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# THE SPLINE RULE FOR INTEGRATION OF FUZZY FUNCTIONS

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**Abstract**

In this paper, we propose the numerical solution for fuzzy integration by the spline rule. The method is discussed in detail and there is followed by convergence theorem. The fuzzy quantities are presented in parametric form. The algorithm is illustrated by solving some numerical tests.

## 1. Introduction

In recent years much attention has been given to develop several numerical methods for solving integrals. It is well known that the integration problems play an important and significant role in various areas such as mathematics, physics, statistics, engineering and social sciences. Since in many applications at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, it is important to develop fuzzy integration and solve them. The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [8].

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Zimmerman [9] has discussed and studied fuzzy integration and then followed by Allahviranloo [1]. In this paper, at the first we obtain some result of cubic spline interpolation in particular case for fuzzy functions, and then we apply the results to solve fuzzy integrals. The organization of the paper is as follows:

In Section 2, we recall some fundamental results and definitions on fuzzy numbers and fuzzy integrations and the integration formulas of Newton Cot's methods for fuzzy integrations and also in Section 3, we make a review of the trapezoidal rule for fuzzy integrations. In Section4, we have a discussion about cubic spline functions which interpolate fuzzy functions. In Section 5, we introduce the spline rule for solving of fuzzy integrations. The proposed method is illustrated by solving some examples in Section 6 and conclusions in the last section.

## 2. Preliminaries

We represent an arbitrary fuzzy number by an ordered pair of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \le r \le 1$, which satisfy the following requirements:

**1.** $\underline{u}(r)$ is a bounded lift continuous non decreasing function over $[0, 1]$.

**2.** $\overline{u}(r)$ is a bounded left continuous non increasing function over $[0, 1]$.

**3.** $\underline{u}(r) \le \overline{u}(r)$, $0 \le r \le 1$.

A crisp number $\alpha$ is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \le r \le 1$. The set of all the fuzzy numbers is denoted by $E^1$.

**Lemma 2.1.** *Let* $u, v \in E^1$ *and* $s$ *be real number. Then for* $0 \le r \le 1$, $u = v$ *if and only if* $\underline{u}(r) = \underline{v}(r)$ *and* $\overline{u}(r) = \overline{v}(r)$,

$$u + v = (\underline{u}(r) + \underline{v}(r),\ \overline{u}(r) + \overline{v}(r)),$$

$$u - v = (\underline{u}(r) - \overline{v}(r),\ \overline{u}(r) - \underline{v}(r)),$$

$$u \cdot v = (\min\{\underline{u}(r).\ \underline{v}(r),\ \underline{u}(r).\ \overline{v}(r),\ \overline{u}(r).\ \underline{v}(r),\ \overline{u}(r).\ \overline{v}(r)\},$$

$\max\{\underline{u}(r).\ \underline{v}(r),\ \underline{u}(r).\ \overline{v}(r),\ \overline{u}(r).\ \underline{v}(r),\ \overline{u}(r).\ \overline{v}(r)\})$,

$su = s(\underline{u}(r),\ \overline{u}(r))$. *See* [6].

$E^1$ with addition and multiplication as defined by Lemma 2.1 is a convex cone which is then embedded isomorphically and isometrically in to a Banach space.

**Definition 2.2.** For arbitrary fuzzy number $u = (\underline{u},\ \overline{u})$ and $v = (\underline{v},\ \overline{v})$ the quantity

$$D(u,\ v) = \sup_{0 \le r \le 1} \{\max[|\ \underline{u}(r) - \underline{v}(r)\ |,\ |\ \overline{u}(r) - \overline{v}(r)\ |]\}$$

is the distance between $u$ and $v$.

It is shown [5] that $E^1,\ D$ is a complete metric space.

**Definition 2.3.** A function $f : R^1 \to E^1$ is called a fuzzy function. If for arbitrary fixed $t_0 \in R^1$ and $\in > 0,\ \delta > 0$ such that

$$|\ t - t_0\ | < \delta \Rightarrow D[f(t),\ f(t_0)] < \varepsilon$$

exist, $f$ is said to be continuous.

Throughout this work we also consider fuzzy function which are defined only over a finite interval $[a,\ b]$. We now follow Goestschel and Voxman [2] and define the integral of a fuzzy function using the Rieman integral concept.

**Definition 2.4.** Let $f : [a,\ b] \to E^1$. For each partition $P = \{t_0,\ t_1,\ ...,\ t_n\}$ of $[a,\ b]$ and for arbitrary $\xi_i : t_{i-1} \le \xi_i \le t_i,\ 1 \le i \le n,$ let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

The definite integral of $f(t)$ over $[a,\ b]$ is

$$\int_a^b f(t)dt = \lim R_p,\ \max_{1 \le i \le n} |\ t_i - t_{i-1}\ | \to 0$$

provided that this limit exists in the metric $D$.

If the fuzzy function $f(t)$ is continuous in the metric $D$, its definite integral exists [2]. Further more,

$$\left(\underline{\int_a^b f(t;\, r)dt}\right) = \int_a^b \underline{f}(t;\, r)dt,$$

$$\left(\overline{\int_a^b f(t;\, r)dt}\right) = \int_a^b \bar{f}(t;\, r)dt. \tag{2.1}$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [2, 4]. More details about properties of the fuzzy integral are given in [2, 3].

Now, we have a review on the Newton Cot's rules in [1]. Let $f$ be a fuzzy function. For any natural number $n$, the Newton cot's formulas

$$\int_a^b f(x)dx = h\sum_{i=1}^{n} f_i\alpha_i + E,\ f_i = f(a + ih),\ h = \frac{b-a}{n}, \tag{2.2}$$

provide approximate values for $\int_a^b f(x)dx$. The parametric form of (2.2) is

as follows:

$$\int_a^b \underline{f}(x;\, r)dx = h\sum_{i=1}^{n} \alpha_i \underline{f}(x_i;\, r) + E_n(\underline{f};\, r),$$

$$\int_a^b \bar{f}(x;\, r)dx = h\sum_{i=1}^{n} \alpha_i \bar{f}(x_i;\, r) + E_n(\bar{f};\, r),\ 0 \le r \le 1. \tag{2.3}$$

The weights $\alpha_i$, $i = 1, ..., n$, are rational number with the property $\sum_{i=1}^{n} \alpha_i = n$. This follows (2.3) when applied to $\underline{f}(x;\, r) = \bar{f}(x;\, r) = 1$. It can be shown that the approximation error may be expressed as follows:

$$E(\underline{f};\, r) = h^{p+1} \cdot K \cdot \underline{f}^{(p)}(\underline{\xi};\, r),\ \underline{\xi} \in (a, b),$$

$$E(\bar{f};\, r) = h^{p+1} \cdot K \cdot \bar{f}^{(p)}(\overline{\xi};\, r),\ \overline{\xi} \in (a, b),\ 0 \le r \le 1. \tag{2.4}$$

Here $(a, b)$ is the open interval from $a$ to $b$. The values of $p$ and $K$ depend only on but not on the integrand $f$. Let

$$I_n(\underline{f};\, r) = h\sum_{i=1}^{n} \alpha_i \underline{f}(x_i;\, r),$$

$$I_n(\bar{f};\, r) = h\sum_{i=1}^{n} \alpha_i \bar{f}(x_i;\, r),\ 0 \le r \le 1. \tag{2.5}$$

Then from (2.3), we have

$$I(\underline{f};\, r) = \int_a^b \underline{f}(x;\, r)dx = I_n(\underline{f};\, r) + E_n(\underline{f};\, r),$$

$$I(\bar{f};\, r) = \int_a^b \bar{f}(x;\, r)dx = I_n(\bar{f};\, r) + E_n(\bar{f};\, r),\ 0 \le r \le 1. \tag{2.6}$$

It has shown in [1] that $I_n(\underline{f};\, r)$, $I_n(\bar{f};\, r)$ converge to $I(\underline{f};\, r)$, $I(\bar{f};\, r)$ respectively.

## 3. The Trapezoidal Integration Rule

From (2.3) we have:

$$I_1(\underline{f};\, r) = \frac{h}{2}[\underline{f}(a) + \underline{f}(b)],$$

$$I_1(\bar{f};\, r) = \frac{h}{2}[\bar{f}(a) + \bar{f}(b)],\ 0 \le r \le 1 \tag{3.1}$$

with error terms:

$$E_1(\underline{f};\, r) = \int_a^b \underline{f}(x,\, r)dx - \left(\frac{b-a}{2}[\underline{f}(a;\, r) + \underline{f}(b;\, r)]\right)$$

$$= -\frac{(b-a)^3}{12}\underline{f}^{(2)}(\underline{\xi};\, r),$$

$$E_1(\bar{f};\, r) = \int_a^b \bar{f}(x,\, r)dx - \left(\frac{b-a}{2}[\bar{f}(a;\, r) + \bar{f}(b;\, r)]\right)$$

$$= -\frac{(b-a)^3}{12}\bar{f}^{(2)}(\bar{\xi};\, r), \tag{3.2}$$

where $\underline{\xi}, \overline{\xi} \in [a, b]$. Also we can write,

$$I_n(\underline{f}; r) = \frac{h}{2}\left[\underline{f}(a; r) + \sum_{i=1}^{n-1} \underline{f}(x_i; r) + \underline{f}(b; r)\right],$$

$$I_n(\bar{f}; r) = \frac{h}{2}\left[\bar{f}(a; r) + \sum_{i=1}^{n-1} \bar{f}(x_i; r) + \bar{f}(b; r)\right], \ 0 \le r \le 1, \tag{3.3}$$

for the error in $I_n(\underline{f}; r)$ and $I_n(\bar{f}; r)$,

$$E_n(\underline{f}; r) = I(\underline{f}) - I_n(\underline{f}) = -\frac{(b-a)h^2}{12}\underline{f}^{(2)}(\underline{\xi}; r),$$

$$E_n(\bar{f}; r) = I(\bar{f}) - I_n(\bar{f})$$

$$= -\frac{(b-a)h^2}{12}\bar{f}^{(2)}(\overline{\xi}; r), \ \underline{\xi}, \overline{\xi} \in [a, b], \ 0 \le r \le 1. \tag{3.4}$$

## 4. The Cubic Spline Functions

It is known that cubic spline functions are popular spline functions, for a variety of reasons. They are smooth functions with which to fit data, and when use for interpolation, they do not have the oscillatory behavior that is characteristic of high-degree polynomial interpolation. If $\Delta = \{x_i \mid i = 0, 1, 2, ..., n\}$ is a fixed partition of the interval $[a, b]$ by knots $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and $\underline{y_i} = \underline{f}(x_i; r)$ and $\overline{y_i} = \bar{f}(x_i; r)$; $i = 0, 1, 2, ..., n$; $0 \le r \le 1$, are $(n+1)$ prescribed real number. In addition let $h_{j+1} := x_{j+1} - x_j$; $j = 0, 1, 2, ..., n-1$; then cubic spline functions $\underline{S_\Delta}(x)$ and $\overline{S_\Delta}(x)$ which interpolate the values of the functions of $\underline{f}$ and $\bar{f}$ at the knots $x_0, x_1, ..., x_n \in \Delta$ respectively, and satisfy $\underline{S''_\Delta}(a; r) = \overline{S''_\Delta}(a; r) = \underline{S''_\Delta}(b) = \overline{S''_\Delta}(b; r) = 0$ is readily characterized by their moment of linear equations. We can obtain the following parametric form of the cubic spline functions in terms of its moments [7]:

$$\underline{S_\Delta}(x) = \underline{\alpha_j} + \underline{\beta_j}(x - x_j) + \underline{\gamma_j}(x - x_j)^2 + \underline{\delta_j}(x - x_j)^3 \text{ for } x \in [x_j, x_{j+1}],$$

$$\overline{S_\Delta}(x) = \overline{\alpha_j} + \overline{\beta_j}(x - x_j) + \overline{\gamma_j}(x - x_j)^2 + \overline{\delta_j}(x - x_j)^3 \text{ for } x \in [x_j, x_{j+1}], \tag{4.1}$$

where for $j = 0, 1, 2, ..., n-1$ :

$$\underline{\alpha_j} = \underline{y_j},\ \overline{\alpha_j} = \overline{y_j},$$

$$\underline{\gamma_j} = \frac{\underline{M_j}}{2},\ \overline{\gamma_j} = \frac{\overline{M_j}}{2},$$

$$\underline{\beta_j} = \frac{\underline{y_{j+1}} - \overline{y_j}}{h_{j+1}} - \frac{2\overline{M_j} + \overline{M_{j+1}}}{6} h_{j+1},\ \overline{\beta_j} = \frac{\overline{y_{j+1}} - \underline{y_j}}{h_{j+1}} - \frac{2\underline{M_j} + \underline{M_{j+1}}}{6} h_{j+1},$$

$$\underline{\delta_j} = \frac{\underline{M_{j+1}} - \overline{M_j}}{6h_{j+1}},\ \overline{\delta_j} = \frac{\overline{M_{j+1}} - \underline{M_j}}{6h_{j+1}},$$

and

$\underline{M_j} = \underline{S''_\Delta}(x_j),\ \overline{M_j} = \overline{S''_\Delta}(x_j);\ j = 1, 2, ..., n$ and $\underline{M_0} = \overline{M_0} = \underline{M_n} = \overline{M_n} = 0,$

are the moments of $\underline{S_\Delta}$ and $\overline{S_\Delta}$ and are solution of the following two linear systems of equations, respectively:

$$\begin{pmatrix} 2 & \lambda_0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \mu_1 & 2 & \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mu_2 & 2 & \lambda_2 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & & \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \mu_n & 2 \end{pmatrix} \begin{pmatrix} \underline{M_0} \\ \underline{M_1} \\ \underline{M_2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{M_n} \end{pmatrix} = \begin{pmatrix} \underline{d_0} \\ \underline{d_1} \\ \underline{d_2} \\ \cdot \\ \cdot \\ \cdot \\ \underline{d_n} \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & \lambda_2 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \mu_1 & 2 & \lambda_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \mu_2 & 2 & \lambda_2 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & & \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \mu_{n-1} & 2 & \lambda_{n-1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \mu_n & 2 \end{pmatrix} \begin{pmatrix} \overline{M_0} \\ \overline{M_1} \\ \overline{M_2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{M_n} \end{pmatrix} = \begin{pmatrix} \overline{d_0} \\ \overline{d_1} \\ \overline{d_2} \\ \cdot \\ \cdot \\ \cdot \\ \overline{d_n} \end{pmatrix},$$

where

$$\lambda_j := \frac{h_{j+1}}{h_j + h_{j+1}},\ \mu_j := 1 - \lambda_j = \frac{h_j}{h_j + h_{j+1}},$$

$$\underline{d_j} := \frac{6}{h_j + h_{j+1}} \left\{ \frac{\underline{y_{j+1}} - \overline{y_j}}{h_{j+1}} - \frac{\overline{y_j} - \underline{y_{j-1}}}{h_j} \right\},$$

$$\overline{d_j} := \frac{6}{h_j + h_{j+1}} \left\{ \frac{\overline{y_{j+1}} - \underline{y_j}}{h_{j+1}} - \frac{\underline{y_j} - \overline{y_{j-1}}}{h_j} \right\},$$

and

$$\lambda_0 = 0,\ \underline{d_0} = \overline{d_0} = 0,\ \underline{d_n} = \overline{d_n} = 0,\ \mu_n = 0.$$

## 5. The Spline Rule on Fuzzy Integration

In this section we would like to integrate $\underline{S_\Delta}(x; r)$ and $\overline{S_\Delta}(x; r)$ in interval $[a, b]$ as approximation to integrate $\underline{f}(x; r)$ and $\bar{f}(x; r)$ in interval $[a, b]$. Then we will have:

$$\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x; r)dx$$

$$= \int_{x_j}^{x_{j+1}} (\underline{\alpha_j} + \underline{\beta_j}(x - x_j) + \underline{\gamma_j}(x - x_j)^2 + \underline{\delta_j}(x - x_j)^3)dx$$

$$= \underline{\alpha_j} h_{j+1} + \underline{\beta_j} \frac{h_{j+1}^2}{2} + \underline{\gamma_j} \frac{h_{j+1}^3}{3} + \underline{\delta_j} \frac{h_{j+1}^4}{4}$$

$$= h_{j+1}\underline{y_j} + \frac{h_{j+1}^2}{2}\left(\frac{\underline{y_{j+1}} - \overline{y_j}}{h_{j+1}} - \frac{2\overline{M_j} + \overline{M_{j+1}}}{6}h_{j+1}\right)$$

$$+ \frac{h_{j+1}^3}{3}\left(\frac{\underline{M_j}}{2}\right) + \frac{h_{j+1}^4}{4}\left(\frac{\underline{M_{j+1}} - \overline{M_j}}{6h_{j+1}}\right)$$

$$= h_{j+1}\left(\frac{\underline{y_{j+1}} - \overline{y_j} + 2\underline{y_j}}{2}\right) + \frac{h_{j+1}^3}{24}(4\underline{M_j} - 5\overline{M_j} + \underline{M_{j+1}} - 2\overline{M_{j+1}}),$$

and

$$\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x;\, r)dx$$

$$= \int_{x_j}^{x_{j+1}} (\overline{\alpha_j} + \overline{\beta_j}(x - x_j) + \overline{\gamma_j}(x - x_j)^2 + \overline{\delta_j}(x - x_j)^3)dx$$

$$= \overline{\alpha_j}h_{j+1} + \overline{\beta_j}\frac{h_{j+1}^2}{2} + \overline{\gamma_j}\frac{h_{j+1}^3}{3} + \overline{\delta_j}\frac{h_{j+1}^4}{4}$$

$$= h_{j+1}\overline{y_j} + \frac{h_{j+1}^2}{2}\left(\frac{\overline{y_{j+1}} - \underline{y_j}}{h_{j+1}} - \frac{2\underline{M_j} + \underline{M_{j+1}}}{6}h_{j+1}\right)$$

$$+ \frac{h_{j+1}^3}{3}\left(\frac{\overline{M_j}}{2}\right) + \frac{h_{j+1}^4}{4}\left(\frac{\overline{M_{j+1}} - \underline{M_j}}{6h_{j+1}}\right)$$

$$= h_{j+1}\left(\frac{\overline{y_{j+1}} - \underline{y_j} + 2\overline{y_j}}{2}\right) + \frac{h_{j+1}^3}{24}(4\overline{M_j} - 5\underline{M_j} + \overline{M_{j+1}} - 2\underline{M_{j+1}}).$$

Now let $D_j = \overline{M_j} - \underline{M_j}$ and $A_j = \overline{y_j} - \underline{y_j}$, then we obtain:

$$\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x;\, r)dx$$

$$= \frac{h_{j+1}}{2}(\underline{y_{j+1}} + y_j) - \frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{24}[-(4D_j + D_{j+1}) - (\overline{M_j} + \overline{M_{j+1}})],$$

$$\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x;\, r)dx$$

$$= \frac{h_{j+1}}{2}(\overline{y_{j+1}} + \underline{y_j}) + \frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{24}[(4D_j + D_{j+1}) - (\underline{M_j} + \underline{M_{j+1}})]. \quad (5.1)$$

So by (3.2) and (5.1) obtain:

$$\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x;\, r)dx = \int_{x_j}^{x_{j+1}} \underline{f}(x;\, r)dx - \frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{12} \underline{f}^{(2)}(\underline{\xi_j};\, r)$$

$$+ \frac{h_{j+1}^3}{24}[-(4D_j + D_{j+1}) - (\overline{M_j} + \overline{M_{j+1}})],$$

$$\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x;\, r)dx = \int_{x_j}^{x_{j+1}} \bar{f}(x;\, r)dx \frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{12} \bar{f}^{(2)}(\overline{\xi_j};\, r)$$

$$+ \frac{h_{j+1}^3}{24}[(4D_j + D_{j+1}) - (\underline{M_j} + \underline{M_{j+1}})], \quad (5.2)$$

where $\underline{\xi_j}, \overline{\xi_j} \in [x_j, x_{j+1}]$. Then we obtain:

$$\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x;\, r)dx - \int_{x_j}^{x_{j+1}} \underline{f}(x;\, r)dx$$

$$= -\frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{12}\left[\underline{f}^{(2)}(\underline{\xi_j};\, r) - \frac{1}{2}(4D_j + D_{j+1}) - \frac{\overline{M_j} + \overline{M_{j+1}}}{2}\right],$$

$$\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x;\, r)dx - \int_{x_j}^{x_{j+1}} \bar{f}(x;\, r)dx$$

$$= \frac{h_{j+1}}{2} A_j + \frac{h_{j+1}^3}{12}\left[\bar{f}^{(2)}(\overline{\xi_j};\, r) + \frac{1}{2}(4D_j + D_{j+1}) - \frac{\underline{M_j} + \underline{M_{j+1}}}{2}\right],$$

where $\underline{\xi_j}, \overline{\xi_j} \in [x_j, x_{j+1}]$. So

$$\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x; r)dx - \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \underline{f}(x; r)dx = -\frac{1}{2}\sum_{j=0}^{n-1}(h_{j+1}A_j)$$

$$+\frac{1}{12}\left\{\sum_{j=0}^{n-1} h_{j+1}^3 \underline{f}^{(2)}(\underline{\xi_j}; r) - \frac{1}{2}\sum_{j=0}^{n-1} h_{j+1}^3(4D_j + D_{j+1})\right.$$

$$\left.-\frac{1}{2}\sum_{j=0}^{n-1} h_{j+1}^3(\overline{M_j} + \overline{M_{j+1}})\right\},$$

$$\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x; r)dx - \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \bar{f}(x; r)dx = \frac{1}{2}\sum_{j=0}^{n-1}(h_{j+1}A_j)$$

$$+\frac{1}{12}\left\{\sum_{j=0}^{n-1} h_{j+1}^3 \bar{f}^{(2)}(\overline{\xi_j}; r) + \frac{1}{2}\sum_{j=0}^{n-1} h_{j+1}^3(4D_j + D_{j+1})\right.$$

$$\left.-\frac{1}{2}\sum_{j=0}^{n-1} h_{j+1}^3(\underline{M_j} + \underline{M_{j+1}})\right\},$$

where $\underline{\xi_j}, \overline{\xi_j} \in [x_j, x_{j+1}]$. Now for uniform partition $\Delta = \{x_j \,|\, x_j = a + jh,$ $j = 0, 1, ..., n\}$, where $h = \dfrac{b-a}{n}$, are obtained:

$$\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x; r)dx$$

$$= \frac{h}{2}(\underline{y_j} + \underline{y_{j+1}}) - \frac{h}{2}A_j + \frac{h^3}{24}[-((4D_j + D_{j+1}) - (\overline{M_j} + \overline{M_{j+1}})],$$

$$\int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x; r)dx$$

$$= \frac{h}{2}(\overline{y_j} + \overline{y_{j+1}}) + \frac{h}{2}A_j + \frac{h^3}{24}[(4D_j + D_{j+1}) - (\underline{M_j} + \underline{M_{j+1}})],$$

where $j = 0, 1, ..., n-1$. Therefore we will have:

$$S_n(\underline{f}; r) = \int_a^b \underline{S_\Delta}(x; r)dx = \underline{T}(h) - \frac{h}{2} A + \underline{\beta}(h)h^3,$$

$$S_n(\bar{f}; r) = \int_a^b \overline{S_\Delta}(x; r)dx = \overline{T}(h) + \frac{h}{2} A + \overline{\beta}(h)h^3, \tag{5.3}$$

where

$$\underline{T}(h) = \frac{h}{2}\left[\underline{y_0} + 2\sum_{j=1}^{n-1} \underline{y_j} + \underline{y_n}\right], \ \overline{T}(h) = \frac{h}{2}\left[\overline{y_0} + 2\sum_{j=1}^{n-1} \overline{y_j} + \overline{y_n}\right],$$

are the trapezoidal sum and

$$\underline{\beta}(h) = -\frac{5}{24}\sum_{j=1}^{n-1} D_j - \frac{1}{12}\sum_{j=1}^{n-1} \overline{M_j}, \ \overline{\beta}(h) = \frac{5}{24}\sum_{j=1}^{n-1} D_j - \frac{1}{12}\sum_{j=1}^{n-1} \underline{M_j},$$

$$A = \sum_{j=1}^{n-1} A_j.$$

Since $\underline{M_0} = \underline{S''_\Delta}(a) = \underline{S''_\Delta}(b) = \underline{M_n} = 0$ and $\overline{M_0} = \overline{S''_\Delta}(a) = \overline{S''_\Delta}(b) = \overline{M_n} = 0$, then $\underline{D_0} = \underline{D_n} = \overline{D_0} = \overline{D_n} = 0$. Therefore for error of the spline rule, we can write:

$$\sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x; r)dx - \int_a^b \underline{f}(x; r)dx$$

$$= -\frac{h}{2} A + \frac{b-a}{12} h^2 \frac{1}{h}\sum_{j=0}^{n-1} \underline{f}^{(2)}(\underline{\xi_j}; r) - \frac{5}{24} h^3 \sum_{j=1}^{n-1} D_j - \frac{h^3}{12}\sum_{j=1}^{n-1} \overline{M_j},$$

$$\sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \overline{S_\Delta}(x; r)dx - \int_a^b \bar{f}(x; r)dx$$

$$= \frac{h}{2} A + \frac{b-a}{12} h^2 \frac{1}{n}\sum_{j=0}^{n-1} \bar{f}^{(2)}(\overline{\xi_j}; r) + \frac{5}{24} h^3 \sum_{j=1}^{n-1} D_j - \frac{h^3}{12}\sum_{j=1}^{n-1} \underline{M_j},$$

where $\underline{\xi}_j, \overline{\xi_j} \in [x_j, x_{j+1}]$, since $f \in C^2[a, b]$ therefore there exist $\underline{\xi}, \overline{\xi} \in [a, b]$ such that

$$\underline{f}^{(2)}(\underline{\xi}; r) = \frac{1}{h}\sum_{j=0}^{n-1} \underline{f}^{(2)}(\underline{\xi}_j; r),\ \overline{f}^{(2)}(\overline{\xi}; r) = \frac{1}{n}\sum_{j=0}^{n-1} \overline{f}^{(2)}(\overline{\xi_j}; r),$$

and thus we obtain:

$$\int_a^b \underline{S_\Delta}(x; r)dx - \int_a^b \underline{f}(x; r)dx$$

$$= -\frac{h}{2}A + \frac{b-a}{12}h^2\underline{f}^{(2)}(\underline{\xi}; r) - \frac{h^3}{12}\left(\frac{5}{2}\sum_{j=1}^{n-1} D_j + \sum_{j=1}^{n-1}\overline{M_j}\right),$$

$$\int_a^b \overline{S_\Delta}(x; r)dx - \int_a^b \overline{f}(x; r)dx$$

$$= \frac{h}{2}A + \frac{b-a}{12}h^2\overline{f}^{(2)}(\overline{\xi}; r) + \frac{h^3}{12}\left(\frac{5}{2}\sum_{j=1}^{n-1} D_j - \sum_{j=1}^{n-1}\underline{M_j}\right),\ \underline{\xi}, \overline{\xi} \in [a, b].$$

Thus, we proved the following theorem:

**Theorem 5.1.** *Let* $\Delta = \{x_j \mid j = 0, 1, ..., n\}$ *is a partition of interval* $[a, b]$ *by knots* $a = x_0 < x_1 < \cdots < x_n = b$, *fuzzy function* $f \in C^2[a, b]$ *and for arbitrarily fixed* $r$, $0 \le r \le 1$, $\underline{y_j} = \underline{f}(x_j; r)$, $\overline{y_j} = \overline{f}(x_j; r)$, $j = 0, 1, ..., n$, *and* $h_{j+1} = x_{j+1} - x_j$, $j = 0, 1, ..., n-1$, *and* $\underline{S_\Delta}(x; r)$, $\overline{S_\Delta}(x; r)$ *are the cubic spline functions which interpolate the values of the functions* $\underline{f}$ *and* $\overline{f}$ *at knots* $x_0, x_1, ..., x_n \in \Delta$ *and satisfies* $\underline{S''_\Delta}(a; r) = \underline{S''_\Delta}(b; r) = \overline{S''_\Delta}(a; r) = \overline{S''_\Delta}(b; r) = 0$. *Then we obtain the following equations for the errors of integrations of* $\underline{f}$ *and* $\overline{f}$ *by cubic spline*:

$$\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \underline{S_\Delta}(x; r)dx - \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} \underline{f}(x; r)dx = -\frac{1}{2}\sum_{j=0}^{n-1}(h_{j+1}A_j)$$

$$+\frac{1}{12}\left\{\sum_{j=0}^{n-1}h_{j+1}^3\underline{f}^{(2)}(\underline{\xi}_j;\,r)-\frac{1}{2}\sum_{j=0}^{n-1}h_{j+1}^3(4D_j+D_{j+1})\right.$$

$$\left.-\frac{1}{2}\sum_{j=0}^{n-1}h_{j+1}^3(\overline{M_j}+\overline{M_{j+1}})\right\},$$

$$\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\overline{S_\Delta}(x;\,r)dx-\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}\bar{f}(x;\,r)dx=\frac{1}{2}\sum_{j=0}^{n-1}(h_{j+1}A_j)$$

$$+\frac{1}{12}\left\{\sum_{j=0}^{n-1}h_{j+1}^3\bar{f}^{(2)}(\overline{\xi}_j;\,r)+\frac{1}{2}\sum_{j=0}^{n-1}h_{j+1}^3(4D_j+D_{j+1})\right.$$

$$\left.-\frac{1}{2}\sum_{j=0}^{n-1}h_{j+1}^3(\underline{M_j}+\overline{M_{j+1}})\right\},$$

*for all* $\Delta$ *partition and for* $\Delta=\{x_j\,|\,x_j=a+jh,\ j=0,1,\ldots,n\}$ *and* $h=\dfrac{b-a}{n}$,

$$\int_a^b\underline{S_\Delta}(x;\,r)dx-\int_a^b\underline{f}(x;\,r)dx$$

$$=-\frac{h}{2}A+\frac{b-a}{12}h^2\underline{f}^{(2)}(\underline{\xi};\,r)-\frac{h^3}{12}\left(\frac{5}{2}\sum_{j=1}^{n-1}D_j+\sum_{j=1}^{n-1}\overline{M_j}\right),$$

$$\int_a^b\overline{S_\Delta}(x;\,r)dx-\int_a^b\bar{f}(x;\,r)dx$$

$$=\frac{h}{2}A+\frac{b-a}{12}h^2\bar{f}^{(2)}(\overline{\xi};\,r)+\frac{h^3}{12}\left(\frac{5}{2}\sum_{j=1}^{n-1}D_j-\sum_{j=1}^{n-1}\underline{M_j}\right),\ \underline{\xi},\,\overline{\xi}\in[a,\,b].$$

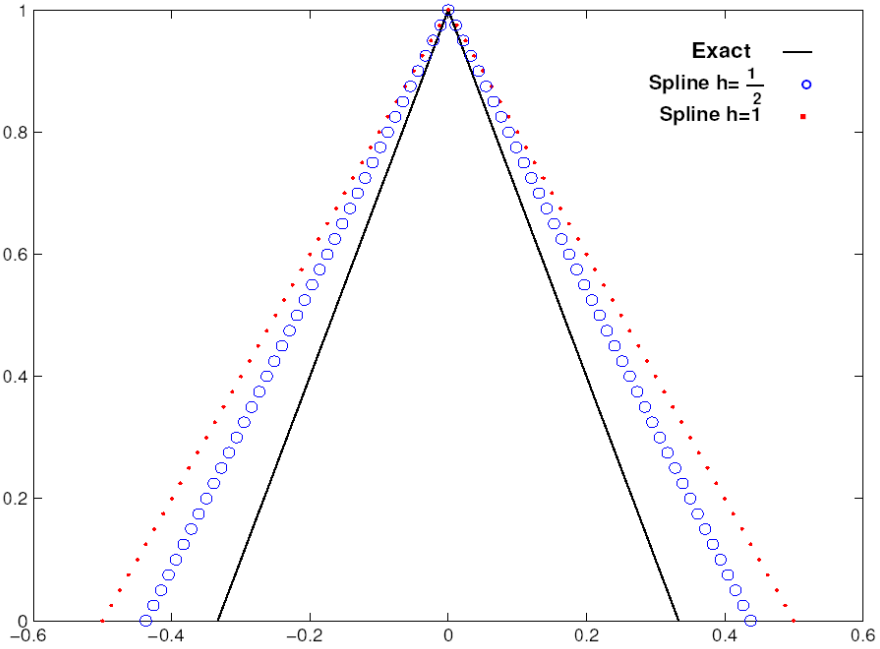


**Figure 1.**

## 6. Numerical Results

In this section we apply the method in Section 5 by solving some numerical examples.

**Example 6.1.** Consider the following fuzzy integral:

$$\int_0^1 \tilde{k}x^2 dx,\ \tilde{k} = (r-1, 1-r)$$

the exact solution of the integral is $\frac{1}{3}(r-1, 1-r)$.

From the spline integration rule in (5.3) with $h = 1$ :

$$S_1(\underline{f};\ r) = \int_0^1 \underline{S_\Delta}(x;\ r)dx = \frac{1}{2}(r-1),\ S_1(\bar{f};\ r)\int_0^1 \overline{S_\Delta}(x;\ r)dx = \frac{1}{2}(1-r),$$

$$\underline{f}^{(2)}(t;\ r) = 2(r-1),\ \bar{f}^{(2)}(t;\ r) = 2(1-r)$$

and

$$E_1(\underline{f};\, r) = -\frac{1}{6}(1-r),\; E_1(\bar{f};\, r) = -\frac{1}{6}(r-1),$$

it is clear that Theorem 5.1 holds. Now with $h = \frac{1}{2}$ :

$$S_2(\underline{f};\, r) = \int_a^b \underline{S_\Delta}(x;\, r)dx = \frac{7}{16}(r-1),\; S_2(\bar{f};\, r)\int_a^b \overline{S_\Delta}(x;\, r)dx = \frac{7}{16}(1-r),$$

$$E_2(\underline{f};\, r) = \frac{3}{48}(r-1),\; E_2(\bar{f};\, r) = \frac{3}{48}(1-r).$$

Theorem 5.1 holds too. The exact solution and the spline rule solution with $h = 1$ and $h = \frac{1}{2}$ are plotted and compared in Figure 1.

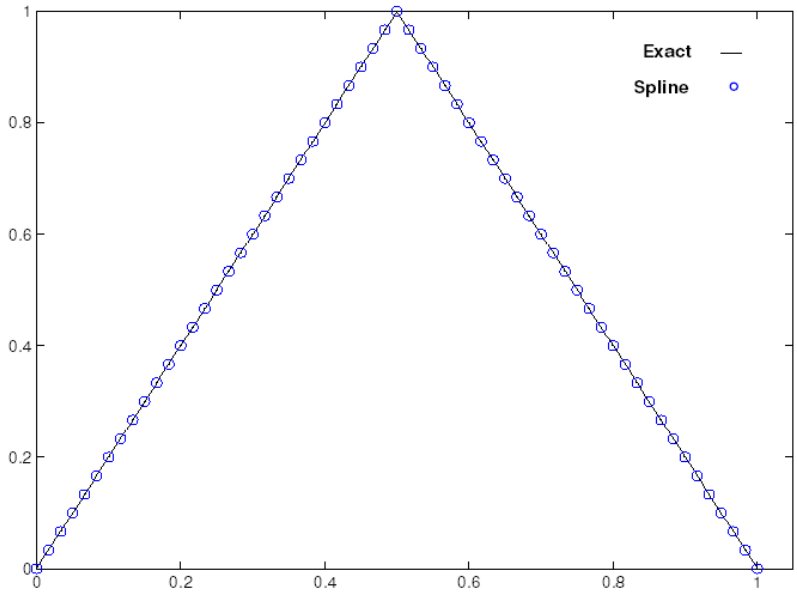


**Figure 2.**

**Example 6.2.** Consider the following fuzzy integral:

$$\int_0^1 \tilde{k}x dx,\; \tilde{k} = (r,\, 2-r),$$

the exact solution of the integral is $\frac{1}{2}(r,\ 2-r)$.

Solution obtained from cubic spline integration rule in (5.3) with $h = 1$ is equal to exact solution, see Figure 2, then:

$$E(\underline{f};\ r) = E(\bar{f};\ r) = 0.$$

## 7. Conclusion

In this paper we applied the cubic spline integration rule to solve fuzzy integral over a finite interval $[a, b]$ Since this integration yields fuzzy number in parametric form, we used the parametric form of method. The integration of triangular fuzzy number is a triangular fuzzy number.

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