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A COMPUTATIONALLY-EFFICIENT SOLUTIONS OF COUPLED MATRIX DIFFERENTIAL EQUATIONS FOR DIAGONAL UNKNOWN MATRICES

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Abstract

In the present paper, we present a simple and effective method to produce solutions to the coupled matrix linear differential equations based on the Hadamard product of matrices. The analysis indicates that the Hadamard structure method can achieve good efficient when the unknown matrices are diagonal. Several matrix systems can be solved by the new approach as a special case.

1. Introduction and Preliminary Results

Coupled matrix differential equations have been widely used in stability theory of differential equations, control theory, communication systems, perturbation analysis and other fields of pure and applied mathematics; and also recently in the context of the analysis and numerical simulation of descriptor systems. For instance, the canonical system

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$$X'(t) = AX(t) + BY(t),\ Y'(t) = CX(t) - A^T Y(t) \tag{1.1}$$

with the boundary conditions $X(0) = E$ and $Y(b) = 0$ has been used to the solution of optimal control problem with the performance index [13].

In addition, many interesting problems lead to the general class of non-homogeneous coupled matrix differential equations:

$$X_1'(t) = A_{11}X_1(t)B_{11} + A_{12}X_2(t)B_{12} + \cdots + A_{1p}X_p(t)B_{1p} + U_1(t),$$

$$X_2'(t) = A_{21}X_1(t)B_{21} + A_{22}X_2(t)B_{22} + \cdots + A_{2p}X_p(t)B_{2p} + U_2(t),$$

$$\vdots$$

$$X_p'(t) = A_{p1}X_1(t)B_{p1} + A_{p2}X_2(t)B_{p2} + \cdots + A_{pp}X_p(t)B_{pp} + U_p(t), \tag{1.2}$$

where $A_{ij}, B_{ij} \in M_n$ are given scalar matrices, $U_i(t) \in M_n$ is a given matrix function, $X_i(t) \in M_n$ are the unknown diagonal matrix functions to be solved and $X_i(0) = C_i$; and where $X_i'(t)$ denotes the derivative of matrix function $X_i(t)$. $(i, j = 1, 2, ..., p)$ (*where* $M_{m,n}$ is the set of all $m \times n$ matrices over the complex number field $\mathbb{C}$ and when $m = n$, we write M_m instead of $M_{m,n}$).

Examples of such situation are: Singular and hybrid system control [3, 12], nonzero sum differential games [4], optimal control problem [13], and coupled matrix Riccati differential systems [7]. Depending on the problem considered, different coupling terms may appear. However, in all the above mentioned cases the systems are difficult to solve.

Let us recalling some concepts that will be used below. Given two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{p,q}$, then the *Kronecker product* of A and B is defined by, e.g., [1, 5, 6, 9, 15, 18]

$$A \otimes B = [a_{ij}B]_{ij} \in M_{mp,nq}. \tag{1.3}$$

While if $A \in M_{m,n}, B \in M_{p,n}$, and let $\{a_i : 1 \le i \le n\}$ and $\{b_i : 1 \le i \le n\}$ be the columns of A and B, respectively, namely

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n],\ B = [b_1 \quad b_2 \quad \cdots \quad b_n].$$

The columns of the Kronecker product $A \otimes B$ are $\{a_i \otimes b_j\}$ for all i, j combinations in lexicographic order namely,

$$A \otimes B = [a_1 \otimes b_1 \ \cdots \ a_1 \otimes b_n \ \ a_2 \otimes b_1 \ \cdots \ a_2 \otimes b_n \ \cdots a_n \otimes b_1 \ \cdots \ a_n \otimes b_n]. \quad (1.4)$$

Thus, the *Khatri-Rao product* of A and B is defined by [10, 14]

$$A \ \Theta \ B = [a_1 \otimes b_1 \quad a_2 \otimes b_2 \quad \cdots \quad a_n \otimes b_n] \quad (1.5)$$

consists of a subset of the columns of $A \otimes B$. Notice that $A \otimes B$ is of order $mp \times n^2$ and $A \ \Theta \ B$ is of order $mp \times n$. This observation can be expressed in the form [10]

$$(A \otimes B)S_n = A \ \Theta \ B, \quad (1.6)$$

where the selection matrix S_n is of order $n^2 \times n$ and

$$S_n = [e_1 \quad e_{n+2} \quad e_{2n+3} \quad \cdots \quad e_{n^2}] \quad (1.7)$$

and e_k is an $n^2 \times 1$ column vector with a unity element in the k-th position and zeros elsewhere $(1 \leq k \leq n^2)$.

Additionally, if both matrices $A = [a_{ij}]$ and $B = [b_{ij}] \in M_{m,n}$ have the same size, then the Hadamard product of A and B is defined by [1, 6, 15, 16]

$$A \circ B = [a_{ij} b_{ij}] \in M_{m,n}. \quad (1.8)$$

This product is much simpler than Kronecker and Khatri-Rao products and it can be connected with isomorphic diagonal matrix representations that can have a certain interest in many fields of pure and applied mathematics, for example, Tauber [15] applied the Hadamard product to solving a partial differential equation coming from an air pollution problem. The Hadamard product is clearly commutative, associative, and distributive with respect to addition. It has been known that $A \circ B$ is a (principal) submatrix of $A \otimes B$ if A and B are (square) of the same size. This can be found in Visick [18] and even in Zhang's book [19]. Liv-Ari [10, Theorem 3.1, pp. 128] gave the following new relations related to Kronecker, Khatri-Rao and Hadamard products:

$$S_n^T(A \ \Theta \ B) = A \circ B, \tag{1.9}$$

$$S_n^T(A \otimes B)S_n = A \circ B. \tag{1.10}$$

The Kronecker product and vector operator affirming their capability of solving some matrix and matrix differential equations. Such equations can be readily converted into the standard linear equation form by using the well-known identity, e.g., [6, 8, 10, 11, 14, 15, 17, 19]

$$Vec(AXB^T) = (B \otimes A)VecX, \tag{1.11}$$

where $Vec(\cdot)$ denotes vectorization by columns of a matrix .

As for the *diagonal extraction operator* $vecd(\cdot)$, we observe that for any square $n \times n$ matrix $Y = [y_{ij}]$,

$$Vecd(Y) = S_n^T VecY. \tag{1.13}$$

If Y is diagonal, then we also have

$$VecY = S_n Vecd(Y), \ Y \text{ is diagonal.} \tag{1.14}$$

Moreover, the columns of the $n^2 \times n$ selection matrix S_n are mutually orthonormal, viz.,

$$S_n^T S_n = I_n. \tag{1.15}$$

Using (1.14) and (1.9), we get the fundamental relation between the Hadamard product and *diagonal extraction operator* $vecd(\cdot)$ which is given by:

$$Vecd(AXB^T) = (B \circ A)Vecd(X), \ X \text{ is diagonal,} \tag{1.16}$$

where $A, B \in M_{m,n}$ and X is $n \times n$ diagonal matrix.

Finally, the need to compute the e^A, $\cosh(A)$ and $\sinh(A)$ are due its appearance in the solutions of coupled matrix differential equations.

In this paper, we extend the use of connection between the Hadamard product and diagonal extraction (vector) operator to produce a computationally-efficient solution of non-homogeneous coupled matrix linear differential equations. The analysis indicates that the Hadamard structure method can achieve good efficient when the unknown matrices are diagonal.

2. Main Result

We will use our knowledge of the solution of the simplest homogeneous matrix differential equation [2, 6, 8]:

$$X'(t) = AX(t),\ X(0) = C, \tag{2.1}$$

where $A \in M_m$, $C \in M_{m,n}$ are given scalar matrices, and $X(t) \in M_{m,n}$ is the unknown matrix function to be solved. In fact the unique solution of (2-1) is given by

$$X(t) = e^{At}C. \tag{2.2}$$

The following Lemma is straightforward and its proof is omitted here.

Lemma 2.1. *Let* $A \in M_m$, $C \in M_{m,n}$ *are given scalar matrices*, $U(t) \in M_{m,n}$ *is a given matrix function and* $X(t) \in M_{m,n}$ *is the unknown matrix. Then the general solution of the non-homogeneous matrix differential equation*:

$$X'(t) = AX(t) + U(t),\ X(0) = C \tag{2.3}$$

is given by

$$X(t) = e^{At}C + e^{At} * U(t), \tag{2.4}$$

where $e^{At} * U(t) = \int_0^t e^{A(t-s)}U(s)ds$ *is well-defined* [2], *which involves the so-called* "*convolution product*" *of matrices* e^{At} *and* $U(t)$.

Theorem 2.2. *Let* $A = [a_{ij}]$, $B = [b_{ij}]$, $C \in M_n$ *are given scalar matrices*, $U(t) \in M_n$ *is a given matrix function and* $X(t) \in M_n$ *is unknown diagonal matrix function. Then the general solution of non-homogeneous matrix differential equation*

$$X'(t) = AX(t) + X(t)B + U(t),\ X(0) = C \tag{2.5}$$

is given by

$$Vecd\{X(t)\} = diag(e^{(a_{11}+b_{11})t}, \ldots, e^{(a_{nn}+b_{nn})t})Vecd\{C\}$$

$$+ diag(e^{(a_{11}+b_{11})t}, \ldots, e^{(a_{nn}+b_{nn})t}) * Vecd\{U(t)\}. \tag{2.6}$$

Proof. Using the identity (1.16) we can transform (2.5) into the vector form

$$Vecd\{X'(t)\} = (I_n \circ A + B^T \circ I_n)Vecd\{X(t)\} + Vecd\{U(t)\}$$

$$= \{(A + B^T) \circ I_n\}Vecd\{X(t)\} + Vecd\{U(t)\}$$

$$= diag(a_{11} + b_{11}, \ldots, a_{nn}+b_{nn})Vecd\{X(t)\} + Vecd\{U(t)\}. \tag{2.7}$$

Now, applying (2.4), then the unique solution of (2.7) is

$$Vecd\{X(t)\} = e^{diag(a_{11}+b_{11}, \ldots, a_{nn}+b_{nn})t}Vecd\{C\}$$

$$+ e^{diag(a_{11}+b_{11}, \ldots, a_{nn}+b_{nn})t} * Vecd\{U(t)\}$$

$$= diag(e^{(a_{11}+b_{11})t}, \ldots, e^{(a_{nn}+b_{nn})t})Vecd\{C\}$$

$$+ diag(e^{(a_{11}+b_{11})t}, \ldots, e^{(a_{nn}+b_{nn})t}) * Vecd\{U(t)\}.$$

If we put $U(t) = 0$ in Theorem 2.2 we obtain the following corollary:

Corollary 2.3. *Let* $A, B, C \in M_n$ *are given scalar matrices. Then general solution of the homogeneous matrix differential equation*:

$$X'(t) = AX(t) + X(t)B,\ X(0) = C,\ X(t) \text{ is diagonal} \tag{2.8}$$

is given by

$$Vecd\{X(t)\} = diag(e^{(a_{11}+b_{11})t}, \ldots, e^{(a_{nn}+b_{nn})t})Vecd\{C\}. \tag{2.9}$$

Now we will discuss the general class of non-homogeneous coupled matrix differential equations which are defined in (1.2):

By using the $Vecd(\cdot)$ – notation of (1-2), we have

$$\begin{bmatrix} Vecd\{X_1'(t)\} \\ Vecd\{X_2'(t)\} \\ \vdots \\ Vecd\{X_p'(t)\} \end{bmatrix}$$

$$= \begin{bmatrix} B_{11}^T \circ A_{11} & B_{12}^T \circ A_{12} & \cdots & B_{1p}^T \circ A_{1p} \\ B_{21}^T \circ A_{21} & B_{22}^T \circ A_{22} & \cdots & B_{2p}^T \circ A_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ B_{p1}^T \circ A_{p1} & B_{p2}^T \circ A_{p2} & \cdots & B_{pp}^T \circ A_{pp} \end{bmatrix} \begin{bmatrix} Vecd\{X_1(t)\} \\ Vecd\{X_2(t)\} \\ \vdots \\ Vecd\{X_p(t)\} \end{bmatrix}$$

$$+ \begin{bmatrix} Vecd\{U_1(t)\} \\ Vecd\{U_2(t)\} \\ \vdots \\ Vecd\{U_p(t)\} \end{bmatrix}. \tag{2.10}$$

Let

$$x'(t) = \begin{bmatrix} Vecd\{X_1'(t)\} \\ Vecd\{X_2'(t)\} \\ \vdots \\ Vecd\{X_p'(t)\} \end{bmatrix}; \; x(t) = \begin{bmatrix} Vecd\{X_1(t)\} \\ Vecd\{X_2(t)\} \\ \vdots \\ Vecd\{X_p(t)\} \end{bmatrix}; \; c = \begin{bmatrix} Vecd\{C_1\} \\ Vecd\{C_2\} \\ \vdots \\ Vecd\{C_p\} \end{bmatrix};$$

$$u(t) = \begin{bmatrix} Vecd\{U_1(t)\} \\ Vecd\{U_2(t)\} \\ \vdots \\ Vecd\{U_p(t)\} \end{bmatrix}; H = \begin{bmatrix} B_{11}^T \circ A_{11} & B_{12}^T \circ A_{12} & \cdots & B_{1p}^T \circ A_{1p} \\ B_{21}^T \circ A_{21} & B_{22}^T \circ A_{22} & \cdots & B_{2p}^T \circ A_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ B_{p1}^T \circ A_{p1} & B_{p2}^T \circ A_{p2} & \cdots & B_{pp}^T \circ A_{pp} \end{bmatrix}. \tag{2.11}$$

Now (2.10) can be written as

$$x'(t) = Hx(t) + u(t), \; x(0) = c$$

and the general solution is given by:

$$x(t) = e^{Ht}c + e^{Ht} * u(t). \tag{2.12}$$

Notice that many special cases can be considered from the above general class coupled matrix differential equations; now we will discuss some important special cases in the next results.

Theorem 2.4. *Let* $A, B, C, D, E, F \in M_n$ *are given scalar matrices such that* $(D^T \circ C)(B^T \circ A) = (B^T \circ A)(D^T \circ C)$; $U(t), V(t) \in M_n$ *are given matrix functions and* $X(t), Y(t) \in M_n$ *are the unknown diagonal matrices. Then the general solution of non-homogeneous coupled matrix differential equations*:

$$X'(t) = AX(t)B + CY(t)D + U(t),\ Y'(t) = CX(t)D + AY(t)B + V(t),$$

$$X(0) = E,\ Y(0) = F \tag{2.13}$$

is given by

$$\begin{aligned} Vecd\{X(t)\} = {} & e^{(B^T \circ A)t}\{[\cosh(D^T \circ C)t]\,Vecd\{E\} + [\sinh(D^T \circ C)t]\,Vecd\{F\}\} \\ & + e^{(B^T \circ A)t}\{[\cosh(D^T \circ C)t] * Vecd\{U(t)\} \\ & + [\sinh(D^T \circ C)t] * Vecd\{V(t)\}\}. \end{aligned}$$

$$\begin{aligned} Vecd\{Y(t)\} = {} & e^{(B^T \circ A)t}\{[\sinh(D^T \circ C)t]\,Vecd\{E\} + [\cosh(D^T \circ C)t]\,Vecd\{F\}\} \\ & + e^{(B^T \circ A)t}\{[\sinh(D^T \circ C)t] * Vecd\{U(t)\} \\ & + [\cosh(D^T \circ C)t] * Vecd\{V(t)\}\}. \end{aligned} \tag{2.14}$$

Proof. Using the identity (1.16) we can transform (2.13) into the vector form:

$$\begin{bmatrix} Vecd\{X'(t)\} \\ Vecd\{Y'(t)\} \end{bmatrix} = \begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix} \begin{bmatrix} Vecd\{X(t)\} \\ Vecd\{Y(t)\} \end{bmatrix} + \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{U(t)\} \end{bmatrix}. \tag{2.15}$$

From (2.12), this system has the following solution:

$$\begin{aligned} \begin{bmatrix} Vecd\{X(t)\} \\ Vecd\{Y(t)\} \end{bmatrix} = {} & e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix} t} \cdot \begin{bmatrix} Vecd\{E\} \\ Vec\{F\} \end{bmatrix} \\ & + e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix} t} * \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\} \end{bmatrix}. \end{aligned} \tag{2.16}$$

Now we will deal with

$$e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix}}. \tag{2.17}$$

Since $(D^T \circ C)(B^T \circ A) = (B^T \circ A)(D^T \circ C)$, then we have

$$\begin{bmatrix} B^T \circ A & 0 \\ 0 & B^T \circ A \end{bmatrix}\begin{bmatrix} 0 & D^T \circ C \\ D^T \circ C & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & D^T \circ C \\ D^T \circ C & 0 \end{bmatrix}\begin{bmatrix} B^T \circ A & 0 \\ 0 & B^T \circ A \end{bmatrix}.$$

Then

$$e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix}} = e^{\begin{bmatrix} B^T \circ A & 0 \\ 0 & B^T \circ A \end{bmatrix} + \begin{bmatrix} 0 & D^T \circ C \\ D^T \circ C & 0 \end{bmatrix}}$$

$$= e^{\begin{bmatrix} B^T \circ A & 0 \\ 0 & B^T \circ A \end{bmatrix}} \quad e^{\begin{bmatrix} 0 & D^T \circ C \\ D^T \circ C & 0 \end{bmatrix}}.$$

But

$$e^{\begin{bmatrix} B^T \circ A & 0 \\ 0 & B^T \circ A \end{bmatrix}} = \begin{bmatrix} e^{(B^T \circ A)} & 0 \\ 0 & e^{(B^T \circ A)} \end{bmatrix};$$

$$e^{\begin{bmatrix} 0 & D^T \circ C \\ D^T \circ C & 0 \end{bmatrix}} = \begin{bmatrix} \dfrac{e^{(D^T \circ C)} + e^{-(D^T \circ C)}}{2} & \dfrac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} \\ \dfrac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} & \dfrac{e^{(D^T \circ C)} + e^{-(D^T \circ C)}}{2} \end{bmatrix}.$$

So

$$e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix}}$$

$$= \begin{bmatrix} e^{(B^T \circ A)} & 0 \\ 0 & e^{(B^T \circ A)} \end{bmatrix} \cdot \begin{bmatrix} \frac{e^{(D^T \circ C)} + e^{-(D^T \circ C)}}{2} & \frac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} \\ \frac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} & \frac{e^{(D^T \circ C)} + e^{-(D^T \circ C)}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{(B^T \circ A)} \left\{ \frac{e^{(D^T \circ C)} + e^{-(D^T \circ C)}}{2} \right\} & e^{(B^T \circ A)} \left\{ \frac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} \right\} \\ e^{(B^T \circ A)} \left\{ \frac{e^{(D^T \circ C)} - e^{-(D^T \circ C)}}{2} \right\} & e^{(B^T \circ A)} \left\{ \frac{e^{D^T \circ C} + e^{-(D^T \circ C)}}{2} \right\} \end{bmatrix}$$

$$= \begin{bmatrix} e^{(B^T \circ A)} \cosh(D^T \circ C) & e^{(B^T \circ A)} \sinh(D^T \circ C) \\ e^{B^T \circ A} \sinh(D^T \circ C) & e^{(B^T \circ A)} \cosh(D^T \circ C) \end{bmatrix}. \tag{2.18}$$

Due to (2-18) we have

$$e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix} t} \cdot \begin{bmatrix} Vecd\{E\} \\ Vecd\{F\} \end{bmatrix}$$

$$= \begin{bmatrix} e^{(B^T \circ A)t} \cosh(D^T \circ C)t & e^{(B^T \circ A)t} \sinh(D^T \circ C)t \\ e^{(B^T \otimes A)t} \sinh(D^T \circ C)t & e^{(B^T \circ A)t} \cosh(D^T \circ C)t \end{bmatrix} \begin{bmatrix} Vecd\{E\} \\ Vecd\{F\} \end{bmatrix}; \tag{2.19}$$

$$e^{\begin{bmatrix} B^T \circ A & D^T \circ C \\ D^T \circ C & B^T \circ A \end{bmatrix} t} * \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\} \end{bmatrix}$$

$$= \begin{bmatrix} e^{(B^T \circ A)t} \cosh(D^T \circ C)t & e^{(B^T \circ A)t} \sinh(D^T \circ C)t \\ e^{(B^T \circ A)t} \sinh(D^T \circ C)t & e^{(B^T \circ A)t} \cosh(D^T \circ C)t \end{bmatrix} * \begin{bmatrix} Vecd\{U(t)\} \\ Vecd\{V(t)\} \end{bmatrix}. \tag{2.20}$$

Now substitute (2.19) and (2.20) in (2.16), we get (2.14).

If we put $U(t) = V(t) = 0$ in Theorem 2.4 we obtain the following corollary:

Corollary 2.5. *Let* $A, B, C, D, E, F \in M_n$ *are given scalar matrices such that* $(D^T \circ C)(B^T \circ A) = (B^T \circ A)(D^T \circ C)$, *and* $X(t), Y(t) \in M_n$ *are the unknown diagonal matrices. Then the general solution of homogeneous coupled matrix differential equations*:

$$X'(t) = AX(t)B + CY(t)D,\ Y'(t) = CX(t)D + AY(t)B,$$

$$X(0) = E,\ Y(0) = F \tag{2.21}$$

is given by

$$Vecd\{X(t)\} = e^{(B^T \circ A)t}\{[\cosh(D^T \circ C)t]Vecd\{E\}$$
$$+ [\sinh(D^T \circ C)t]Vecd\{F\}\};$$
$$Vecd\{Y(t)\} = e^{(B^T \circ A)t}\{[\sinh(D^T \circ C)t]Vecd\{E\}$$
$$+ [\cosh(D^T \circ C)t]Vecd\{F\}\}. \tag{2.22}$$

Corollary 2.6. *Let* $B = [b_{ij}],\ D = [d_{ij}],\ E,\ F \in M_n$ *are given scalar matrices and* $X(t),\ Y(t) \in M_n$ *are the unknown diagonal matrices. Then the general solution of homogeneous coupled matrix differential equations*:

$$X'(t) = X(t)B + Y(t)D,\ Y'(t) = X(t)D + Y(t)B,\ X(0) = E,\ Y(0) = F \tag{2.23}$$

is given by

$$Vecd\{X(t)\} = \{diag(e^{b_{11}t} \cosh d_{11}t,\ ...,\ e^{b_{nn}t} \cosh d_{nn}t)\}Vecd\{E\}$$
$$+ \{diag(e^{b_{11}t} \sinh d_{11}t,\ ...,\ e^{b_{nn}t} \sinh d_{nn}t)\}Vecd\{F\};$$
$$Vecd\{Y(t)\} = \{diag(e^{b_{11}t} \sinh d_{11}t,\ ...,\ e^{b_{nn}t} \sinh d_{nn}t)\}Vecd\{E\}$$
$$+ \{diag(e^{b_{11}t} \cosh d_{11}t,\ ...,\ e^{b_{nn}t} \cosh d_{nn}t)\}Vecd\{F\}. \tag{2.24}$$

Proof. For any matrix $A = [a_{ij}] \in M_n$, it is easy to show that

$$\cosh(A^T \circ I)t = diag(\cosh(a_{11}t),\ \cosh(a_{22}t),\ ...,\ ,\cosh(a_{nn}t)); \tag{2.25}$$

$$\sinh(A^T \circ I)t = diag(\sinh(a_{11}t),\ \sinh(a_{22}t),\ ...,\ \sinh(a_{nn}t)). \tag{2.26}$$

Now put $A = C = I_n$ in Theorem 2.5 we have

$$Vecd\{X(t)\} = e^{(B^T \otimes I_n)t}\{[\cosh(D^T \circ I_n)t]Vecd\{E\}$$
$$+ [\sinh(D^T \circ I_n)t]Vecd\{F\}\}$$

$$= \{diag(e^{b_{11}t}, \ldots, e^{b_{nn}t}) \cdot diag(\cosh d_{11}t, \ldots., \cosh d_{nn}t)\} Vecd\{E\}$$

$$+ \{diag(e^{b_{11}t}, \ldots, e^{b_{nn}t}) \cdot diag(\sinh d_{11}t, \ldots, \sinh d_{nn}t)\} Vecd\{F\}$$

$$= \{diag(e^{b_{11}t} \cosh d_{11}t, \ldots, e^{b_{nn}t} \cosh d_{nn}t)\} Vecd\{E\}$$

$$+ \{diag(e^{b_{11}t} \sinh d_{11}t, \ldots, e^{b_{nn}t} \sinh d_{nn}t)\} Vecd\{F\}.$$

Similarly,

$$Vecd\{Y(t)\} = \{diag(e^{b_{11}t} \sinh d_{11}t, \ldots, e^{b_{nn}t} \sinh d_{nn}t)\} Vecd\{E\}$$

$$+ \{diag(e^{b_{11}t} \cosh d_{11}t, \ldots, e^{b_{nn}t} \cosh d_{nn}t)\} Vecd\{F\}.$$

3. Concluding Remarks

We used the fundamental relation between the Hadamard product and diagonal extraction (vector) operator in (1.16) and (1.11) to derive our main results in Section 2 and, subsequently, to construct a computationally-efficient solution of coupled matrix differential equations. In fact, the Hadamard product and operator $Vecd(\cdot)$ affirming their capability of solving matrix differential equations fast when the unknown matrices are diagonal. To demonstrate the usefulness of applying some properties of the Hadamard product, suppose we have to solve, for example, the following system:

$$BXA^T = C, \tag{3.1}$$

where $A, B \in M_n$ are given scalar matrices and $X \in M_n$ is unknown matrix to be solved. Then it is not hard by using $Vecd(\cdot)$ – notation and Hadamard product to establish the following equivalence:

$$(A \circ B) Vecd(X) = Vecd(C), \; X \text{ is diagonal.} \tag{3.2}$$

If we ignore the Hadamard product structure, then we need to solve the following both matrix equations:

- $$BY = C \tag{3.3}$$

Here, Y can be obtained in $O(n^3)$ arithmetic operations (flops) by using LU factorization of matrix B (Forward Substitution).

- $XA^T = Y$ (3.4)

Here X can be obtained also in $O(n^3)$ operations (flops) by using LU factorization of matrix A (Back Substitution).

Now without exploiting the Hadamard product structure, an $n^2 \times n^2$ system defined in (3.1) would normally (by Gaussian elimination) require $O(n^6)$ operations to solve. While if X is $n \times n$ diagonal matrix and use the Hadamard product structure:

$$(A \circ B)Vecd(X) = Vecd(C),$$

the calculations show that $Vecd(X)$ can be obtained only in $O(n)$ operations by using LU factorization of $A \circ B$.

We can say that the system of the form: $(A \circ B)Vecd(X) = Vecd(C)$ can be solved fast more than LU-decomposition or Kronecker structure, only the very smaller, lower and upper triangular matrices $L_{A \circ B}$ and $U_{A \circ B}$ are needed.

For example, consider A and B are 3×3 matrices, C is 9×1 vector and X is 3×3 diagonal matrix. To demonstrate the usefulness of applying Hadamard product and $Vecd(\cdot)$ – notation, we return to the system problem: $(A \circ B)Vecd(X) = Vecd(C)$. If $A \circ B$ is non-singular and regarding with LU factorizations of $A \circ B = L_{A \circ B}U_{A \circ B}$, then a solution of system exists and can be written as:

$$U_{A \circ B}Vecd(X) = y,\ L_{A \circ B}\ y = Vecd(C). \tag{3.5}$$

First, the lower triangular system $L_{A \circ B}\ y = Vecd(C)$ can be solved by forward substitution as the following:

$$\begin{bmatrix} a_{11}b_{11} & 0 & 0 \\ a_{21}b_{21} & a_{22}b_{22} & 0 \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{22} \\ c_{33} \end{bmatrix},$$

which can be solved in $O(n) = O(3)$ operations as follows:

- $$a_{11}b_{11}y_1 = c_{11} \Rightarrow y_1 = \frac{c_{11}}{a_{11}b_{11}}. \tag{3.6}$$

- $$a_{21}b_{21}y_1 + a_{22}b_{22}y_2 = c_{22} \Rightarrow y_2 = \frac{a_{11}b_{11}c_{22} - a_{21}b_{21}c_{11}}{a_{11}b_{11}a_{22}b_{22}}. \tag{3.7}$$

- $a_{31}b_{31}y_1 + a_{32}b_{32}y_2 + a_{33}b_{33}y_3 = c_{33}$

$$\Rightarrow y_3 = \frac{a_{11}b_{11}a_{22}b_{22}c_{33} - [a_{31}b_{31}a_{22}b_{22} - a_{32}b_{32}a_{21}b_{21}]c_{11} - a_{32}b_{32}a_{11}b_{11}c_{22}}{a_{11}b_{11}a_{22}b_{22}a_{33}b_{33}}. \tag{3.8}$$

Conclusion

The solution of coupled matrix differential equations is studied and some important special cases are discussed. The analysis indicates that solving for $Vecd(\cdot)$ is more efficient when the unknown matrices are diagonal. Although the algorithms are presented for non-homogeneous coupled matrix differential equations, the idea adopted can be easily extended to study coupled matrix nonlinear differential equations, e.g., the coupled matrix Riccati differential equations.

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