right] \\ - \dfrac{\lambda}{8}\coth^2\left[\dfrac{1}{4}\sqrt{\lambda}(x + \lambda t + \xi_0)\right], \\ w_{14}(x,\,t) = 0, \end{cases} \end{cases} \tag{3.17}$$

and with respect to Eq. (3.13)

$$
\begin{cases}
\begin{cases}
u_{15}(x,t) = \dfrac{(2+\sqrt{10})\lambda}{40} - \dfrac{(10-\sqrt{10})\lambda}{80}\tan^2\left[\dfrac{1}{4}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right] \\
+\dfrac{(-11+2\sqrt{10})\lambda}{8(10-\sqrt{10})}\cot^2\left[\dfrac{1}{4}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right], \\
w_{15}(x,t) = -\dfrac{3\lambda^2}{20(-10+\sqrt{10})^2} \\
\left\{\left(-179+62\sqrt{10}+(-11+2\sqrt{10})\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right)\right. \\
\left.\csc^2\left[\dfrac{1}{2}\sqrt{\dfrac{10\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases} \\
\begin{cases}
u_{16}(x,t) = \dfrac{(2+\sqrt{10})\lambda}{40} - \dfrac{(10-\sqrt{10})\lambda}{80}\cot^2\left[\dfrac{1}{4}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right] \\
+\dfrac{(-11+2\sqrt{10})\lambda}{8(10-\sqrt{10})}\tan^2\left[\dfrac{1}{4}\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right], \\
w_{16}(x,t) = -\dfrac{3\lambda^2}{20(-10+\sqrt{10})^2} \\
\left\{\left(-179+62\sqrt{10}+(-11+2\sqrt{10})\cos\left[\sqrt{\dfrac{10-\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right)\right. \\
\left.\csc^2\left[\dfrac{1}{2}\sqrt{\dfrac{10\sqrt{10}}{10}}\sqrt{\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases} \\
\begin{cases}
u_{17}(x,t) = \dfrac{(2-\sqrt{10})\lambda}{40} + \dfrac{(10+\sqrt{10})\lambda}{80}\tanh^2\left[\dfrac{1}{4}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right] \\
-\dfrac{(-11-2\sqrt{10})\lambda}{8(10+\sqrt{10})}\coth^2\left[\dfrac{1}{4}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right], \\
w_{17}(x,t) = -\dfrac{3\lambda^2}{20(10+\sqrt{10})^2}, \\
\left\{\left(179+62\sqrt{10}+(11+2\sqrt{10})\cosh\left[\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right)\right. \\
\left.\operatorname{csch}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right\},
\end{cases} \\
\begin{cases}
u_{18}(x,t) = \dfrac{(2-\sqrt{10})\lambda}{40} + \dfrac{(10+\sqrt{10})\lambda}{80}\coth^2\left[\dfrac{1}{4}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right] \\
-\dfrac{(-11-2\sqrt{10})\lambda}{8(10+\sqrt{10})}\tanh^2\left[\dfrac{1}{4}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right], \\
w_{18}(x,t) = -\dfrac{3\lambda^2}{20(10+\sqrt{10})^2} \\
\left\{\left(179+62\sqrt{10}+(11+2\sqrt{10})\cosh\left[\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right)\right. \\
\left.\operatorname{csch}^2\left[\dfrac{1}{2}\sqrt{\dfrac{10+\sqrt{10}}{10}}\sqrt{-\lambda}(x+\lambda t+\xi_0)\right]\right\}.
\end{cases}
\end{cases}
\tag{3.18}
$$

Remark. The solutions corresponding to Eq. (3.10) are the same as given in Eq. (3.14).

4. Conclusions

In this paper, we have derived a lot of different solutions to new integrable system given by Eq. (1.12) from which solutions to KdV6 equation (Eq. (1.1)) can be derived in a more general form than the soliton solution of the potential Korteweg-de Vries equation. Using the generalized tanh-coth method [3], periodic and soliton solutions have been derived. The method used here, can be considered as a powerful technique to analyze several forms of the generalized KdV6 equation (Eq. (1.2)).

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