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ON THE POINTS AT DIFFERENT DISTANCES FROM ALL LATTICE POINTS

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Abstract

Schoenberg has given a characterization of the set of points at different distances from all rational lattice points. It was motivated from a solution of Sierpiński on a problem of Steinhaus, and the problem was originally asked to the integral lattice points. In this article, we study how far Schoenberg's method can be applied to the points at different distances from all integral lattice points, and show a complete description in case of the plane.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the sets of the integers, the rational numbers, the real numbers and the irrational numbers, respectively. We identify an n-dimensional vector $\boldsymbol{x} = (x_1, \cdots, x_n)$ with a point in the n-dimensional Euclidean space $\mathbb{R}^n$, and often denote it by (x_i). Then, $\mathbb{Z}^n = \{(a_i) \in \mathbb{R}^n \mid a_i \in \mathbb{Z}\}$ (resp., $\mathbb{Q}^n = \{(q_i) \in \mathbb{R}^n \mid q_i \in \mathbb{Q}\}$) is the set

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of integral (resp., rational) lattice points. Let $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$ be the inner product of vectors $\boldsymbol{u} = (u_i)$ and $\boldsymbol{v} = (v_i)$ in $\mathbb{R}^n$. Then, the distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by $\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}$. We shall only consider points in $\mathbb{R}^n$ with $n \geq 2$, and assume $n \geq 2$ throughout the article.

A problem of Steinhaus asks if, for an arbitrarily given integer $k > 0$, there is a circle that encircles just k numbers of integral lattice points in the plane (cf. [1]). Sierpiński (cf. [3]) answered the problem by showing that the point $\boldsymbol{p} = (1/3, \sqrt{2}) \in \mathbb{R}^2$ is at different distances from all integral points in the plane. Thus, the required circle is obtained taking the center at $\boldsymbol{p}$ and an appropriate radius. Then, one can ask what is the set of points which qualify as the centers of such circles just like $\boldsymbol{p}$, and also how about the points in $\mathbb{R}^n$. We can call such points at different distances from all lattice points *the Sierpiński points*, and the problem to determine the set of the Sierpiński points *the revised problem*. The purpose of this paper is to study the revised problem, and in case of $\mathbb{R}^2$ we give a complete description for the set of the Sierpiński points.

Schoenberg [2] has given a result to the revised problem for the rational lattice points $\mathbb{Q}^n$ instead of $\mathbb{Z}^n$. His result is quite elegant, and seems to be applied even to the integral lattice. We recall it briefly using our notations. Let $\prod_n$ be the set of rational hyperplanes in $\mathbb{R}^n$, that is,

$$\prod\nolimits_n = \{\pi : \boldsymbol{q} \cdot \boldsymbol{x} = r \mid \boldsymbol{q} \in \mathbb{Q}^n \text{ with } \boldsymbol{q} \neq 0,\ r \in \mathbb{Q}\}.$$

Notice that, multiplying a common multiple of the denominators, we can replace $\boldsymbol{q}$ and r in the definition of $\prod_n$ by some $c \in \mathbb{Z}^n$ and $d \in \mathbb{Z}$, respectively. We set

$$\Phi_n = \bigcup_{\pi \in \Pi_n} \pi \quad \text{and} \quad \Omega_n = \mathbb{R}^n \setminus \Phi_n.$$

Then, the following has been shown.

Theorem 1.1 (Schoenberg)**.** *The set* Ω_n *is equal to the set of points at different distances from all points in* $\mathbb{Q}^n$.

Returning to the integral lattice points, we set

$$S_n = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| \neq \|\boldsymbol{x} - \boldsymbol{b}\| \text{ for any } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n \text{with } \boldsymbol{a} \neq \boldsymbol{b}\};$$

$$T_n = \mathbb{R}^n - S_n$$

$$= \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| = \|\boldsymbol{x} - \boldsymbol{b}\| \text{ for some } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n \text{ with } \boldsymbol{a} \neq \boldsymbol{b}\}.$$

Thus, the set of the Sierpiński points is S_n, and our task is to characterize the sets S_n and T_n. Obviously, we have the following corollary of Theorem 1.1:

Corollary 1.2. $S_n \supset \Omega_n$ *and* $T_n \subset \Phi_n$.

As the first step, we determine the clear parts of T_n and S_n. For that purpose, we call an integral vector $\boldsymbol{f} = (f_i) \in \mathbb{Z}^n$ *a fundamental vector* if $\boldsymbol{f} \neq 0$ and $\gcd\{f_1, \cdots, f_n\} = 1$, where $\gcd\{- -\}$ denotes the greatest common divisor of integers in $\{- -\}$. By a basic property of integer, for any fundamental vector $\boldsymbol{f} = (f_i)$ there exists an integral vector $\boldsymbol{w} = (w_i) \in \mathbb{Z}^n$ with the property that

$$\boldsymbol{f} \cdot \boldsymbol{w} = \sum_{i=1}^{n} f_i w_i = 1. \tag{1}$$

We denote by F_n the set of fundamental vectors in $\mathbb{Z}^n$. Then, we can divide $\mathbb{R}^n$ into three subsets defined by

$$G_{n,1} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{f} \cdot \boldsymbol{x} \in \mathbb{Z} \text{ for } \exists \boldsymbol{f} \in F_n\};$$

$$G_{n,2} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{f} \cdot \boldsymbol{x} \in \mathbb{I} \text{ for } \forall \boldsymbol{f} \in F_n\};$$

$$G_{n,3} = \mathbb{R}^n \setminus (G_{n,1} \cup G_{n,2})$$

$$= \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{f} \cdot \boldsymbol{x} \in \mathbb{Q} \text{ for } \exists \boldsymbol{f} \in F_n \text{ and } \boldsymbol{g} \cdot \boldsymbol{x} \notin \mathbb{Z} \text{ for } \forall \boldsymbol{g} \in F_n\}.$$

Then, we have the following:

Theorem 1.3. $G_{n,1} \subset T_n$ *and* $G_{n,2} \subset S_n$.

Here, we notice that $\Phi_n = G_{n,1} \cup G_{n,3}$ and $\Omega_n = G_{n,2}$ by the definitions, and thus the second inclusion in the theorem is the same as the first relation in Corollary 1.2. Thus, the revised problem is open for the elements in $G_{n,3}$.

To represent some subsets of $G_{n,1}$, we use the following notations. Let A and B be subsets of $\mathbb{R}$. Then, we define $A^k * B^{n-k}$ to be the subset of $\mathbb{R}^n$ composed of all points whose some k components belong to A and the remaining $n-k$ components belong to B. In particular,

$$\mathbb{Q}^2 * \mathbb{R}^{n-2} = \{ \boldsymbol{x} = (x_i) \in \mathbb{R}^n \mid x_i, x_j \in \mathbb{Q} \text{ for some } 1 \le i < j \le n \}.$$

Then, we have the following:

Proposition 1.4. $\mathbb{Q}^2 * \mathbb{R}^{n-2} \subset G_{n,1}$ *and thus* $\mathbb{Q}^2 * \mathbb{R}^{n-2} \subset T_n$.

Examples of elements in $G_{n,2} = \Omega_n$ are seen in [2]. More elementary examples of $G_{n,2}$ are given as follows, whose proof is in debt to Yoshiyuki Kuramoto who instructed it to the first author.

Proposition 1.5. *Assume that* $a_1, \cdots, a_n$ *are positive integers with the following properties for* $1 \le i \le n$:

(i) a_i *has no factor of the square of an integer other than* 1*;*

(ii) a_i *has a prime factor which does not divide any* a_j *with* $1 \le j \ne i \le n$.

Then, for any $(q_i) \in \mathbb{Q}^n$ *with* $q_i \ne 0 (1 \le i \le n)$ *and* $(r_i) \in \mathbb{Q}^n$,

$$(q_1\sqrt{a_1} + r_1, \cdots, q_n\sqrt{a_n} + r_n) \in G_{n,2}.$$

As an example about $G_{n,3}$, we have

$$\boldsymbol{w}_m = (1/m, \sqrt{2}, \cdots, \sqrt{p_i}, \cdots, \sqrt{p_{n-1}}) \in G_{n,3}$$

for any integer m with $m \geq 2$, where p_i is the i-th prime integer. In fact, the fundamental vector $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$ satisfies $\boldsymbol{e}_1 \cdot \boldsymbol{w}_m = 1/m \in \mathbb{Q}$, and $\boldsymbol{f} \cdot \boldsymbol{w}_m \notin \mathbb{Z}$ for any fundamental vector $\boldsymbol{f} \in F_n$ by the same reason as Proposition 1.5. It follows $\boldsymbol{w}_2 \in T_n$ from the next lemma, and $\boldsymbol{w}_m \in S_n$ for $m \geq 3$ from Theorem 1.7 below.

Let $\mathbb{Z}[1/2] = \{ k/2 \in \mathbb{Q} \mid k \in \mathbb{Z} \}$. Then, $\mathbb{Z} \subset \mathbb{Z}[1/2]$ obviously, and we have the following.

Lemma 1.6. $\mathbb{Z}[1/2] * \mathbb{R}^{n-1} \subset T_n$.

By Proposition 1.4 and Lemma 1.6, among the vectors which has at least one rational component, the revised problem is only open for the elements in $(\mathbb{Q} \setminus \mathbb{Z}[1/2]) * \mathbb{I}^{n-1}$. We can divide this set into the following three subsets $J_{n,i}$ for $i = 1, 2$ and 3:

$$J_{n,i} = (\mathbb{Q} \setminus \mathbb{Z}[1/2]) * (G_{n-1,i} \cap \mathbb{I}^{n-1}),$$

where we set $G_{1,1} = \mathbb{Z}, G_{1,2} = \mathbb{I}$ and $\mathrm{G}_{1,3} = \mathbb{Q} \setminus \mathbb{Z}$ when $n = 2$. Then, we have the following.

Theorem 1.7. $J_{n,1} \subset T_n$ *and* $J_{n,2} \subset S_n$.

Thus, summarizing the above results, the revised problem is only open for the elements in the set

$$G_{n,3} \cap \mathbb{I}^n \quad \text{or} \quad J_{n,3}.$$

In case of the plane $\mathbb{R}^2$, we shall give a complete answer to the revised problem. In this case, $J_{2,3} = \varnothing$ since $G_{1,3} \cap \mathbb{I} = \varnothing$, and thus we have only to settle the problem for $\mathrm{G}_{2,3} \cap \mathbb{I}^2$. Let $(x, y) \in G_{2,3}$. Then, there are integers a, b and c satisfying

$$ax + by = c, \quad (a, b) \neq (0, 0) \quad \text{and} \quad \gcd\{a, b, c\} = 1. \tag{2}$$

We shall show in Lemma 2.2 that $(a, b, c) \in \mathbb{Z}^3$ satisfying (2) is unique up to sign for (x, y) if at least one of x and y is an irrational number. Furtheremore, we notice that $\gcd\{a, b\} \geq 2$ when $(x, y) \in G_{2,3}$. Then, the following theorem completes the revised problem for $\mathbb{R}^2$.

Theorem 1.8. *Let* $\boldsymbol{x} \in G_{2,3} \cap \mathbb{I}^2$, *and let* a *and* b *be the integers satisfying* (2) *for* $\boldsymbol{x}$. *Then,* $\boldsymbol{x} \in T_2$ *if* $\gcd\{a, b\} = 2$ *and* $a + b \equiv 2 (\mathrm{mod}\ 4)$, *and* $\boldsymbol{x} \in S_2$ *if* $\gcd\{a, b\} \geq 3$ *or* $a + b \equiv 0 (\mathrm{mod}\ 4)$.

Since $(x, y, z) \in T_3$ if $(x, y) \in T_2$, we have the following:

Corollary 1.9. *Let* $\boldsymbol{x} \in J_{3,3} = (Q \setminus \mathbb{Z}[1/2]) * (G_{2,3} \cap \mathbb{I}^2)$, *and* $\boldsymbol{x}' \in \mathbb{R}^2$ *the vector whose components are equal to those of* $G_{2,3} \cap \mathbb{I}^2$ *part of* $\boldsymbol{x}$. *Then,* $x \in T_3$ *if* $\gcd\{a', b'\} = 2$ *and* $a' + b' \equiv 2 (\mathrm{mod}\ 4)$, *where* a' *and* b' *are the integers satisfying* (2) *for* $\boldsymbol{x}'$.

2. Proofs of Results

First, we characterize the set $T_n = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| = \|\boldsymbol{x} - b\|$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} \neq \boldsymbol{b}\}$ by the same way as in [1].

Proposition 2.1. *Let* $u \in \mathbb{R}^n$. *Then, a necessary and sufficient condition for* $\boldsymbol{u} \in T_n$ *is that there exist* $\boldsymbol{c} = (c_i) \in \mathbb{Z}^n$ *and* $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n$ *satisfying the following relations:*

$$2\boldsymbol{c} \cdot \boldsymbol{u} = \boldsymbol{c} \cdot \boldsymbol{d}, \quad \boldsymbol{c} \neq 0 \quad \textit{and} \quad c_i \equiv d_i \ (\mathrm{mod}\ 2) \quad \textit{for} \quad 1 \leq i \leq n. \tag{3}$$

Proof. Assume that $\boldsymbol{u} \in T_n$. Then, there exist $\boldsymbol{a} \in \mathbb{Z}^n$ and $\boldsymbol{b} \in \mathbb{Z}^n$ with $\boldsymbol{a} \neq \boldsymbol{b}$ satisfying $\|\boldsymbol{u} - \boldsymbol{a}\| = \|\boldsymbol{u} - \boldsymbol{b}\|$. Thus, $\boldsymbol{u}$ is a point on the hyperplane that bisects vertically the points $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^n$. Thus, $\boldsymbol{u}$ satisfies the equation

$$(\boldsymbol{a} - \boldsymbol{b}) \cdot (\boldsymbol{u} - (\boldsymbol{a} + \boldsymbol{b}) / 2) = 0.$$

We set $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} \in \mathbb{Z}^n$ and $\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{b} \in \mathbb{Z}^n$. Then, $\boldsymbol{c} \neq 0$, and $2\boldsymbol{c} \cdot \boldsymbol{u} = \boldsymbol{c} \cdot \boldsymbol{d}$. Thus, $\boldsymbol{c}$ and $\boldsymbol{d}$ are the required integral vectors since the last condition in (3) is satisfied by the relation $\boldsymbol{c} + \boldsymbol{d} = 2\boldsymbol{a}$.

Conversely, assume that $\boldsymbol{c} = (c_i) \in \mathbb{Z}^n$ and $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n$ satisfy (3) for $\boldsymbol{u} \in \mathbb{R}^n$. Since $c_i \equiv d_i \pmod 2$ for $1 \leq i \leq n$, we set $\boldsymbol{a} = (\boldsymbol{c} + \boldsymbol{d}) / 2 \in \mathbb{Z}^n$ and $\boldsymbol{b} = (\boldsymbol{d} - \boldsymbol{c}) / 2 \in \mathbb{Z}^n$. Then, $\boldsymbol{a} \neq \boldsymbol{b}$ since $\boldsymbol{c} \neq 0$, and $\|\boldsymbol{u} - \boldsymbol{a}\| = \|\boldsymbol{u} - \boldsymbol{b}\|$ follows from the first equaity in (3). Hence, we have $\boldsymbol{u} \in T_n$ as is required.

Now, we shall prove Theorem 1.3.

Proof of Theorem 1.3. Let $\boldsymbol{x} \in \mathbb{R}^n$ be any element of $G_{n,1}$. Then, there exist a fundamental vector $\boldsymbol{f} \in F_n$ and an integer b satisfying $\boldsymbol{f} \cdot \boldsymbol{x} = b$. Then, there is a vector $\boldsymbol{w} \in \mathbb{Z}^n$ satisfying $\boldsymbol{f} \cdot \boldsymbol{w} = 1$ by (1). Thus, we have $2(2\boldsymbol{f}) \cdot \boldsymbol{x} = 4b = (2\boldsymbol{f}) \cdot (2b\boldsymbol{w})$, and hence $\boldsymbol{c} = 2\boldsymbol{f}$ and $\boldsymbol{d} = 2b\boldsymbol{w}$ satisfy the relations (3) in Proposition 2.1 for $\boldsymbol{x}$. Actually, $\boldsymbol{x}$ is at the same distance from $b\boldsymbol{w} + \boldsymbol{f}$ and $b\boldsymbol{w} - \boldsymbol{f}$, and $\boldsymbol{x} \in T_n$ as is required.

Next, let $\boldsymbol{x} \in \mathbb{R}^n$ be any element of $G_{n,2,}$ and suppose that $\boldsymbol{x} \in T_n$. Then, by Proposition 2.1, there exist $\boldsymbol{c} = (c_i) \in \mathbb{Z}^n$ with $\boldsymbol{c} \neq 0$ and $h \in \mathbb{Z}$ satisfying $\boldsymbol{c} \cdot \boldsymbol{x} = h$. Let $k = \gcd\{c_1, \cdots, c_n\}$. Thus, $\boldsymbol{c}' = \boldsymbol{c} / k$ is a fundamental vector and we have $\boldsymbol{c}' \cdot \boldsymbol{x} = h / k \in \mathbb{Q}$, which contradicts the hypothesis $\boldsymbol{x} \in G_{n,2}$. Hence, $\boldsymbol{x} \in S_n$, and we have the required result.

The proofs of Proposition 1.4, Proposition 1.5 and Lemma 1.6 are as follows:

Proof of Proposition 1.4. Let $\boldsymbol{x} = (x_i) \in \mathbb{Q}^2 * \mathbb{R}^{n-2}$. We may assume that $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}$, and represent them as fractions $x_1 = n_1 / m_1$ and $x_2 = n_2 / m_2$. If $n_1 = 0$, then the fundamental vector $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$ satisfies $\boldsymbol{e}_1 \cdot \boldsymbol{x} = 0 \in \mathbb{Z}$, and thus $\boldsymbol{x} \in G_{n,1}$. When

$n_1 \neq 0$, since $(m_1 n_2)x_1 + (-m_2 n_1)x_2 + 0x_3 + \cdots + 0x_n = 0 \in \mathbb{Z}$, the fundamental vector $\boldsymbol{f} = (m_1 n_2 / k, -m_2 n_1 / k, 0, \cdots, 0)$, where $k = \gcd\{m_1 n_2, m_2 n_1\}$, satisfies $\boldsymbol{f} \cdot \boldsymbol{x} = 0 \in \mathbb{Z}$. Thus, we have $\boldsymbol{x} \in G_{n,1}$ as is required.

Proof of Proposition 1.5. We set $s_i = q_i \sqrt{a_i} + r_i \, (1 \leq i \leq n)$ for any positive integers $\{a_1, \cdots, a_n\}$ satisfying the properties (i) and (ii) and any rational numbers q_i and r_i with $q_i \neq 0$. Let $F(t_1, \cdots, t_m)$ denote the extended field of a field F obtained by adding $t_1, \cdots, t_m$. Then, for $1 \leq i \leq n$, let

$$K_i = \mathbb{Q}(s_1, \cdots, s_i) = K_{i-1}(s_i),$$

where we set $K_0 = \mathbb{Q}$. We shall show

$$s_k \notin K_{k-1} = \mathbb{Q}(s_1, \cdots, s_{k-1}) \tag{4}$$

for $1 \leq k \leq n$. Then, it implies that there are no $(u_i) \in \mathbb{Q}^n$ with $(u_i) \neq 0$ satisfying $\sum_{i=1}^{n} u_i s_i = v$ for some $v \in \mathbb{Q}$, which shows $(s_i) \in G_{n,2}$ as is required.

Now, we prove (4) by the induction on k. When $k = 1$, (4) holds since $s_1 \in \mathbb{I}$. Thus, as the inductive hypothesis, we assume that $k \geq 2$ and $s_{k-1} \notin K_{k-2} = \mathbb{Q}(s_1, \cdots, s_{k-2})$ holds for any $\{s_1, \cdots, s_{k-2}, s_{k-1}\}$ satisfying the conditions. Then, $s_k \notin K_{k-2}$ holds since $\{s_1, \cdots, s_{k-2}, s_k\}$ also satisfies the conditions.

Now, let $k \geq 2$, and suppose that $s_k \in K_{k-1} = K_{k-2}(s_{k-1})$. Then, there are $c, d \in K_{k-2}$ satisfying $s_k = c + d s_{k-1}$, and thus

$$\sqrt{a_k} = e + f\sqrt{a_{k-1}}$$

for some $e, f \in K_{k-2}$. Squaring this equation, we obtain $2ef\sqrt{a_{k-1}} = a_k - e^2 - f^2 a_{k-1}$. If $ef \neq 0$, then $\sqrt{a_{k-1}} \in K_{k-2}$, which contradicts $s_{k-1} \notin K_{k-2}$. Also, if $f = 0$, then $\sqrt{a_k} = e \in K_{k-2}$, which contradicts $s_k \notin K_{k-2}$. Hence, we have $e = 0$. Then, $\sqrt{a_k} = f\sqrt{a_{k-1}}$, and $\sqrt{a_k a_{k-1}} = fa_{k-1} \in K_{k-2}$. However, $a_k a_{k-1} = h^2 b$ for positive integers h and b with the property that the set $\{a_1, \cdots, a_{k-2}, b\}$ satisfies the conditions (i) and (ii). Hence, $\sqrt{a_k a_{k-1}} = h\sqrt{b} \notin K_{k-2}$ by the inductive hypothesis, which contradicts the above. Thus, we conclude that $s_k \notin K_{k-1}$, which completes the proof.

Proof of Lemma 1.6. Let $\boldsymbol{x} = (x_i)$ be any element of $\mathbb{Z}[1/2] * \mathbb{R}^{n-1}$. Then, we may assume that $x_1 = k/2$ for some $k \in \mathbb{Z}$. Set $\varepsilon = 1$ (resp., 2) if k is odd (resp., even), $\boldsymbol{a} = (k/2 + \varepsilon/2, 0, \cdots, 0)$ and $\boldsymbol{b} = (k/2 - \varepsilon/2, 0, \cdots, 0)$. Then, $\boldsymbol{a}$ and $\boldsymbol{b}$ are integral vectors with $\boldsymbol{a} \neq \boldsymbol{b}$, and satisfy $\|\boldsymbol{x} - \boldsymbol{a}\| = \|\boldsymbol{x} - \boldsymbol{b}\|$. Thus, $x \in T_n$ as is required.

Next, we prove Theorem 1.7.

Proof of Theorem 1.7. To show the first required inclusion, let $\boldsymbol{x} = (x_i) \in \mathbb{R}^n$ be any element of $J_{n,1} = (\mathbb{Q} \setminus \mathbb{Z}[1/2]) * (G_{n-1,1} \cap \mathbb{I}^{n-1})$. Then, we may assume that $(x_2, \cdots, x_n) \in G_{n-1,1}$. Thus, there exists a fundamental vector $(f_2, \cdots, f_n) \in \mathbb{Z}^{n-1}$ which satisfies $\sum_{i=2}^{n} f_i x_i \in \mathbb{Z}$. Then, $(0, f_2, \cdots, f_n) \in \mathbb{Z}^n$ is also a fundamental vector, and $0x_1 + \sum_{i=2}^{n} f_i x_i \in \mathbb{Z}$. Hence, we have $\boldsymbol{x} \in G_{n,1,}$ and thus $\boldsymbol{x} \in T_n$ by Theorem 1.3, as is required.

Next, to prove the second required inclusion, let $\boldsymbol{x} = (x_i) \in \mathbb{R}^n$ be any element of $J_{n,2} = (\mathbb{Q} \setminus Z[1/2]) * (G_{n-1,2} \cap \mathbb{I}^{n-1})$. Then, we may assume that $x_1 = n_1/m_1 \in \mathbb{Q} \setminus \mathbb{Z}[1/2]$, where n_1/m_1 is an irreducible fraction with $m_1 \geq 3$ and $(x_2, \cdots, x_n) \in G_{n-1,2} \cap \mathbb{I}^{n-1}$. Under these conditions, we suppose that $\boldsymbol{x} \in T_n$. Then, there exist $\boldsymbol{c} = (c_i) \in \mathbb{Z}^n$ and $\boldsymbol{d} = (d_i) \in \mathbb{Z}^n$ satisfying (3) in Proposition 2.1 for $\boldsymbol{x}$. Then, multiplying m_1 on both sides of $2\boldsymbol{c} \cdot \boldsymbol{x} = \boldsymbol{c} \cdot \boldsymbol{d}$, we have

$$2c_1n_1 + 2m_1c_2x_2 + \cdots + 2m_1c_nx_n = m_1\sum_{i=1}^{n} c_id_i. \tag{5}$$

If $(c_2, \cdots, c_n) \neq 0$, then this equation contradicts the hypothesis that $(x_2, \cdots, x_n) \in G_{n-1,2}$ since $m_1 \neq 0$ and $\sum_{i=2}^{n}(2m_1c_i)x_i = m_1\sum_{i=1}^{n} c_id_i - 2c_1n_1 \in \mathbb{Z}$. Hence, $(c_2, \cdots, c_n) = 0$ and thus $c_1 \neq 0$. Then, it follows $2n_1 = m_1d_1$ from (5), which is a contradiction since m_1 is prime to n_1 and $m_1 \geq 3$. Therefore, we conclude that $\boldsymbol{x} \in S_n$, which completes the proof.

As is stated in the previous section, we shall use the following lemma in the proof of Theorem 1.8.

Lemma 2.2. *Let $(x, y) \in \mathbb{R}^2$, and assume that x or y is an irrational number. If an integral vector $(a, b, c) \in \mathbb{Z}^3$ satisfy*

$$ax + by = c, \quad (a, b) \neq (0, 0) \quad \textit{and} \quad \gcd\{a, b, c\} = 1, \tag{6}$$

then (a, b, c) is unique for (x, y) up to sign.

Proof. Assume that there exists another $(a', b', c') \in \mathbb{Z}^3$ satisfying the same conditions. If (a, b) and (a', b') are linearly independent in $\mathbb{R}^2$, then we have

$$x = \begin{vmatrix} c & b \\ c' & b' \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \quad \text{and} \quad y = \begin{vmatrix} a & c \\ a' & c' \end{vmatrix} \Big/ \begin{vmatrix} a & b \\ a' & b' \end{vmatrix},$$

which contradicts the hypothesis that x or y is an irrational number. Thus, there exists $r \in \mathbb{R}$ satisfying $(a', b') = r(a, b)$. Then, r is a rational number and $c' = a'x + b'y = r(ax + by) = rc$. Hence, $(a', b', c') = r(a, b, c)$, and we have $r = \pm 1$ since $\gcd\{a', b', c'\} = 1$ and $\gcd\{a, b, c\} = 1$. Thus, $(a', b', c') = \pm(a, b, c)$ as is required.

Now, we prove Theorem 1.8.

Proof of Theorem 1.8. Let $\boldsymbol{x} = (x, y) \in G_{2,3} \cap \mathbb{I}^2$. Then, there exists $(a, b, c) \in \mathbb{Z}^3$ satisfying (6) in Lemma 2.2 for (x, y), and $\gcd\{a, b\} \geq 2$ holds since $(x, y) \in G_{2,3}$.

First, we assume that $\gcd\{a, b\} = 2$ and $a + b \equiv 2 \pmod 4$. Then, c is an odd integer since $\gcd\{a, b, c\} = 1$. Set $a' = a/2$ and $b' = b/2$ in $\mathbb{Z}$. Then, $a' + b' \equiv 1 \pmod 2$, and we may assume that a' is even and b' is odd. Since $\gcd\{a', b'\} = 1$, there exist $(u', v') \in \mathbb{Z}^2$ satisfying $a'u' + b'v' = 1$. Then, v' must be odd. We set $(u, v) = (u', v')$ if u' is even, and $(u, v) = (u' + b', v' - a')$ if u' is odd. Then, u is even and v is odd, and $a'u + b'v = 1$ holds. Hence, we have $2(a'x + b'y) = c = a'(cu) + b'(cv)$, where a' and cu are both even and b' and cv are both odd. Therefore, $(c_1, c_2) = (a', b')$ and $(d_1, d_2) = (cu, cv)$ satisfy (3) in Proposition 2.1 for (x, y). Hence, $\boldsymbol{x} = (x, y)$ is an element of T_2 as is required.

Next, we consider the remaining cases. Suppose that $\boldsymbol{x} = (x, y) \in T_2$. Then, by Proposition 2.1, there exist $(c_1, c_2) \in \mathbb{Z}^2$ and $(d_1, d_2) \in \mathbb{Z}^2$ satisfying

$$2(c_1 x + c_2 y) = c_1 d_1 + c_2 d_2, \quad (c_1, c_2) \neq (0, 0),$$

$$c_i \equiv d_i \pmod 2 \quad \text{for} \quad i = 1, 2.$$

Set $c_0 = \gcd\{c_1, c_2\}$, $c_1' = c_1 / c_0$ and $c_2' = c_2 / c_0$ in $\mathbb{Z}$. We further set $k = \gcd\{2c_1', 2c_2', c_1' d_1 + c_2' d_2\}$. Then, since $2(c_1' x + c_2' y) = c_1' d_1 + c_2' d_2$ holds, we have the following by Lemma 2.2:

$$2c_1' = \pm ka, \quad 2c_2' = \pm kb \quad \text{and} \quad c_1' d_1 + c_2' d_2 = \pm kc. \tag{7}$$

Now, consider the case when $\gcd\{a, b\} = 2$ and $a + b \equiv 0 \pmod 4$. Then, $a' = a/2$ and $b' = b/2$ are both odd integers and $c_1' = \pm ka'$ and $c_2' = \pm kb'$ by (7). Thus, $k = \pm 1$ since $\gcd\{c_1', c_2'\} = 1$, and hence c_1' and c_2' are both odd integers. Then, according as c_0 is even or odd, d_1 and d_2 are both even or both odd integers since $d_i \equiv c_i \pmod 2$ as above.

Hence, $c = \pm(c_1' d_1 + c_2' d_2)$ is even, and thus $\gcd\{a, b, c\} = 2$ which contradicts the assumption $\gcd\{a, b, c\} = 1$. Therefore, we conclude that $\boldsymbol{x} = (x, y) \notin T_2$, and thus $\boldsymbol{x} \in S_2$ as is required.

Lastly, when $\gcd\{a, b\} = e \geq 3$, using (7) we have $\gcd\{2c_1', 2c_2'\} = \gcd\{ka, kb\} = ke \geq 3$ which contradicts $\gcd\{2c_1', 2c_2'\} = 2$. Hence, $\boldsymbol{x} = (x, y) \in S_2$ holds in this case too, and we have completed the proof.

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