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ON THE DISTRIBUTION SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS ON THE REAL LINE

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Abstract

Given the following ordinary differential equation:

$$[p_m(x)\partial^m + p_{m-1}(x)\partial^{m-1} + \ldots + p_1(x)\partial + p_0(x)](\mu) = 0, \tag{0.1}$$

where μ is a distribution, $\{p_\nu(x)\}$ are polynomials, which in general may have complex coefficients, and ∂ is the first order derivation operator with respect to the variable x.

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We prove using analytic $\mathcal{D}$-module theory tools that the dimension of the solution space in the space of distributions is $m + \sum e_\nu$, where the $e'_\nu s$ are the multiplicities of the real roots a_ν of the leading polynomial coefficient $p_m(x)$. This result is an extension of a similar result highlighted by Mandai [6].

1. Introduction

Mandai [6] on the section of a review of some results for ordinary differential equations with polynomial coefficients considers an ordinary differential operator with regular singularity at the origin $t = 0$ with weight $w = m - k$ given by

$$P = t^k \partial_t^m + \sum_{j=1}^{k} a_j t^{k-j} \partial_t^{m-j} + \sum_{l=0}^{m} b_l(t) t^{d(l)} \partial_t^l, \tag{1.1}$$

where $0 \le k \le m$, $a_j \in \mathbb{C}$, $d(l) := \max\{0, l - m + k + 1\}$, and $b_l \in C^\infty(-T_0, T_0)$. He demonstrates that the dimension of the kernel of the operator P of the form (1.1) is equal to $m + k$. This result is based on the following:

Let

$$\mathcal{C}(\lambda) := \{t^{-\lambda+w} P(t^\lambda)\}\big|_{t=0} = (\lambda)_m + \sum_{j=1}^{k} a_j (\lambda)_{m-j} \in \mathbb{C}[\lambda]. \tag{1.2}$$

The polynomial in (1.2) is called the *indicial polynomial* of P. A root of $\mathcal{C}(\lambda) = 0$ is called the *characteristic exponent* (*index*) of P. We can decompose $\mathcal{C}(\lambda)$ as

$$\mathcal{C}(\lambda) = (\lambda)_w \overline{\mathcal{C}}(\lambda - w),$$

where

$$\overline{\mathcal{C}}[P](\lambda) = \overline{\mathcal{C}}(\lambda) := (\lambda)_k + \sum_{j=1}^{k} a_j (\lambda)_{k-j} \in \mathbb{C}[\lambda].$$

Mandai [6] assumes that

$$\overline{C}(\lambda) = \prod_{l=1}^{d}(\lambda - \lambda_l)^{\eta_l}, \quad d \in \mathbb{N} \text{ and all } \lambda_l \text{ are distinct,} \tag{1.3}$$

and considers distribution solutions of the forms

$$G(z) = G(z;t) := \frac{t_+^z}{\Gamma(z+1)},$$

$$G^{(j)}(z) := \partial_z^j(G(z)),$$

with the derivative $\partial_t^h(G(z)) = G(z-h)$, $(h \in \mathbb{N})$ and $G(-d) = \partial_t^d(G(0)) = \delta^{(d-1)}(t)$, for $d = 1, 2, 3, \cdots$. He proves that these yield k distribution solutions with half line support, whereas the m solutions come from power series solutions therefore totaling to $m + k$ distribution solutions to $P(\mu) = 0$ with P as in (1.1).

In this paper, we shall prove a similar result without condition (1.3) and extend it to a generic *locally Fuchsian* differential operator with the leading polynomial coefficient $p_m(x)$ having its real zeros on the real line by using methods from $\mathcal{D}$-module theory.

2. Preliminaries

If f is a distribution, then its value at the test function ϕ is denoted by $< f, \phi >$. We sometimes denote this by $\int f\phi$. The associated derivative to the tempered distribution/general distribution f is defined by

$$< f', \phi > = \int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi' = - < f, \phi'(x) >,$$

and higher order k-th derivatives are defined by

$$< f^{(k)}, \phi > = \int_{\mathbb{R}} f^{(k)}\phi = (-1)^{(k)}\int_{\mathbb{R}} f\phi^{(k)} = (-1)^{(k)} < f, \phi^{(k)}(x) >.$$

Definition 1. A distribution solution to (0.1) means a solution to the differential equation, where all the derivatives are considered in a distribution sense.

2.1. Some homological algebra definitions

Let $\{M_n, d_n\}$, $n \in \mathbb{Z}$ be a family of modules M_n and module homomorphisms, d_n defined by

$$d_n : M_n \to M_{n-1},$$

indexed by the integers, where d_n is a boundary operator in degree n.

Definition 2. A complex $\{M_n, d_n\}$, $n \in \mathbb{Z}$ is an exact sequence, if $Im(d_{n+1}) = Ker(d_{n-1})$ for all n. Particularly, an exact sequence with at most three non-trivial elements is a short exact sequence and is written as

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0.$$

Note 2.1. Given that $f : M \to N$ and $g : N \to P$, if the modules satisfy the short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$, then $P \simeq \dfrac{N}{Im(f)} = Coker(f)$ and the function g is therefore surjective.

Definition 3. $Hom_A(M, N)$ is a set of functions $\{\varphi_i\}$ over the ring A such that

$$\varphi(ax + by) = a\varphi(x) + b\varphi(y),$$

for all $a, b \in A$ and $x, y \in M$.

Definition 4. Let X be a topological space, a skyscraper sheaf denoted by $\mathcal{F}(U)$, with U a union of the disjoint open sets U_i, $i \in I$, is a sheaf such that for every $x \in X$,

$$\mathcal{F}(U) = \begin{cases} 0, & \text{if } x \notin U_i, \\ \mathcal{F}(U_i), & \text{if } x \in U_i. \end{cases}$$

Definition 5. The operator P is *locally Fuchsian* at $z = a$, if it can be written in the form

$$P = \nabla^m + r_{m-1}(z)\nabla^{m-1} + \ldots + r_1(z)\nabla + r_0(z), \tag{2.1}$$

where $\{r_\nu(z)\}$ are germs of holomorphic functions at $z = a$ and $\nabla = (z - a)\partial$.

Remark 1. P is said to be *locally Fuchsian*, if it is *locally Fuchsian* at every point.

2.2. Special case of a locally Fuchsian differential operator at the origin

Let P be a *locally Fuchsian* differential operator as in (2.1), at the origin. Set

$$\overline{P} = \nabla^m + r_{m-1}(0)\nabla^{m-1} + \ldots + r_1(0)\nabla + r_0(0), \tag{2.2}$$

then (2.2) represents a constant coefficient *locally Fuchsian* differential operator.

By the fundamental theorem of algebra, we can rewrite (2.2) as

$$\overline{P} = \prod (\nabla - \alpha_\nu)^{e_\nu},$$

where $\{\alpha_\nu\}$ is a set of distinct complex numbers and $e_1 + e_2 + \ldots + e_k = m$.

Let D be a small open disc of radius δ centered at the origin such that every $r_\nu(z)$ is holomorphic in D. In the simply connected set $D_* = D \setminus [0, \delta)$, for each pair (ν, j) with $0 \le j \le e_\nu - 1$, there exists single valued branches of multivalued analytic functions of the form

$$\rho_{\alpha_\nu, j}(z) = z^{\alpha_\nu} \cdot (\log z)^j.$$

With this notation, we give a result on the general properties of the ∇ operator.

Theorem 1. *The solution set* $\{f \in \mathcal{O}(D_*) : P(f) = 0\}$ *with* P *as in* (2.2) *is an m-dimensional vector space, which has a basis consisting of functions of the form*

$$\rho_{\alpha_\nu, j}(z) : 1 \le \nu \le k : 0 \le j \le e_{\nu-1}.$$

Proof. See Saito et al. [8]. □

2.3. Boundary value distributions ($\mathfrak{Db}$) on the real line

This subsection provides the necessary and sufficient condition for the existence of a boundary value distribution attached to an analytic function $f(z)$ as one approaches the real axis locally from the two half planes.

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing and infinitely differentiable functions on the real line $(C^\infty(\mathbb{R}))$ and $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on the real line. If $f(z)$ is an analytic function defined in a domain near the real axis such as the open rectangle defined by

$$\{(x, y) : 0 < x < a : 0 < y < b\},$$

where a and b are small finite real numbers, then it is said to have moderate or temperate growth when we approach the real axis if each compact subinterval $\alpha \le x \le \beta$ of (a, b), there exists some integer $N \ge 0$ and a constant C such that

$$|f(x + iy)| \le C \cdot y^{-N}.$$

This moderate growth is a necessary and sufficient condition for the complex valued function $f(z)$ to have a boundary value $f(x \pm i0)$ given by a distribution $\mu(f) \in \mathcal{S}'(\mathbb{R})$ defined on the real axis by the limit

$$f(x \pm i0) = \lim_{\epsilon \to 0} \int_0^a g(x) f(x \pm i\epsilon) dx : g(x) \in C_0^\infty(0, a), \tag{2.3}$$

(see [3], Theorem 3.1.13).

Definition 6. The limit integral in (2.3) yields a linear functional on test functions, which give a distribution denoted by $\mathfrak{b}(f)$ and called the *boundary value distribution* of the analytic function $f(z)$.

2.4. Schwartz reflection principle

Let $\square_+ = \{x + iy,\ 0 < x < A : 0 < y < B\}$ and $\square_- = \{x + iy,\ 0 < x < A : -B < y < 0\}$ be opposite rectangles in the upper and, respectively, lower half plane on the real line with $A, B > 0$ and let $g \in \mathcal{O}(\square_+)$ and $h \in \mathcal{O}(\square_-)$ satisfy the moderate growth condition. Then $g(z)$ and $h(z)$ will have boundary values $\mathfrak{b}g$ and $\mathfrak{b}h$, respectively, (see [3], Theorem 3.1.11).

Theorem 2. *Let $g \in \mathcal{O}(\square_+)$ and $h \in \mathcal{O}(\square_-)$ be a pair such that $\mathfrak{b}g = \mathfrak{b}h$ holds as distributions. Then g and h are analytic continuations of each other, i.e., there exists an analytic function F defined in $\{-B < y < B : 0 < x < A\}$ such that $F = g$ in $\square_+$ and $F = h$ in $\square_-$.*

Proof. See Hörmander [3]. □

Remark 2. These boundary values may be thought of as the local cohomology

$$\mathcal{H}^1_M(\mathcal{O}_X) = \mathcal{O}_U / \mathcal{O}_X,$$

where $X = M + U$ and X is a disk. The point of using local cohomology is that it is a cohomological functor, i.e., if $L \longrightarrow N \longrightarrow K$, with L, N, and K as $\mathcal{O}_X$ modules, then there is an exact sequence

$$\Gamma_M(L) \longrightarrow \Gamma_M(N) \longrightarrow \Gamma_M(K) \longrightarrow \mathcal{H}^1_M(L) \longrightarrow \ldots.$$

For the elementary properties we use here refer to Kashiwara and Schapira [5].

2.5. Malgrange's index formula on the space of holomorphic functions $(\mathcal{O})$

The following theorem gives a priori idea about the number of solutions that exist for a particular differential equation over the space of holomorphic functions $(\mathcal{O})$:

Theorem 3. *Let P be a differential operator with polynomial coefficients of some order $m \geq 1$ given in the form*

$$P(x, \partial_x) = x^k \partial^m + p_{m-1}(x)\partial^{m-1} + \ldots + p_1(x)\partial + p_0(x).$$

Assume that $k \geq 0$, where the case $k = 0$ may occur and all the functions $\{p_i(x)\}$ belong to $\mathcal{O}$. Considering the action of P on $\mathcal{O}$, we get two complex vector spaces given by the kernel and the cokernel of P on $\mathcal{O}$, where

$$Ker_P(\mathcal{O}) = \{g \in \mathcal{O} : P(g) = 0\},$$

and

$$Coker_P = \frac{\mathcal{O}}{P(\mathcal{O})}.$$

It turns out that both vector spaces are finite dimensional and satisfy the following index formula due to Malgrange

$$\dim(Ker_P) - \dim(Coker_P) = m - k.$$

2.6. Sato's sheaf theoretic equation

If the leading polynomial coefficient $p_m(x)$ has real zeros, then we will assume that $P(x, \partial_x)$ can be re-arranged so that it is *locally Fuchsian* at each real zero of $p_m(x)$. With this assumption, one says that $P(x, \partial)$ is *locally Fuchsian* on the whole real axis. Since the study shall concentrate on describing solutions locally at the zeros of $p_m(x)$ on the real line, then we shall use sheaf theory.

Let $\mathcal{D}$ be the sheaf of holomorphic differential operators over $\mathbb{C}$, $\mathcal{O}$ be the sheaf of holomorphic functions in the unit disk centered at a zero of $p_m(x)$, and $M = \{z : \Im\mathrm{m}\ z = 0\}$ be the real line considered as a closed subset of the complex z-plane.

Considering the following differential equation:

$$P(u) = 0, \tag{2.4}$$

where $P \in \mathcal{D}$ and u is an unknown function. Classically, we may associate Equation (2.4) to the left $\mathcal{D}$-module $\frac{\mathcal{D}}{\mathcal{D}P}$ in the following way (see Hotta et al. [4]):

The set $Hom_{\mathcal{D}}(\frac{\mathcal{D}}{\mathcal{D}P}, \mathcal{O})$ of $\mathcal{D}$-linear homomorphisms from $\frac{\mathcal{D}}{\mathcal{D}P}$ to $\mathcal{O}$ is isomorphic to the solution space of P, that is,

$$Hom_{\mathcal{D}}\left(\frac{\mathcal{D}}{\mathcal{D}P}, \mathcal{O}\right) \simeq \{\varphi \in Hom_{\mathcal{D}}(\mathcal{D}, \mathcal{O}) \text{ such that } \varphi(P) = 0\},$$

hence we see by $Hom_{\mathcal{D}}(\mathcal{D}, \mathcal{O}) \simeq \mathcal{O}(\varphi \longmapsto \varphi(1))$ that

$$Hom_{\mathcal{D}}\left(\frac{\mathcal{D}}{\mathcal{D}P}, \mathcal{O}\right) \simeq \{f \in \mathcal{O} \text{ such that } P(f) = 0\},$$

since $P(f) = P(\varphi(1)) = \varphi(P \cdot 1) = \varphi(P) = 0$.

This is the reason why the solution complex $Sol(\mathcal{D} / \mathcal{D}P)$ is isomorphic to $Hom_{\mathcal{D}}(\frac{\mathcal{D}}{\mathcal{D}P}, \mathcal{O})$. Using homological algebra, we get the derived version of the holomorphic solution complex

$$Sol(\mathcal{D} / \mathcal{D}P) = \mathbf{R}Hom_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O}). \tag{2.5}$$

The short exact sequence,

$$\mathcal{D} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{D} / \mathcal{D}P,$$

where the first map is multiplication by P gives a free resolution of $\mathcal{D} / \mathcal{D}P$. Hence $\mathbf{R}Hom_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O})$ is the complex

$$Hom_{\mathcal{D}}(\mathcal{D}, \mathcal{O}) \longrightarrow Hom_{\mathcal{D}}(\mathcal{D}, \mathcal{O}),$$

or $\mathcal{O} \longrightarrow \mathcal{O}$, where the map is acting by the differential operator P. This then means that the holomorphic solution complex has two cohomology modules

$$Hom_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O}) = \{f \in \mathcal{O} : P(f) = 0\} = Ker_P,$$

$$\mathcal{E}xt^1(\mathcal{D} / \mathcal{D}P, \mathcal{O}) = \mathcal{O} / P(\mathcal{O}) = Coker_P.$$

3. Main Results and Proofs

3.1. Introduction

In this subsection, we shall state and prove the main theorem about the dimension of the complex vector space $Ker_P(\mathfrak{D}\mathfrak{b})$ under the condition that P is *locally Fuchsian* at each real zero of the leading polynomial $p_m(x)$. This proof will be done on the premise that the *locally Fuchsian* differential operator P is locally surjective on the space of distributions. Below we recall the set up of the problem.

Let $\{a_\nu\}$ be the set of real zeros of $p_m(x)$ with some multiplicity $e_\nu \geq 1$, so that the polynomial coefficient $p_m(x) = q(x)(x - a_\nu)^{e_\nu}$ at each a_ν, where $q(a_\nu) \neq 0$. At each zero a_ν, the operator P in (0.1) can be re-written as

$$P(x, \partial) = q(x) \cdot [(x - a_\nu)^{e_\nu} \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_1(x)\partial + r_0(x)],$$

where $\{r_j(x) = \dfrac{p_j(x)}{q(x)}\}$ are rational functions with no pole at a_ν. The differential operator

$$P_*(x, \partial) = (x - a_\nu)^{e_\nu} \partial^m + r_{m-1}(x)\partial^{m-1} + \ldots + r_1(x)\partial + r_0(x),$$

can be identified with a germ of differential operators with coefficients in the local ring $\mathcal{O}(a_\nu)$ of germs of analytic functions. At each a_ν, we define a subfamily of the ring $\mathcal{D}$ of differential operators that are *locally Fuchsian*. It means that we impose the condition that P is *locally Fuchsian* at each a_ν, such that P is of the form

$$(x - a_\nu)^{m-e_\nu} \cdot P(x, \partial) = \nabla^m + g_{m-1}(x)\nabla^{m-1} + \ldots + g_1(x)\nabla + g_0(x), \quad (3.1)$$

where $\nabla = (x - a_\nu)\partial$ is the first order Fuchsian operator and the rational functions $g_j(x)$ have no poles at each $\{x = a_\nu\}$.

Theorem 4. *The locally Fuchsian differential operator* (3.1) *is locally surjective on the space of distributions* $\mathfrak{D}\mathfrak{b}$.

Proof. The holomorphic solution complex (2.5) to $P(z, \partial)$ has cohomology modules

$$\mathcal{E}^0 = Hom_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O}) : \mathcal{E}^1 = \mathcal{E}xt^1_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O}).$$

Since $P(z, \partial_z)$ is surjective on the stalks of $\mathcal{O}$, where the leading polynomial is non zero, it follows that $\mathcal{E}^1$ is a skyscraper sheaf supported by zeros of $p_m(x)$. The restriction of $\mathcal{E}^1$ to the real line M is therefore given by

$$\mathcal{E}^1|M = \oplus \mathbb{C}^{\mu_\nu}_{\{a_\nu\}} : \mu_\nu = \dim_{\mathbb{C}} \left\{ \frac{\mathcal{O}(a_\nu)}{P(\mathcal{O}(a_\nu))} \right\},$$

so the support of $\mathcal{E}^1|M$ consist of those real zeros of $p_m(x)$, where $P(z, \partial_z)$ fails to be surjective on the local ring of germs of analytic functions.

Next, Cauchy's theorem entails that the sheaf $\mathcal{E}^0$ is locally free of rank m in $\mathbb{C} \setminus p_m^{-1}(0)$. With $A > 0$ chosen so that $p_m(x) \neq 0$ in $\square_+ = \{0 < \Im \mathrm{m} z < A\}$ and $\square_- = \{-A < \Im \mathrm{m} z < 0\}$, it follows that $\mathcal{E}^0$ restricts to a free sheaf of rank m in each of these strips. Since the strips are simply connected domains, the higher local cohomology sheaves $\mathcal{H}^p_M(\mathcal{E}^0) = 0$, when $p \geq 2$ and since $\mathcal{E}^0$ appears as a sub sheaf of $\mathcal{O}$ it has no sections supported by M, i.e., the sheaf $\mathcal{H}^0_M(\mathcal{E}^0) = 0$.

Consider the complex $\mathcal{O} \xrightarrow{P} \mathcal{O}$, which is used to calculate $\mathbf{R}Hom_{\mathcal{D}}(\mathcal{D} / \mathcal{D}P, \mathcal{O})$. Applying local cohomology to each of the two short exact sequences

$$Ker_P \hookrightarrow \mathcal{O} \twoheadrightarrow im_P,$$

$$im_P \hookrightarrow \mathcal{O} \twoheadrightarrow \mathcal{O} / im_P,$$

and using the previous vanishing results, we get

$$0 \longrightarrow \mathcal{H}^1_M(Ker_P) \longrightarrow \mathcal{H}^1_M(\mathcal{O}) = \mathfrak{Db} \xrightarrow{g} \mathcal{H}^1_M(im_P) \longrightarrow 0, \qquad (3.2)$$

and

$$\Gamma_M(\mathcal{O}/im_P) \hookrightarrow \mathcal{H}^1_M(im_P) = \mathfrak{Db} \xrightarrow{f} \mathcal{H}^1_M(\mathcal{O}) \longrightarrow 0. \qquad (3.3)$$

The composition $f \circ g$ is the map

$$\mathcal{H}^1_M(\mathcal{O}) = \mathfrak{Db} \longrightarrow \mathcal{H}^1_M(\mathcal{O}) = \mathfrak{Db},$$

induced by applying P to the sheaf of distributions. Now using the two sequences (3.2) and (3.3), we see that $f \circ g$ is locally surjective, since both g and f are. This means that P is locally surjective on the sheaf of distributions.

4. Some Result on the Local Study of Distribution Solutions at $z = 0$

Consider the differential operator $P(x, \partial_x)$ with polynomial coefficients $q_i(x) \in \mathcal{O}$ defined locally at $z = 0$ by

$$P(x, \partial_x) = x^k \partial^m + q_{m-1}(x)\partial^{m-1} + \ldots + q_0(x).$$

Let $D\backslash K$ be a simply connected set, where K is the set $[0, \epsilon]$ with $\epsilon \approx 0$ and assume that $k \leq m$ and that near the point $x = 0$, P is locally Fuchsian in the sense of the previous section, i.e.,

$$x^{m-k}P(x, \partial) = \nabla^m + r_{m-1}(x)\nabla^{m-1} + \ldots + r_1(x)\nabla + r_0(x), \qquad (4.1)$$

where $r_i(x) \in \mathcal{O}$. If we consider $Q(x, \partial)(\mu) = 0$, where Q is the right hand side operator in (4.1), then we get an m-tuple of analytic functions (solutions) $\{\phi_\nu(x)\}$ of the form $\psi(x) \cdot x^\alpha (\log x)^j$, where $\psi(x)$ is any continuous function by a Theorem 1.3.3 (Saito et al. [8]), which is a generalization of Theorem 1.

Each of these restrict to a function on the interval $0 < x < \delta$ and we consider them as distributions with support on the half line $\{x \geq 0\}$ and refer to them as Euler distributions denoted by $\{\phi_\nu(x)_+\}$.

Let the action of P on these distributions be given by

$$\gamma_\nu = P(\phi_\nu(x)_+). \tag{4.2}$$

The distribution γ_ν is zero outside the origin. Consider the vector space of all Dirac distributions at $x = 0$ denoted by $\mathfrak{Dir}$ and defined by

$$\mathfrak{Dir} = \oplus\mathbb{C} \cdot \delta_0^{(k)} : k \geq 0,$$

where $\delta_0^{(k)} = \partial^k(\delta_0)$ is the k-th order derivative of the Dirac function for each positive integer k.

Restricting the action of P on $\mathfrak{Dir}$, we can take its range and consider the quotient space

$$W = \frac{\mathfrak{Dir}}{P(\mathfrak{Dir})}. \tag{4.3}$$

At the same time, we have the P-kernel on $\mathfrak{Dir}$. Using linear algebra, one has the index formula

$$|W| - |Ker_P(\mathfrak{Dir})| = m - k. \tag{4.4}$$

Next, let V be the m-dimensional vector space spanned by the γ distributions in (4.2). We have the induced linear map

$$\overline{P} : V \to W,$$

where we take the image of the Dirac distributions γ in the quotient space W. We state below a crucial result, which is the basis for the proof of Theorem 6.

Theorem 5. *The induced linear map $\overline{P}$ is surjective.*

Proof. For the general local surjectivity of P refer to Theorem 4. In this particular case of the induced linear map $\overline{P}$, we prove that for every non zero $\mu \in W$, there exists a distribution $\gamma \in V$ such that under the induced linear map, $\overline{P}(\gamma) = \mu.$. We note that if such a distribution γ does

not exist, then $\overline{P}(\gamma) = 0$ for all $\gamma \in V$ thus all $\gamma \in Ker\ \overline{P}(\mathfrak{Dir})$, which is a contradiction as $Ker\ \overline{P}(\mathfrak{Dir}) = P(\mathfrak{Dir})$ and $\gamma \notin P(\mathfrak{Dir})$. □

4.1. Leray-Sato formula

From the discussion in Subsection 2.6, where the action of P on $\mathcal{O}$ yields two vector spaces. Considering a point, say 0 (assumed to be one of the zeros of $p_m(x)$), and letting $|\cdot|$ denote the dimension of a finite dimensional complex vector space it follows that,

$$|Ker_P(\mathfrak{Db}(0))| = |\frac{\mathcal{O}(0)}{P(\mathcal{O}(0))}| + |\mathcal{H}^1_M(\mathcal{F}(0))|, \tag{4.5}$$

where $\mathcal{F}^0$ is the sheaf of solutions to the equation $P(f) = 0$ and $\mathcal{F}(0)$ means the stalk of the sheaf $\mathcal{F}$ at a point 0.

We may generalize (4.5) to Theorem 6 as follows:

Theorem 6. *If P is locally Fuchsian at every real zero a_ν of $p_m(x)$, i.e., of the form in* (3.1)*, then $Ker_P(\mathfrak{Db})$ is a complex vector space, whose dimension is equal to $m + e_1 + e_2 + \ldots + e_k$.*

The proof of the above theorem will follow in Subsection 4.4.

4.2. An illustration of Theorem 6

Example 4.1. Consider the second order differential operator

$$P = x^2\partial^2 + x\partial + x^2 - 1.$$

In terms of the Fuchsian operator $\nabla = x\partial$, we have

$$P = \nabla^2 + x^2 - 1. \tag{4.6}$$

The equation $P(f(x)) = 0$ has a holomorphic solution given by the entire function $f(x)$ with a series expansion

$$f(x) = x + c_3x^3 + c_5x^5 + \ldots,$$

whose coefficients can be found by recursive equations. The action of P on $f(x)\cdot H_{\pm}$, i.e., $P(f(x)\cdot H_{+})$ and $P(f(x)\cdot H_{-})$ gives that the $Ker_P(\mathfrak{Db}(\mathbb{R}))$ contains the distributions $f(x)\cdot H_{+}$ and $f(x)\cdot H_{-}$. Theorem 6 applied to P predicts that there exists two other linearly independent solutions, one of them being the Dirac measure since $\nabla(\delta_0) = -\delta_0$ and $\nabla^2(\delta_0) = \delta_0$, thus $P(\delta_0) = 0$. For the fourth distribution solution, we consider the function $g(x)$ defined on $\{x > 0\}$ by

$$g(x) = \log x \cdot f(x).$$

The action of P on $g(x)$ while noting that $\nabla(\log x) = 1$ gives

$$P(g(x)) = 2\nabla(f),$$

$\nabla^2(\frac{1}{x}) = \frac{1}{x}$ on $\{x > 0\}$ implies $P(\frac{1}{x}) = x$. From the action of P on $g(x)$ and $\frac{1}{x}$, there exists a function $\phi(x)$ with a series expansion $d_3x^3 + d_5x^5 + \ldots$, such that

$$P(2x^{-1} + \phi(x)) = 2\nabla(f) : x > 0.$$

It follows that the fourth solution is the boundary value distribution

$$\mu = \log(x + i0)\cdot f(x) - 2(x + i0)^{-1} - \phi(x).$$

Thus,

$$Ker_P(\mathfrak{Db}) = \{\delta_0,\ f(x)\cdot H_{+},\ f(x)\cdot H_{-},\ \mu\}.$$

The prediction in Theorem 8 is also fulfilled, since

$$Ker_P(\mathfrak{Db}(+)) = \{\delta_0,\ f(x)\cdot H_{+}\}.$$

The local surjectivity in Theorems 4 and 5 gives:

Theorem 7. *With P as in* (4.1), *the P-kernel on the space of germs of distributions at $x = 0$ has dimension $m + k$.*

Let us first give an illustration of the above theorem.

Example 4.2. Consider the operator

$$P = z\partial^2 + \partial.$$

This operator has the constant functions as its holomorphic solution and it is surjective as an operator from $\mathcal{O}$ to $\mathcal{O}$ so Malgrange's index formula gives that this constant function generates the holomorphic P-kernel. In the upper half-plane, we have the analytic function $\phi(z)$, which solves P given by

$$\phi(z) = \int_i^z \frac{1}{\zeta}\, d\zeta,$$

where as we approach the real line from the upper half plane, we have the boundary value distribution $\phi(x + i0)$ and in the lower half plane, we have $\psi(x - i0)$, therefore the three dimensional P-kernel on $\mathfrak{Db}$ has a basis given by the analytic constant functions and the boundary value distributions $\phi(x + i0)$ and $\psi(x - i0)$.

Proof (of Theorem 7)**.**

Consider the action of P on the local ring of germs of analytic functions at the origin. Malgrange's theorem on the index of P on $\mathcal{O}$ asserts that

$$|Ker_P(\mathcal{O}(0))| - \left|\frac{\mathcal{O}(0)}{P(\mathcal{O})(0)}\right| = m - k. \tag{4.7}$$

If we let $|Ker_P(\mathcal{O}(0))| = s$, then it follows from sheaf theory that

$$|\mathcal{H}^1_M(\mathcal{F}(0))| = s + 2(m - s) = 2m - s.$$

Using this, together with the Leray-Sato formula (4.5), we have that

$$|Ker_P(\mathfrak{Db}(0))| = 2m - s + k + s - m = m + k. \qquad \square$$

4.3. Distributions supported on the half-line $\{x \geq 0\}$

Let $\mathfrak{Db}(+)$ be the space of distributions on the real line $\mathfrak{Im}z = 0$ with support on the half-line $\{x \geq 0\}$, i.e.,

$$\mathfrak{Db}(+) = \begin{cases} \mu \in \mathfrak{Db} : \operatorname{supp}(\mu) = \{x \geq 0\}, \\ 0 \qquad\quad : \operatorname{supp}(\mu) = \{x < 0\}, \end{cases}$$

then the following theorem gives the dimension of the kernel of P on the space of distributions with support on the half-line $\{x \geq 0\}$ denoted by $|Ker_P(\mathfrak{Db}(+))|$.

Theorem 8. $|Ker_P(\mathfrak{Db}(+))|$ *to the locally Fuchsian operator in* (4.1) *is k.*

Proof. A modification of the Sato-Leray formula in Subsection 4.1, gives the following dimension equation:

$$|Ker_P(\mathfrak{Db}(+))| = \left|\frac{\mathcal{O}}{P(\mathcal{O})}\right| + \mu, \tag{4.8}$$

where μ is the number of non-holomorphic solutions to P. This follows since $\mu = |\mathcal{H}^1_M(\mathcal{E}^0)(0)|$ and $\mathcal{H}^1_M(\mathcal{E}^0)$ is locally identified with the analytic solutions in the upper half plane that do not propagate over the real axis (see Subsection 2.4). This implies that the dimension of the kernel of P on $\mathcal{O}$ denoted by $|Ker_P(\mathcal{O})|$ is $m - \mu$.

By Malgrange's index formula, the following relation is true:

$$|Ker_P(\mathcal{O})| - \left|\frac{\mathcal{O}}{P(\mathcal{O})}\right| = m - k. \tag{4.9}$$

Since $|Ker_P(\mathcal{O})| = m - \mu$, then substituting this in Equations (4.8) and (4.9) gives the dimension requested in Theorem 8. □

4.4. The proof of Theorem 6

The strategy in the proof is to pass to the complex z-plane, where P gives the holomorphic differential operator $P(z, \partial_z)$ acting on complex analytic functions $\{\phi_i(z)\}$ defined in open subsets of $\mathbb{C}$. The construction of boundary values attached to the analytic functions $\{\phi_i(z)\}$ is one other tool in the description of the dimension of the distribution solution space $(\mathfrak{Db})$.

Let $\mathcal{F}$ be the m-dimensional subspace of $\mathfrak{Db}$ as in Section 4. Let a_ν be a fixed real zero of $p_m(x)$, V_{a_ν} be the space of solutions f that vanish for $x < a_\nu$ and do not vanish in any neighborhood of a_ν.

Claim. The spaces V_{a_ν} are linearly independent, that is, if $\sum_\nu f_\nu = 0$, $f_\nu \in V_{a_\nu}$, then $f_\nu = 0$ for all ν.

Proof of Claim. We prove this by contradiction, assume $\sum_\nu f_\nu = 0$ and that at least one of the f_ν is different from zero. Let a_{ν_0} be the largest of the real numbers a_ν for which $f_\nu \neq 0$. Then, we write

$$\sum_{\nu \neq \nu_0} f_\nu + f_{\nu_0} = 0. \tag{4.10}$$

If Equation (4.10) is true, then f_{ν_0} could be expressed as the boundary value from the upper half-plane. But f_ν is known to vanish for $x < a_\nu$ so f_{ν_0} will vanish in a neighborhood of a_{ν_0}. But by the Theorem 15.23 (Rudin [7]), a boundary value of an analytic function, which vanishes in an open set must vanish identically. Hence f_{ν_0} must vanish, which is a contradiction and thus proves the claim.

This claim therefore implies that

$$|Ker_P(\mathfrak{Db})| \geq m + \sum e_\nu. \tag{4.11}$$

Recalling that the real zeros a_ν of $p_m(x)$ occur with respective multiplicities e_ν. Then every solution must be real analytic except at the zeros. Assume that the real zeros are arranged in ascending order $a_1 < a_2 < \ldots < a_k$.

Let L be the space of all solutions, L_0 be the space of real analytic solutions, and L_ν be the space of real analytic solutions at the respective

real zeros a_ν. From Cauchy's classic result, we know that $\dim(L_0) = m$. The $\dim(L / L_1) = e_1$. Moreover, since the solution space $L_\nu, \nu \geq 2$, can only have a singularity at the respective singularities a_ν, then

$$|Ker_P(\mathfrak{Db})| = \dim(L) \leq m + \sum e_\nu. \tag{4.12}$$

Comparing Equations (4.11) and (4.12), then the proof of Theorem 6 follows.

4.5. Local distribution solutions

Let m be a positive integer and consider a Fuchsian differential operator

$$P = \nabla^m + p_{m-1}(x)\nabla^{m-1} + \ldots + p_0(x), \tag{4.13}$$

where $\{p_\nu(x)\}$ are holomorphic functions in some disc D centered at the origin. For the P-kernel on the space of distribution solutions $\mathfrak{Db}$, the following theorem that is a corollary to Theorem 6 about its dimension is true:

Corollary 1. *The complex vector space of boundary value distribution solutions to the operator P defined in* (4.13) *denoted by $Ker_P(\mathfrak{Db})$ is* $2m$ *dimensional.*

Example 4.3. Consider an operator P of the second order defined by $P := z^2\partial^2 + z\partial - 1$ and seek boundary value distributions μ such that

$$z^2\partial^2 + z\partial - 1(\mu(f)) = 0,$$

where $f \in \mathbb{C}^\infty(\mathbb{R})$. We show that the space of distributions μ defined on the real axis satisfying $P(\mu(f)) = 0$ is a 4-dimensional vector space.

Note 4.1. (a) The operators x and ∂ are related by $\partial \bullet x = x\partial + 1$.

(b) $x^2\partial^2 = x\partial^2 \bullet x - 2x\partial$.

(c) $x^k\delta_0 = 0, \quad \forall k \geq 1$.

Among the solutions is the Dirac delta distribution δ_0 since;

$$[x^2\partial^2 + x\partial - 1](\delta_0) = [x\partial^2 \bullet x - 2x\partial](\delta_0) + [\partial \bullet x - 1](\delta_0) - 1(\delta_0).$$

Using Note 4.1, then $P(\delta_0) = 0$.

The holomorphic function $\mu = x$ is the other solution, this exists because the operator P is not surjective on $\mathcal{O}$, the non-analytic function $\phi(z) = \frac{1}{z}$ contributes the two linearly independent solutions $\frac{1}{x + i0}$ and $\frac{1}{x - i0}$.

Therefore, the solution space has the four distributions $\left\{\frac{1}{x + i0}, \delta_0, x, \frac{1}{x_+} + 2\delta_0\right\}$.

The proof of Corollary 1

Proof. Let $m = \sum e_\nu$ in (4.11) and (4.12). □

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