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EXACT SOLUTIONS FOR THE GENERAL FIFTH ORDER KDV EQUATION BY THE EXTENDED TANH METHOD

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Abstract

In this paper we show some exact solutions for the general fifth order KdV equation $u_t + \omega_{xxxxx} + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0$. These solutions are obtained by the extended tanh method.

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1. Introduction

A large variety of physical, chemical, and biological phenomena is governed by nonlinear evolution equations. The analytical study of nonlinear partial differential equations was of great interest during the last decades. Investigations of traveling wave solutions of nonlinear equations play an important role in the study of nonlinear physical phenomena. The importance of obtaining the exact solutions, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions.

The nonlinear generalized KdV equation of fifth order (fKdV equation) reads

$$u_t + \omega u_{xxxxx} + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x = 0, \qquad (1.1)$$

where α, β, γ and ω are arbitrary real parameters.

This equation describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice and it is an important mathematical model with wide applications in quantum mechanics and nonlinear optics. Some important particular cases of Eq. (1.1) are :

- Kaup-Kupershmidt equation (KK equation) [3, 6, 7, 8, 10]

$$u_t + u_{xxxxx} + 10 u u_{xxx} + 25 u_x u_{xx} + 20 u^2 u_x = 0. \qquad (1.2)$$

- Sawada-Kotera equation (SK equation) [11, 2]

$$u_t + u_{xxxxx} + 5 u u_{xxx} + 5 u_x u_{xx} + 5 u^2 u_x = 0. \qquad (1.3)$$

- Caudrey-Dodd-Gibbon equation

$$u_t + u_{xxxxx} + 30 u u_{xxx} + 30 u_x u_{xx} + 180 u^2 u_x = 0. \qquad (1.4)$$

- Lax equation [9]

$$u_t + u_{xxxxx} + 10 u u_{xxx} + 20 u_x u_{xx} + 30 u^2 u_x = 0. \qquad (1.5)$$

- Ito equation [4, 5]

$$u_t + u_{xxxxx} + 3 u u_{xxx} + 6 u_x u_{xx} + 2 u^2 u_x = 0. \qquad (1.6)$$

As the constants α, β and γ change, the properties of the equation (1.1) drastically change. For instance, the Lax equation with $\alpha = 10$, $\beta = 20$, and $\gamma = 30$, and the SK equation where $\alpha = \beta = \gamma = 5$, are completely integrable. These two equations have *N*-soliton solutions and an infinite set of conserved densities. Another example is the KK equation with $\alpha = 10$, $\beta = 25$, and $\gamma = 20$, which is known to be integrable [7], and has bilinear representations [6, 7], but for which the explicit form of the *N*-soliton solutions is not known. A fourth equation in this class is the Ito equation, with $\alpha = 3$, $\beta = 6$, and $\gamma = 2$, which is not completely integrable, but has a limited number of special conserved densities [5].

We will find solutions of Eq. (1.1) for $\omega \neq 0$ and $\gamma \neq 0$.

2. The Extended Tanh Method

The extended tanh method [1] may be described in the following three steps:

Step 1. We search exact solutions of equation (1.1) in the form

$$\begin{cases} u(x, t) = v(\xi) \\ \xi = x + \lambda t, \end{cases} \tag{2.7}$$

As a result we have that the equation (1.1) is reduced to the nonlinear ordinary differential equation (ODE) of fifth order

$$\gamma v'(\xi)v(\xi)^2 + \alpha v^{(3)}(\xi)v(\xi) + \lambda v'(\xi) + \beta v'(\xi)v''(\xi) + \omega v^{(5)}(\xi) = 0. \tag{2.8}$$

Step 2. We seek solutions of (2.8) in the form

$$v(\xi) = a_0 + \sum_{i=1}^{m} (\varphi(\xi)^i + \varphi(\xi)^{-i}), \tag{2.9}$$

where

$$\varphi'(\xi) = k + \varphi^2(\xi), \tag{2.10}$$

where k is a parameter to be determined later.

The Riccati equation (2.10) has the general solutions:

a) If $k<0:\varphi(\xi)=-\sqrt{-k}\ \tanh(\sqrt{-k}\xi)$ and $\varphi(\xi)=-\sqrt{-k}\coth(\sqrt{-k}\xi)$.

b) If $k=0:\varphi(\xi)=-\frac{1}{\xi}$.

c) If $k>0:\varphi(\xi)=\sqrt{k}\ \tan(\sqrt{k}\xi)$ and $\varphi(\xi)=\sqrt{k}\ \cot(\sqrt{k}\xi)$.

Substituting (2.9), along with (2.10) into (2.8) and collecting all terms with the same power in $\varphi(\xi)$, we get a polynomial in the variable $\varphi=\varphi(\xi)$. This polynomial has the form

$$a\varphi(\xi)^{3m+1}+b\varphi(\xi)^{2m+3}+c\varphi(\xi)^{m+5}+\text{ lower degree terms.} \tag{2.11}$$

The parameter m can be found by balancing the highest order term in (2.11). We assume that $m\geq 1$ to avoid trivial solutions. The degrees of the highest terms are $m+5$ (the degree of the term $c\varphi(\xi)^{m+5}$), $2m+3$ (the degree of the term $b\varphi(\xi)^{2m+3}$) and $3m+1$ (the degree of the term $a\varphi(\xi)^{3m+1}$). The only integer value of m for which $3m+1=2m+3$ or $3m+1=m+5$ or $2m+3=m+5$ is $m=2$. Equating in (2.11) the coefficients of every power of $\varphi(\xi)$ to zero, we obtain an algebraic system in the variables $k,\ \lambda,\ a_0,\ a_1,\ b_1$.

Step 3. (This is the more difficult step) Solving the system in Step 2 for $k,\ \lambda,\ a_0,\ a_1,\ b_1,\ldots,$ we get $a_1=b_1=0$, so when $m=2$ solutions have the form

$$v(\xi)=a_0+a_2\varphi^2(\xi)+\frac{b_2}{\varphi^2(\xi)}.$$

We get the following solutions, where

$A=2\alpha+\beta+\sqrt{(2\alpha+\beta)^2-40\gamma\omega}$, $B=\frac{12\gamma\omega-A\beta}{8\gamma}$ and $C=\frac{(3A-10\beta)\omega}{2A}$:

- $a_0=-\frac{2Ak}{\gamma}$, $a_2=0$, $b_0=-\frac{3Ak^2}{\gamma}$, $\lambda=16Bk^2$:

$$u_1(x,t)=-\frac{Ak}{\gamma}(2+3\cot^2(\sqrt{k}(x+16Bk^2t))).$$

$$u_2(x,\,t) = \frac{Ak}{\gamma}\left(1 + 3\,\mathrm{csch}^2\sqrt{-k}(x + 16Bk^2t)\right))$$

• $a_0 = -\dfrac{80k\omega}{A}$, $a_2 = 0$, $b_0 = -\dfrac{120k^2\omega}{A}$, $\lambda = 16Ck^2$:

$$u_3(x,\,t) = -\frac{40k\omega}{A}\left(2 + 3\cot^2(\sqrt{k}(x + 256Ck^2t))\right)$$

$$u_4(x,\,t) = \frac{40k\omega}{A}\left(1 + 3\,\mathrm{csch}^2(\sqrt{-k}(x + 256Ck^2t))\right)$$

• $a_0 = -\dfrac{2Ak}{\gamma}$, $a_2 = -\dfrac{3A}{\gamma}$, $b_0 = 0$, $\lambda = 16Bk^2$:

$$u_5(x,\,t) = -\frac{Ak}{\gamma}\left(2 + 3\tan^2(\sqrt{k}(x + 16Bk^2t))\right)$$

$$u_6(x,\,t) = -\frac{Ak}{\gamma}\left(2 - 3\tanh^2\sqrt{-k}(x + 16Bk^2t)\right)).$$

• $a_0 = -\dfrac{80k\omega}{A}$, $a_2 = -\dfrac{120\omega}{A}$, $b_0 = 0$, $\lambda = 16Ck^2$:

$$u_7(x,\,t) = -\frac{40k\omega}{A}\left(2 + 3\tan^2(\sqrt{k}(x + 256Ck^2t))\right).$$

$$u_8(x,t) = -\frac{40k\omega}{A}\left(2 - 3\tanh^2(\sqrt{-k}(x + 256Ck^2t))\right).$$

• $a_0 = -\dfrac{2Ak}{\gamma}$, $a_2 = -\dfrac{3A}{\gamma}$, $b_2 = -\dfrac{3Ak^2}{\gamma}$, $\lambda = 256Bk^2$:

$$u_9(x,t) = -\frac{Ak}{\gamma}\left(2 + 3\tan^2(\sqrt{k}(x + 256Bk^2t)) + 3\cot^2(\sqrt{k}(x + 256Bk^2t))\right).$$

$$u_{10}(x,\,t) = -\frac{Ak}{\gamma}\left(2 - 3\tanh^2(\sqrt{-k}(x + 256Bk^2t)) - 3\coth^2\right.$$

$$\left.(\sqrt{-k}(x + 256Bk^2t))\right)$$

• $a_0 = -\dfrac{80k\omega}{A}$, $a_2 = -\dfrac{120\omega}{A}$, $b_2 = -\dfrac{120k^2\omega}{A}$, $\lambda = 256Ck^2$:

$$u_{11}(x,\,t) = -\frac{40k\omega}{A}\left(2 + 3\tan^2(\sqrt{k}(x + 256Ck^2t))\right.$$

$$+ 3\cot^2(\sqrt{k}(x + 256Ck^2t))).$$

$$u_{12}(x, t) = -\frac{40k\omega}{A}(2 - 3\tanh^2(\sqrt{-k}(x + 256Ck^2t))$$

$$- 3\coth^2(\sqrt{-k}(x + 256Ck^2t)).$$

3. Conclusions

In this paper, by using the extended method and the help of a symbolic computation engine, we obtain some exact solutions for the equation (1.1). At the same time, we may find solutions for *any* particular case of this equation, for example, (1.2)-(1.6). In our opinion, this result is new in the literature.

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