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THE EXISTENCE OF POSITIVE SOLUTIONS FOR SECOND-ORDER SINGULAR SEMIPOSITONE BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, by using the fixed point index theorem, the existence results of positive solutions are obtained for second-order boundary value problem

$$\begin{cases} u'' + f(t, u) - g(t, u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

where $f \in C[(0, 1) \times R^+ \to R^+]$, $g \in C[(0, 1) \times R^+ \to R]$, that is, f, g are singular at $t = 0$ and $t = 1$, and the sign of g may change.

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1. Introduction

In this paper, we consider the existence results of the following second-order singular semipositone boundary value problem:

$$\begin{cases} u'' + f(t, u) - g(t, u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where f, g are singular at $t = 0, 1$, and the sign of g may change.

In recent years, singular boundary value problems in the case of $g \equiv 0$ (i.e., positone problems) have been studied extensively (see, for example, [1, 2, 5, 6, 8] and references therein). Naturally we hope there are the same excellent results on singular semipositone boundary value problems.

Recently, Li [7] has considered the following semipositone problems without singularity

$$\begin{cases} u'' + f(t, u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $f \in C(I \times R^+)$ $I = [0, 1]$, $R^+ = [0, +\infty)$. The author obtained the existence of positive solutions by using the fixed point theorem of cone expansion and compression.

As far as we know, only a handful of papers have considered the singular semipositone boundary value problems. The purpose of this paper is to deal with the singular semipositone boundary value problems. The main features of this paper are as follows. Firstly , BVP (1.1) possesses singularity, i.e., f, g are singular at $t = 0$ and $t = 1$. Secondly, the sign of g may change. Thirdly, the tool used here is the fixed point index theorem.

2. Preliminaries

Concerning the boundary value problem (1.1), we make the following hypotheses:

(H_1) $f, g : (0, 1) \times (0, \infty) \to R$ are continuous and there exists $q(t) \in L(0, 1)$ such that

$$f(t, u) \geq 0,\ g(t, u) > -q(t),\ \forall (t, u) \in (0, 1) \times (0, \infty),$$

$$0 < \int_0^1 s^2 (1 - s)^2 q(s) ds < \infty;$$

(H_2) There exist $p_1 \in C[(0, 1) \to R^+]$, $p_2 \in C[(0, 1) \to R]$, $q_1 \in C[R^+ \to R^+]$, $q_2 \in C[R^+ \to R]$ such that

$$f(t, u) \leq p_1(t) q_1(u),\ g(t, u) \leq p_2(t) q_2(u),$$

$$0 < \int_0^1 e(s) q(s) ds < \infty,\ 0 < \int_0^1 e(s)(p_1(s) + p_2(s) + q(s)) ds < \infty,$$

where $e(s) = s(1 - s)$;

(H_3) There exists $r : r > 2\int_0^1 q(s) ds$ such that

$$\frac{r}{\max\limits_{0 \leq \|u\| \leq r} \{q_1(u), |q_2(u)|, 1\}} > \int_0^1 e(s)(p_1(s) + p_2(s) + q(s)) ds;$$

(H_4)

$$\underline{\lim}_{u \to \infty} \inf_{0 < t < 1} \frac{f(t, u)}{u} > M, \text{ where } M^{-1} = \frac{1}{2} \max_{0 \leq t \leq 1} \int_\theta^{1-\theta} G(t, s) ds,\ \theta \in (0, \tfrac{1}{2}).$$

We define $E = C[0, 1]$ be the Banach space, $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$ be the norm of E, and $G(t, s)$ be the Green's function of the following second-order boundary value problem:

$$\begin{cases} u'' = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

that is,

$$G(t, s) = \begin{cases} t(1 - s),\ 0 \leq t \leq s \leq 1, \\ s(1 - t),\ 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

It is clear that $G(t, s) \geq 0$, $G(t, s) \leq e(t) = t(1 - t)$, $\forall (t, s) \in (0, 1)$.

We define a cone $K \subset E$ by

$$K = \{u \mid u \in E,\ u(t) \geq e(t)\|u\|,\ t \in [0, 1]\}.$$

Let

$$[u]^* = \begin{cases} u, & u \geq 0, \\ 0, & u < 0. \end{cases} \tag{2.3}$$

Next we define a mapping $A : K \to E$ by

$$Au(t) = \int_0^1 G(t, s)(f(s, [u - \omega]^*) + g(s, [u - \omega]^*) + q(s))ds, \tag{2.4}$$

where

$$\omega(t) = \int_0^1 G(t, s)q(s)ds. \tag{2.5}$$

It is easy to check that u is the positive solution of the second-order singular semipositone boundary value problem (1.1) if and only if $u > 0$ and $u + \omega$ is the fixed point of A.

The following lemmas are crucial to the proof of our theorem

Lemma 2.1 [4]**.** *Let K be a cone of the real Banach space E, Ω be a bounded set of E, $\theta \in \Omega$, and $A : \overline{\Omega} \cap K \to K$ be completely continuous. Suppose that*

$$Au \neq mu, \quad \forall u \in \partial\Omega \cap K,\ m \geq 1,$$

then $i(A, \Omega \cap K, K) = 1$.

Lemma 2.2 [4]**.** *Let K be a cone of the real Banach space E, Ω be a bounded set of E, $\theta \in \Omega$, and $A : \overline{\Omega} \cap K \to K$ be completely continuous. Suppose that there exists $B : \partial\Omega \cap K \to K$ which is completely continuous such that*

(1) $\inf_{u \in \partial\Omega \cap K} \|Bu\| > 0$,

(2) $u - Au \neq mBu$, $\forall u \in \partial\Omega \cap K$, $m \geq 0$,

then $i(A, \Omega \cap K, K) = 0$.

Lemma 2.3. *If* (H_1) *holds, then* $A : K \to K$ *is completely continuous.*

Proof. Firstly, we will prove $A : K \to K$.

We define a function $\tau(s) : [0, 1] \to [0, 1]$ such that $G(\tau(s), s) = \max_{0 \le t \le 1} G(t, s)$, then we have

$$\frac{G(t, s)}{G(\tau(s), s)} = \begin{cases} \dfrac{t}{\tau(s)}, & t, \tau(s) \le s, \\ \dfrac{t(1-s)}{s(1-\tau(s))}, & t \le s \le \tau(s), \\ \dfrac{1-t}{(1-\tau(s))}, & s \le t, \tau(s), \\ \dfrac{s(1-t)}{\tau(s)(1-s)}, & \tau(s) \le s \le t \end{cases} \ge t(1-t) \equiv e(t),$$

thus, for $\forall u \in K$, we get

$$Au(t) = \int_0^1 G(t, s)(f(s, ([u-\omega]^*)) + g(s, ([u-\omega]^*)) + q(s))ds$$

$$= \int_0^1 \frac{G(t, s)}{G(\tau(s), s)} G(\tau(s), s)(f(s, ([u-\omega]^*)) + g(s, ([u-\omega]^*)) + q(s))ds$$

$$\ge e(t)\|Au\|, \ t \in [0, 1],$$

which implies $A : K \to K$.

Secondly, we prove that A is completely continuous.

For any bounded set $D \subset K$, there exists $L > 0$ such that $\|u\| \le L, \forall u \subset D$.

Thus,

$$Au(t) = \int_0^1 G(t, s)(f(s, [u-\omega]^*) + g(s, [u-\omega]^*) + q(s))ds$$

$$\le \int_0^1 G(t, s)(p_1(s)q_1([u-\omega]^*) + p_2(s)q_2([u-\omega]^*) + q(s))ds$$

$$\leq \int_0^1 G(t, s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq L} \{q_1(u), |q_2(u)|, 1\} < \infty, \quad (2.6)$$

then $A(D)$ is uniformly bounded.

From (H_2) we know that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\int_0^{\delta_1} e(s)(p_1(s) + p_2(s) + q(s))ds \leq \frac{\varepsilon}{\max_{0 \leq u \leq L} \{q_1(u), |q_2(u)|, 1\}}, \quad (2.7)$$

$$\int_{1-\delta_1}^1 e(s)(p_1(s) + p_2(s) + q(s))ds \leq \frac{\varepsilon}{\max_{0 \leq u \leq L} \{q_1(u), |q_2(u)|, 1\}}. \quad (2.8)$$

Let $a_0 = \max_{\delta_1 \leq s \leq 1-\delta_1} (p_1(s) + p_2(s) + q(s))$. Noticing that $G(t, s)$ is uniformly bounded for any $(t, s) \in [0, 1] \times [0, 1]$, we know that there exists a positive δ such that

$$|G(t_1, s) - G(t_2, s)| \leq \frac{\varepsilon}{3a_0 \cdot \max_{0 \leq u \leq L} \{q_1(u), |q_2(u)|, 1\}}. \; t_1, t_2 \in [0, 1], |t_1 - t_2| \leq \delta. \quad (2.9)$$

From (2.7) to (2.9) and the fact $G(t, s)k \leq e(s)$ for any $(t, s) \in [0, 1] \times [0, 1]$, we have

$$\begin{aligned}
&| Au(t)_1 - Au(t)_2 | \\
&\leq \int_0^{\delta_1} | G(t_1, s) - G(t_2, s) | (p_1(s)q_1([u - \omega]^*) + p_2(s)q_2([u - \omega]^*) + q(s))ds \\
&\quad + \int_{1-\delta_1}^1 | G(t_1, s) - G(t_2, s) | (p_1(s)q_1([u - \omega]^*) + p_2(s)q_2([u - \omega]^*) + q(s))ds \\
&\quad + \int_{\delta_1}^{1-\delta_1} | G(t_1, s) - G(t_2, s) | (p_1(s)q_1([u - \omega]^*) + p_2(s)q_2([u - \omega]^*) + q(s))ds \\
&\leq 2\int_0^{\delta_1} e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \frac{\varepsilon}{\max_{0 \leq u \leq L} \{q_1(u), |q_2(u)|, 1\}}
\end{aligned}$$

$$+ 2\int_0^{\delta_1} e(s)(p_1(s) + p_2(s) + q(s))ds. \frac{\varepsilon}{\max\limits_{0\le u\le L}\{q_1(u), |q_2(u)|, 1\}}$$

$$+ a_0\int_{\delta_1}^{1-\delta_1} |G(t_1, s) - G(t_2, s)|ds \cdot \frac{\varepsilon}{3a_0 \cdot \max\limits_{0\le u\le L}\{q_1(u), |q_2(u)|, 1\}}$$

$$< \varepsilon. \quad \forall u \in D, |t_1 - t_2| < \delta,$$

which implies that $A(D)$ is equicontinuous, according to Ascoli-Arzela theorem we get that A is compact, and it is easy to verify that A is continuous, thus A is completely continuous.

3. The Existence of Positive Solution

For convenience, we set

$$\mu^* = \frac{2}{L}, \ \mu_* = \frac{1}{(b+1)\varphi^{-1}(\int_0^1 a(r)dr)}.$$

The main results of this section are the following three Theorems.

Theorem 1. *Suppose* $(\mathrm{H}_1) - (\mathrm{H}_4)$ *hold, then BVP* (1.1) *has at least one positive solution.*

Proof. Firstly, we prove: $u \neq \alpha Au$, $\alpha \in [0, 1]$, $u \in \partial\Omega_r$, $\Omega_r = \{u \in \Omega | \|u\| < r\}$. If otherwise, there exist $\alpha_0 \in [0, 1]$, $u_0 \in \partial\Omega_r$ such that

$$u_0 = \alpha_0 Au_0,$$

then we have

$$u_0(t) \ge \|u_0\|e(t) = r \cdot e(t).$$

On the other hand,

$$\omega(t) = \int_0^1 G(t, s)q(s)ds \le \int_0^1 q(s)ds \cdot e(t) \le \frac{\int_0^1 q(s)ds}{r} u_0(t)ds, \ t \in [0, 1]. \quad (3.1)$$

Thus

$$u_0(t) - \omega(t) \geq \left(1 - \frac{\int_0^1 q(s)ds}{r}\right) u_0(t) \geq \frac{1}{2} u_0(t) \geq 0,\ t \in [0, 1], \tag{3.2}$$

and so

$$\begin{aligned} u_0 &= \alpha_0 A u_0 \\ &= \alpha_0 \int_0^1 G(t, s)(f(s, [u_0 - \omega]^*) + g(s, [u_0 - \omega]^*) + q(s))ds \\ &\leq \int_0^1 G(t, s)(f(s, [u_0 - \omega]^*) + g(s, [u_0 - \omega]^*) + q(s))ds \\ &\leq \int_0^1 G(t, s)(p_1(s)q_1([u_0 - \omega]) + p_2(s)q_2([u_0 - \omega]) + q(s))ds \\ &\leq \int_0^1 G(t, s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq r} \{q_1(u), |q_2(u)|, 1\} \\ &\leq \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq r} \{q_1(u), |q_2(u)|, 1\}, \end{aligned}$$

then $\|u_0\| = r \leq \int_0^1 e(s)(p_1(s) + p_2(s) + q(s))ds \cdot \max_{0 \leq u \leq r} \{q_1(u), |q_2(u)|, 1\}$, which is contradiction to (H_3). By Lemma 2.1 we know

$$i(A, \Omega_r \cap K, K) = 1. \tag{3.3}$$

By (H_4), for $\forall \varepsilon > 0$, there exists $R' > r > 0$ such that $f(t, u) > (M + \varepsilon)u \quad \forall \varepsilon > 0,\ u > R'$. Let $\Omega_R = \{u \in \Omega \mid \|u\| < R\}$, where $R \geq 2\frac{(b + a)(d + c)}{(b + a\theta)(d + c\theta)} R'$. Let $Bu(t) \equiv 1, \quad \forall t \in [0, 1]$. It is clear that $\inf_{u \in \partial\Omega_R \cap K} \|Bu\| > 0$, and $B : \partial\Omega \cap K \to K$ is completely continuous.

Secondly, we prove that $u - Au \neq mBu \quad \forall u \in \partial\Omega \cap K,\ m \geq 0$.

Otherwise, there exist $v_0 \in \partial\Omega_R,\ m_0 \geq 0$, such that $v_0 - Av_0 = m_0 Bv_0$. Let $\xi = \min\{v_0(t),\ t \in [\theta, 1 - \theta],\ \theta \in (0, \frac{1}{2})$. Since

$R > r > 0$, noticing that (3.2) holds, then for any $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} v_0(t) &= \int_0^1 G(t, s)(f(s, [v_0 - \omega]^*) + g(s, [v_0 - \omega]^*) + q(s))ds + m_0 \\ &\geq \int_\theta^{1-\theta} G(t, s)(f(s, [v_0 - \omega])ds + m_0 \\ &\geq \int_\theta^{1-\theta} G(t, s)\frac{1}{2}(M + \varepsilon)ds \cdot \xi + m_0 \\ &= M^{-1}(M + \varepsilon)\xi + m_0 \\ &> \xi, \end{aligned}$$

which contradicts $\xi = \min\{v_0(t), t \in [\theta, 1-\theta], \theta \in (0, \frac{1}{2})\}$, then for $\forall u \in \partial\Omega_R \cap K$, $m \geq 0$, we have $u - Au \neq mBu$. By Lemma 2.2, we get

$$i(A, \Omega_R \cap K, K) = 0. \tag{3.4}$$

It follows from (3.3)-(3.4) that $i(A, (\overline{\Omega_R} \setminus \Omega_r) \cap K, K) = -1$. According to the fixed point index point index theorem (see [3]), we get that A has a fixed piont $\tilde{u}$, $r \leq \|\tilde{u}\| \leq R$.

Let $v = \tilde{u} - \omega$. Since $\|\tilde{u}\| \geq r$, then $\tilde{u}(t) - \omega(t) \geq (1 - \frac{\int_0^1 q(s)ds}{r}\tilde{u}(t)$ $\geq \frac{1}{2}\tilde{u}(t) \geq 0$. Thus $v = \tilde{u} - \omega$ is a positive solution of BVP (1.1).

Remark. By the same way, we can obtain the existence results of positive solutions for fourth-order singular semipositone boundary value problem

$$\begin{cases} u^{(4)} = f(t, u) + g(t, u), \ 0 < t < 1, \\ u(0) = u'(0) = u(1) = u'(1) = 0, \end{cases}$$

where $f \in C[(0, 1) \times R^+ \to R^+]$, $g \in C[(0, 1) \times R^+ \to R]$, that is, f, g are singular at $t = 0$ and $t = 1$, and g is semipositone.

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