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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH LIPSCHITZ CONTINUITY

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Abstract

In this paper, we discuss existence and uniqueness of solutions of initial-value problems for nonlinear fractional differential equations with Riemann-Liouville differential derivatives. In principle, we may reduce such an equation to an integral equation with regular or weakly singular kernel and then apply to it basic techniques of fixed points theorems. Proceeding in this direction, we present a list of theorems of existence and uniqueness of continuous solutions on $[0, a]$ or $(0, a]$, and give some examples to illustrate that these theorems can be used directly as a method for solving fractional differential equations.

1. Introduction

We consider a nonlinear fractional differential equation, which takes the form

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$$ {}_0D_t^p u(t) = f(t, u(t)), \quad t > 0, $$

where $0 \leq m - 1 < p < m$, m is a positive integer number and ${}_0D_t^p$ is the fractional derivative (the Riemann-Liouville, the Miller-Ross, the Caputo, or the Grünwald-Letnikove fractional derivative) considered in $\mathbb{R}^+$. Such equations have recently proved to be valuable tools in the modelling of many physical and biological phenomena [5, 6, 8, 11]. The case $0 < p < 1$ seems to be particularly important, but there are also some applications for $p > 1$. In principle, one may reduce such an equation to an integral equation with regular (if $p > 1$) or weakly singular (if $0 < p < 1$) kernel and apply to it basic techniques of nonlinear analysis (see, for example, fixed points theorems, [1]). We proceed in this direction here, presenting some theorems of existence and uniqueness in the sense of Riemann-Liouville derivatives.

It is necessary to mention that some similar results, for a fractional differential equation, have been obtained. For example, in 1999, Podlubny discussed the existence and uniqueness of a solution of an initial-value problem in terms of the Miller-Ross sequential fractional derivatives under the condition of the continuity of the function $f(t, u)$ (see [10], Subsection 3.2); in 1996, Delbosco and Rodinou obtained a unique theorem by assuming $f(t, u) = u^{\alpha}$ (see [3], Theorem 4.5); and in 2002, Diethelm and Ford gave existence and uniqueness theorems in Caputo derivatives sense (see [4], Theorems 2.1 and 2.2). However, theorems in this paper assume only the continuity of the function $t^{\sigma} f(t, u)$, $(0 \leq \sigma < 1)$, which is obviously weaker than those needed in above theses.

The paper is organized as follows. In Section 2, we recall the definitions of fractional integration operator and fractional differentiation operator and related results used in the text. Section 3 contains results for solutions, which are continuous at the origin, and Section 4 considers continuous solutions on $(0, a]$.

2. Definitions and Preliminary Results

In this section, we shall give some definitions and basic results, which will be used to prove theorems in Sections 3 and 4. For more results, we refer the reader to Podlubny [10] or other texts on basic fractional calculus [2, 3, 9].

The notation of following function spaces are given by Delbosco et al. [2, 3].

Let us denote by $C^0(\mathbb{R}^+)$ the space of all continuous real functions defined on $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$ and by $L^1_{loc}(\mathbb{R}^+)$ the space of all continuous real functions defined on $\mathbb{R}^+$, which are Lebesgue integrable on every bounded subinterval of $\mathbb{R}^+$. Consider also the space $C^0(\mathbb{R}^+_0)$ of all continuous real functions defined on $\mathbb{R}^+_0 = \{x \in \mathbb{R}, x \geq 0\}$, which later on we shall identify, by abuse of notation, with the class of all $f \in C^0(\mathbb{R}^+)$ such that $f(0+) = \lim_{t \to 0+} f(t) \in \mathbb{R}$. Furthermore, we denote by $C^0_r(\mathbb{R}^+_0)$, $r \geq 0$ the space of all functions $f \in C^0(\mathbb{R}^+)$ such that $x^r f \in C^0(\mathbb{R}^+_0)$. We define similarly $C^0_r([0, a])$, which turns out to be a Banach space when endowed with the norm

$$\| f \|_r = \max_{x \in [0, a]} x^r |f(x)|. \tag{2.1}$$

We have $C^0_0(\mathbb{R}^+_0) = C^0(\mathbb{R}^+_0)$ and $C^0_0([0, a]) = C^0([0, a])$, the Banach space of all continuous functions on $[0, a]$ with Chebyshev norm $\| f \| = \max_{x \in [0, a]} |f(x)|$.

Definition 2.1 [10]**.** The Riemann-Liouville fractional integral of order $p > 0$ of a function $f : (t_0, +\infty) \to \mathbb{R}$ is given by

$${}_{t_0}D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_{t_0}^{t} (t - \tau)^{p-1} f(\tau) d\tau,$$

provided that the right side is pointwise defined on $(t_0, +\infty)$.

For the simplicity, we assume without loss of generality that $t_0 = 0$, and henceforth, we refrain from explicitly mentioning this parameter. For instance, ${}_0D_t^{-p}f(t)$ exists for all $p > 0$, when $f \in C^0(\mathbb{R}^+)\bigcap L^1_{loc}(\mathbb{R}^+)$; note also that when $f \in C^0(\mathbb{R}_0^+)$, then ${}_0D_t^{-p}f(t) \in C^0(\mathbb{R}_0^+)$ and moreover $[{}_0D_t^{-p}f(t)]_{t=0} = 0$.

Example 1 (see [10], Equation (2.56))**.**

$$ {}_0D_t^{-p}(t^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+p+1)}t^{\mu+p}, \quad p > 0,\ \mu > -1. \tag{2.2} $$

Recall that the law of composition ${}_aD_t^{-p}({}_aD_t^{-q}f(t)) = {}_aD_t^{-p-q}f(t)$ holds for all real values $p, q > 0$.

A more precise analysis of the operator ${}_aD_t^{-p}f(t)$ can be given in the frame of $C_r^0(\mathbb{R}_0^+)$, $r \geq 0$.

Proposition 2.2. *Let* $p > 0$; *if* $f \in C_r^0(\mathbb{R}_0^+)$ *with* $r < \min\{1, p\}$, *then* ${}_0D_t^{-p}f(t) \in C^0(\mathbb{R}_0^+)$, *with* $[{}_0D_t^{-p}f(t)]_{t=0} = 0$. *If* $f \in C_p^0(\mathbb{R}_0^+)$ *with* $p < 1$, *then* ${}_0D_t^{-p}f(t)$ *is bounded at the origin*, *whereas if* $f \in C_r^0(\mathbb{R}_0^+)$ *with* $p < r < 1$, *then we may expect* ${}_0D_t^{-p}f(t)$ *to be unbounded at the origin* (*see* [3], *Section* 2).

Definition 2.3 [10]**.** The Riemann-Liouville differential operator of fractional order $p > 0$ of a continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$ {}_0D_t^{p}f(t) = \frac{1}{\Gamma(m-p)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-p-1}f(\tau)d\tau, \quad t > 0, $$

where m is the integer number defined by $m - 1 < p < m$.

Example 2. As a basic example, we quote for $\lambda > -1$

$$ {}_0D_t^{p}(t^{\lambda}) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-p+1)}t^{\lambda-p}, \quad p > 0, \tag{2.3} $$

which is formally identical to the expression (2.2) (see [10], Equation (2.117)).

In view of $\Gamma(-n) = \infty$, $(n = 0, 1, \cdots)$, formula (2.3) gives in particular

$$ {}_0D_t^{p-k}(t^{p-j}) = \begin{cases} \dfrac{\Gamma(1+p-j)}{\Gamma(1+k-j)} t^{k-j}, & k > j; \\ \Gamma(1+p-j), & k = j; \\ 0, & k < j, \end{cases} \tag{2.4} $$

where $m - 1 < p < m$, $k, j = 1, 2, \cdots, m$.

Let us now consider some properties of the Riemann-Liouville fractional derivatives. The first property of the Riemann-Liouville fractional derivative is the following laws of composition.

Proposition 2.4 (see [10], Equation (2.106)). *For* $p > 0$ *and* $t > 0$

$$ {}_aD_t^p({}_aD_t^{-p} f(t)) = f(t), \tag{2.5} $$

which means that the Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order p.

As with conventional integer-order differentiation and integration, fractional differentiation and integration do not commute.

Proposition 2.5 (see [10], Equation (2.108)). *If the fractional derivative* ${}_0D_t^p f(t)$, $(m - 1 < p < m)$, *of a function* $f(t)$ *is integrable, then*

$$ {}_0D_t^{-p}({}_0D_t^p f(t)) = f(t) - \sum_{j=1}^{m} \left[{}_0D_t^{p-j} f(t)\right]_{t=0} \frac{t^{p-j}}{\Gamma(p-j+1)}. \tag{2.6} $$

Remark 2.6. The existence of all terms in (2.6) follows from the integrability of ${}_0D_t^p f(t)$, i.e., ${}_0D_t^p f(t) \in C^0(\mathbb{R}^+) \bigcap L^1_{loc}(\mathbb{R}^+)$, because due to this condition the fractional derivatives ${}_0D_t^{p-j} f(t)$, $(j = 1, 2, \cdots, m)$ are all bounded at $t = 0$.

An important particular case must be mentioned.

Corollary 2.7. *If* $f(t) \in C_r^0(\mathbb{R}_0^+)$ *with* $r < m - p$ *and* ${}_0D_t^p f(t) \in C^0(\mathbb{R}^+) \cap L_{loc}^1(\mathbb{R}^+)$, *then* $[{}_0D_t^{p-m} f(t)]_{t=0} = 0$ *and*

$$ {}_0D_t^{-p}({}_0D_t^p f(t)) = f(t) - \sum_{j=1}^{m-1} \left[{}_0D_t^{p-j} f(t)\right]_{t=0} \frac{t^{p-j}}{\Gamma(p-j+1)}. \tag{2.7} $$

Theorem 2.8. *Let* $m - 1 < p < m$. *If we assume* $u \in C^0(\mathbb{R}^+) \bigcap L_{loc}^1(\mathbb{R}^+)$, *then the fractional differential equation*

$$ {}_0D_t^p u(t) = 0 \tag{2.8} $$

has

$$ u(t) = \sum_{j=1}^{m} c_j t^{p-j}, \quad c_j \in \mathbb{R}, \quad (j = 1, 2, \cdots, m) \tag{2.9} $$

as unique solutions.

Proof. Applying the integration operator ${}_0D_t^{-p}$ to the two sides of the Equation (2.8) and by using the formula (2.6), we obtain

$$ {}_0D_t^{-p}({}_0D_t^p u(t)) = u(t) - \sum_{j=1}^{m} \left[{}_0D_t^{p-j} u(t)\right]_{t=0} \frac{t^{p-j}}{\Gamma(p-j+1)} = 0, $$

$$ u(t) = \sum_{j=1}^{m} \left[{}_0D_t^{p-j} u(t)\right]_{t=0} \frac{t^{p-j}}{\Gamma(p-j+1)}. \tag{2.10} $$

Substituting directly into the formula (2.10)

$$ c_j = \frac{1}{\Gamma(p-j+1)} \left[{}_0D_t^{p-j} u(t)\right]_{t=0} < \infty, \quad (j = 1, 2, \cdots, m), $$

we obtain the final expression of the solutions of the fractional differential equation (2.8):

$$u(t) = \sum_{j=1}^{m} c_j t^{p-j}, \quad c_j \in \mathbb{R}, \quad (j = 1, 2, \cdots, m),$$

which gives the proof.

From Corollary 2.7 and Theorem 2.8, we have

Corollary 2.9. *If* $u(t) \in C_r^0(\mathbb{R}_0^+)$ *with* $0 \le r < m - p$, *then the Equation* (2.8) *has* $u(t) = \sum_{j=1}^{m-1} c_j t^{p-j}$ *as unique solutions.*

3. Continuous Solutions on $[0, a]$

Let us consider the following initial-value problem of fractional differential equation:

$$ {}_0D_t^p u(t) = f(t, u(t)), \tag{3.1}$$

$$\left[{}_0D_t^{p-j} u(t)\right]_{t=0} = c_j, \quad (j = 1, 2, \ldots, m), \tag{3.2}$$

where $m - 1 < p < m$ and $f : [0, a] \times \mathbb{R}$, $0 < a \le +\infty$, is a given function, continuous in $(0, a) \times \mathbb{R}$.

We may apply the results of Section 2, in particular, Example 2, Propositions 2.4 and 2.5, and Theorem 2.8, to reduce the initial-value problem (3.1)-(3.2) to an integral equation.

Theorem 3.1. *Assume that* $u(t)$ *is in* $C_0(\mathbb{R}^+) \bigcap L_{loc}^1(\mathbb{R})$ *with a fractional derivative of order* $m - 1 < p < m$ *that belongs to* $C_0(\mathbb{R}^+) \bigcap L_{loc}^1(\mathbb{R})$. *Furthermore, assume* $t^\sigma f(t, u)$ *is continuous on* $[0, a]$, *where* $0 \le \sigma < \min\{1, p\}$. *Then, the initial-value problem* (3.1)-(3.2) *is equivalent to the nonlinear Volterra integral equation of the second kind,*

$$u(t) = \sum_{j=1}^{m} \frac{c_j}{\Gamma(p - j + 1)} t^{p-j} + \frac{1}{\Gamma(p)} \int_0^t (t - \tau)^{p-1} f(\tau, u(\tau)) d\tau. \tag{3.3}$$

In other words, every solution of the Volterra equation (3.3) *is also a solution of our original initial-value problem* (3.1)-(3.2), *and vice versa.*

Theorem 3.1 can be easily proved by using formulas (2.4)-(2.6) and (2.9).

Let us denote $G = [0, a] \times \mathbb{R}$ and define a closed region $\overline{R}(h, K) \subset G$ as a set of points $(t, u) \in G$, which satisfy the following inequalities:

$$0 \leq t \leq h, \quad \left| u(t) - \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} \right| \leq K,$$

where h and K are constant.

The following theorem shows the existence and uniqueness of solutions, which are continuous on closed interval $[0, a]$.

Theorem 3.2. *Let* $m-1 < p < m$, $0 \leq \sigma < \min\{1, p\}$, $\|f\|_\sigma \neq 0$, *and* $c_m = 0$. *Assume that the function* $f : G \to \mathbb{R}$ *be continuous on* $(0, a] \times \mathbb{R}$. *Furthermore, assume that* $t^\sigma f(t, u)$ *is a continuous function on* G *and fulfills a Lipschitz condition with respect to the second variable*; *i.e.*,

$$t^\sigma |f(t, u) - f(t, v)| \leq L|u - v|, \tag{3.4}$$

for some positive constant L *independent of* $u, v \in \mathbb{R}$, $t \in (0, a]$. *Let*

$$h = \min\left\{a, \left(\frac{K\Gamma(1-\sigma+p)}{\|f\|_\sigma \Gamma(1-\sigma)}\right)^{\frac{1}{p-\sigma}}\right\}. \tag{3.5}$$

Then there exists in the closed region $\overline{R}(h, K)$ *a unique and continuous solution* $u(t)$ *solving the initial-value problem* (3.1)-(3.2).

Proof. Noting that $c_m = 0$ and according to Theorem 3.1 and Corollary 2.9, we are reduced to consider the following nonlinear integral equation:

$$u(t) = \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau, u(\tau)) d\tau, \tag{3.6}$$

which is equivalent to the initial-value problem (3.1)-(3.2).

We thus introduce the set

$$U = \left\{ u(t) \in C^0[0, h] : \left\| u(t) - \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} \right\| \leq K \right\}.$$

Obviously, this is a closed subset of the Banach space $C^0[0, h]$ of all continuous functions on $[0, h]$. Since the function $u_0(t) = \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j}$ is in U, we also see that U is not empty. On U, we define the operator T by

$$(Tu)(t) = \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau, u(\tau)) d\tau.$$

Using this operator, the initial-value problem can be written as $u = Tu$, and in order to prove our desired uniqueness result, we have to show that T has a unique fixed point. Let us therefore investigate the properties of the operator T.

In view of the assumption of continuity on $t^\sigma f(t, u)$, we can easily prove that $(Tu)(t)$ is a continuous function for every $u(t) \in U$. Moreover, for $u \in U$ and $t \in [0, h]$, we find

$$\begin{aligned} |(Tu)(t) - u_0(t)| &= \frac{1}{\Gamma(p)} \left| \int_0^t (t-\tau)^{p-1} f(\tau, u(\tau)) d\tau \right| \leq \frac{\|f\|_\sigma}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{-\sigma} d\tau \\ &= \frac{\Gamma(1-\sigma)}{\Gamma(p+1-\sigma)} \|f\|_\sigma t^{p-\sigma} \leq \frac{\Gamma(1-\sigma)}{\Gamma(p+1-\sigma)} \|f\|_\sigma h^{p-\sigma} \\ &\leq \frac{\Gamma(1-\sigma)}{\Gamma(p+1-\sigma)} \|f\|_\sigma \frac{K\Gamma(1-\sigma+p)}{\|f\|_\sigma \Gamma(1-\sigma)} \leq K. \end{aligned}$$

Thus, we have shown that $Tu \in U$ if $u \in U$; i.e., T maps the set U to itself.

The next step is to prove that T^n is a contraction operator for n sufficiently large. Actually, we have for $u, v \in U$

$$|T^n u(t) - T^n v(t)| \leq \frac{(ML)^n}{\Gamma(n(p-\sigma)+1)} t^{n(p-\sigma)} \|u - v\|, \tag{3.7}$$

where M depends only on p and σ. In fact,

$$\begin{aligned} |Tu(t) - Tv(t)| &\leq \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau \\ &\leq \frac{\Gamma(1-\sigma)L}{\Gamma(p-\sigma+1)} t^{p-\sigma} \|u - v\|. \end{aligned} \tag{3.8}$$

Therefore (3.7) is proved for $n = 1$, if $M \geq \Gamma(1-\sigma)$. Assuming by induction that (3.7) is valid for n, we obtain similarly

$$\begin{aligned} |T^{n+1} u(t) - T^{n+1} v(t)| &\leq \frac{M^n L^{n+1} \|u - v\|}{\Gamma(n(p-\sigma)+1)\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{n(p-\sigma)-\sigma} d\tau \\ &= \frac{\Gamma(n(p-\sigma)-\sigma+1) M^n L^{n+1}}{\Gamma(n(p-\sigma)+1)\Gamma((n+1)(p-\sigma)+1)} t^{(n+1)(p-\sigma)} \|u - v\|, \end{aligned}$$

and then (3.7) follows for $n+1$, if M is given by

$$M = \max_n M_n, \; M_1 = \Gamma(1-\sigma), \; M_n = \frac{\Gamma(n(p-\sigma)-\sigma+1)}{\Gamma(n(p-\sigma)+1)}, \; (n = 2, 3, \cdots). \tag{3.9}$$

Note that (3.9) defines actually a finite M, since $M_n \leq 1$ for $n \geq (1+\sigma) / (p-\sigma)$ (i.e., $n(p-\sigma) - \sigma + 1 \geq 2$). As a consequence of inequality (3.7), we find, taking Chebyshev norm on interval $[0, h]$,

$$\|T^n u(t) - T^n v(t)\| \leq \frac{(ML)^n}{\Gamma(n(p-\sigma)+1)} h^{n(p-\sigma)} \|u - v\|. \tag{3.10}$$

This, however, is a well-known result; the limit

$$\sum_{n=0}^{\infty} \frac{(ML)^n}{\Gamma(n(p-\sigma)+1)} h^{n(p-\sigma)} = E_{p-\sigma,1}(MLh^{p-\sigma})$$

is the Mittag-Leffler function (see [10], Subsection 1.2), evaluated at $LMh^{p-\sigma}$. Taking n sufficiently large in (3.10), we have $(MLh^{p-\sigma})^n / \Gamma(n(p-\sigma)+1) \leq q < 1$ and therefore,

$$\|T^n u - T^n v\| \leq q\|u - v\|,$$

which ends the proof.

Corollary 3.3. (1) *Taking K sufficiently large in* (3.5), *i.e.*,

$$K \geq \frac{a^{p-\sigma}\|f\|_\sigma \Gamma(1-\sigma)}{\Gamma(1-\sigma+p)},$$

we have $h = a$ *and Theorem* 3.2 *shows that the uniqueness and global existence can be obtained.*

(2) *Noting that* $f(t, u(t)) \in C_\sigma^0[0, h]$ *and taking into account* $p > \sigma$, *we take the limit of* (3.6) *as* $t \to 0+$

$$u(0) = \lim_{t\to 0+} u(t) = 0+ \lim_{t\to 0+} {}_0D_t^{-p} f(t, u(t)) = 0.$$

We may consider the limit case when in Theorem 3.2, we have $p = \sigma < 1$.

Corollary 3.4. *If* $p = \sigma < 1$ *and* $L < 1/\Gamma(1-p)$, *then from* (3.8), *we obtain*

$$\|Tu - Tv\| \leq \frac{\Gamma(1-p)L}{\Gamma(1)}\|u - v\| \leq q\|u - v\|, \quad q = L\Gamma(1-p) < 1,$$

which ensues the global existence and uniqueness of solution for the initial-value problem (3.1)-(3.2). *In this case, the solution* $u(t)$ *is bounded at the origin.*

Remark 3.5. Note that Theorem 3.2 not only asserts that the continuous solution defined on $[0, a]$ is unique; it actually gives us (at least theoretically) a means of determining this solution by a Picard-type iteration process

$$u_0(t) = \sum_{j=1}^{m-1} \frac{c_j}{\Gamma(p-j+1)} t^{p-j}, \tag{3.11}$$

$$u_{k+1}(t) = u_0(t) + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau, u_k(\tau)) d\tau, \ (k = 0, 1, \cdots). \tag{3.12}$$

Example 3. Let us consider the following initial-value problem in terms of the Riemann-Liouville fractional derivatives:

$${}_0D_t^p u(t) = t^{-\mu} u(t), \quad \left[{}_0D_t^{p-1} u(t)\right]_{t=0} = c_1, \quad \left[{}_0D_t^{p-2} u(t)\right]_{t=0} = c_2 = 0,$$

where $1 < p < 2$ and $0 < \mu < 1$. Obviously, the above initial-value problem satisfies all conditions in Theorem 3.2 and $\sigma = \mu$; so we can determine the solution by a Picard-type iteration process

$$u_0(t) = \frac{c_1}{\Gamma(p)} t^{p-1},$$

$$u_1(t) = \frac{c_1}{\Gamma(p)} t^{p-1} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{-\mu} u_0(\tau) d\tau$$

$$= \frac{c_1}{\Gamma(p)} t^{p-1} + \frac{c_1}{\Gamma(p)} \frac{\Gamma(p-\mu)}{\Gamma(2p-\mu)} t^{2p-\mu-1},$$

$$u_2(t) = \frac{c_1}{\Gamma(p)} t^{p-1} + \frac{c_1}{\Gamma(p)} \frac{\Gamma(p-\mu)}{\Gamma(2p-\mu)} t^{2p-\mu-1} + \frac{c_1}{\Gamma(p)} \frac{\Gamma(p-\mu)}{\Gamma(2p-\mu)} \frac{\Gamma(2p-2\mu)}{\Gamma(3p-2\mu)} t^{3p-2\mu-1},$$

$$u_n(t) = \frac{c_1}{\Gamma(p)} t^{p-1} + \frac{c_1}{\Gamma(p)} \frac{\Gamma(p-\mu)}{\Gamma(2p-\mu)} t^{2p-\mu-1} + \ldots$$

$$+ \frac{c_1}{\Gamma(p)} \frac{\Gamma(p-\mu)}{\Gamma(2p-\mu)} \frac{\Gamma(2p-2\mu)}{\Gamma(3p-2\mu)} \cdots \frac{\Gamma(np-n\mu)}{\Gamma((n+1)p-n\mu)} t^{(n+1)p-n\mu-1}. \tag{3.13}$$

Taking the limit of (3.13) as $n \to \infty$, we obtain the solution of the initial-value problem

$$u(t) = \lim_{n\to\infty} u_n(t) = \frac{c_1}{\Gamma(p)} t^{p-1} + \frac{c_1}{\Gamma(p)} \sum_{n=1}^{\infty} \left[\prod_{k=1}^{n} \frac{\Gamma(kp - k\mu)}{\Gamma((k+1)p - k\mu)} \right] t^{(n+1)p - n\mu - 1}.$$

Figure 1 shows computed solutions $u_0(t), u_1(t), \cdots, u_5(t)$ in the closed interval $[0, 3]$ with $a = 3$, $p = 1.5$, $\mu = 0.5$, and $c_1 = 1$. One can see that the solution $u_4(t)$ and $u_5(t)$ are very closed to each other, so we may take $u(t) \approx u_5(t)$.

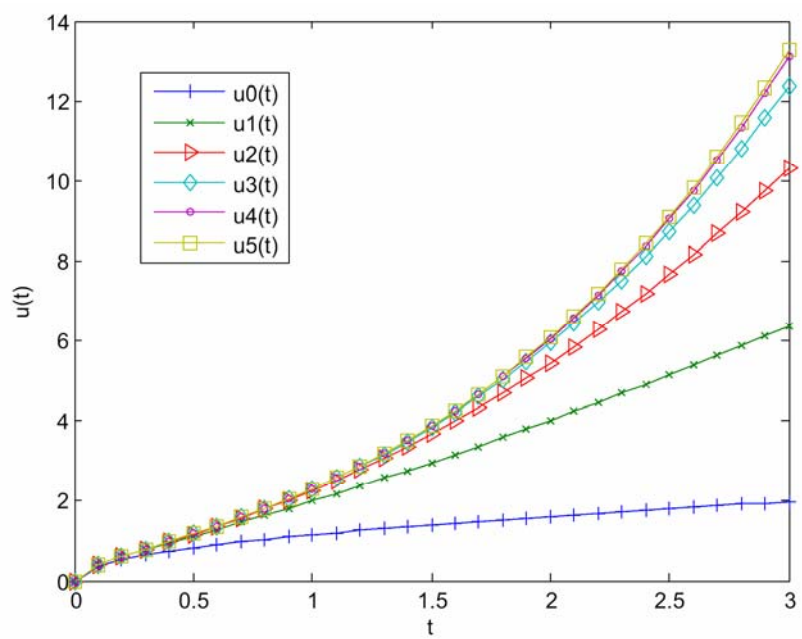

Figure 1. Computed solutions of initial-value problem described in Example 3 obtained with Picard-type iteration process with $a = 3$, $p = 1.5$, $\mu = 0.5$, and $c_1 = 1$.

4. Continuous Solutions on $(0, a]$

In this section, we consider unique solution $u(t) \in C^0_{m-p}[0, h]$, which may be unbounded at the origin. Let us denote $G = [0, a] \times \mathbb{R}$ and define $R(h, K) \subset G$ as a set of points $(t, u) \in G$, which satisfy the following inequalities:

$$0 < t < h, \quad \left| t^{m-p} u(t) - \sum_{j=1}^{m} \frac{c_j}{\Gamma(p-j+1)} t^{m-j} \right| \leq K,$$

where h and K are constant.

Theorem 4.1. *Let* $m - 1 < p < m$, $0 \leq \sigma < \min\{p, 1 - (m - p)\}$, $\|f\|_\sigma \neq 0$, *and* $c_m \neq 0$. *Assume that the function* $f : G \to \mathbb{R}$ *be continuous on* $(0, a] \times \mathbb{R}$. *Furthermore, assume that* $t^\sigma f(t, u)$ *is a continuous function on G and fulfills a Lipschitz condition with respect to the second variable*; *i.e.*,

$$t^\sigma |f(t, u) - f(t, v)| \leq L|u - v|, \tag{4.1}$$

for some positive constant L independent of $u, v \in \mathbb{R}$, $t \in (0, a]$. *Let*

$$h = \min\left\{a, \left(\frac{K\Gamma(1 - \sigma + p)}{\|f\|_\sigma \Gamma(1 - \sigma)}\right)^{\frac{1}{m-\sigma}}\right\}. \tag{4.2}$$

Then there exists in $R(h, K)$ *a unique solution* $u(t) \in C^0_{m-p}[0, h]$ *solving the initial-value problem* (3.1)-(3.2).

Proof. Firstly, We introduce the set

$$U = \left\{ u(t) \in C^0_{m-p}[0, h] : \left\| t^{m-p} u(t) - \sum_{j=1}^{m} \frac{c_j}{\Gamma(p-j+1)} t^{m-j} \right\| \leq K \right\}.$$

Obviously, this is a closed subset of the Banach space $C^0_{m-p}[0, h]$ endowed with the norm defined by (2.1). Since the function $u_0(t) = \sum_{j=1}^{m} \frac{c_j}{\Gamma(p-j+1)} t^{p-j}$ is in U, we also see that U is not empty.

As in the proof of Theorem 3.2, our problem is equivalent to the problem of determination of fixed point of the continuous operator

$$(Tu)(t) = \sum_{j=1}^{m} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau, u(\tau)) d\tau. \tag{4.3}$$

It is immediate to verify that

(i) $T : U \to U$ is well defined. Indeed, we have

$$\left| t^{m-p}(Tu)(t) - \sum_{j=1}^{m} \frac{c_j}{\Gamma(p-j+1)} t^{m-j} \right|$$

$$\leq \frac{t^{m-p}}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{-\sigma} |\tau^{\sigma} f(\tau, u(\tau))| d\tau$$

$$\leq \frac{t^{m-p} \|f\|_{\sigma} \Gamma(1-\sigma)}{\Gamma(1+p-\sigma)} t^{p-\sigma} \leq \frac{\Gamma(1-\sigma)\|f\|_{\sigma}}{\Gamma(1+p-\sigma)} h^{m-\sigma}$$

$$\leq \frac{\Gamma(1-\sigma)\|f\|_{\sigma}}{\Gamma(1+p-\sigma)} \frac{K\Gamma(1-\sigma+p)}{\|f\|_{\sigma}\Gamma(1-\sigma)} \leq K.$$

(ii) T^n is a contraction operator for n sufficiently large. Indeed, we have

$$t^{m-p}|Tu(t) - Tv(t)| \leq \frac{t^{m-p}}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} |f(\tau, u(\tau)) - f(\tau, v(\tau))| d\tau$$

$$\leq \frac{Lt^{m-p}}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{-\sigma} |u(\tau) - v(\tau)| d\tau$$

$$\leq \frac{Lt^{m-p} \|u-v\|_{m-p}}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \tau^{p-m-\sigma} d\tau$$

$$\leq \frac{L\Gamma(1+p-m-\sigma)t^{p-\sigma}}{\Gamma(1+p-m+(p-\sigma))}\|u-v\|_{m-p},$$

and

$$\begin{aligned}
&t^{m-p}|T^2u(t)-T^2v(t)| \\
&\quad\leq \frac{t^{m-p}}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}|f(\tau, Tu(\tau))-f(\tau, Tv(\tau))|d\tau \\
&\quad\leq \frac{Lt^{m-p}}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}\tau^{-\sigma}|Tu(\tau)-Tv(\tau)|d\tau \\
&\quad\leq \frac{L^2t^{m-p}\Gamma(1+p-m-\sigma)\|u-v\|_{m-p}}{\Gamma(1+p-m+(p-\sigma))\Gamma(p)}\int_0^t (t-\tau)^{p-1}\tau^{p-m-\sigma}\tau^{p-\sigma}d\tau \\
&\quad\leq \frac{L^2\Gamma(1+p-m-\sigma)\Gamma(1+p-m+(p-\sigma)-\sigma)t^{2(p-\sigma)}}{\Gamma(1+p-m+(p-\sigma))\Gamma(1+p-m+2(p-\sigma))}\|u-v\|_{m-p}.
\end{aligned}$$

By induction, arguing as in the Theorem 3.2, it is obtained that

$$t^{m-p}|T^nu(t)-T^nv(t)| \leq \frac{L^nM^nt^{n(p-\sigma)}}{\Gamma(1+p-m+n(p-\sigma))}\|u-v\|_{m-p}, \tag{4.4}$$

for all $u, v \in U$, where $M = \max_n M_n < +\infty$,

$$M_1 = \Gamma(1+p-m-\sigma),\ M_n = \frac{\Gamma(1+p-m+(n-1)(p-\sigma)-\sigma)}{\Gamma(1+p-m+n(p-\sigma))}, (n = 2, 3, \cdots).$$

Thus, we get

$$\|T^nu(t)-T^nv(t)\|_{m-p} \leq \frac{L^nM^nh^{n(p-\sigma)}}{\Gamma(1+p-m+n(p-\sigma))}\|u-v\|_{m-p}.$$

Since $L^nM^nh^{n(p-\sigma)}/\Gamma(1+p-m+n(p-\sigma)) \to 0$ as n tends to $+\infty$ (see [10], Mittag-Leffler function), then for n sufficiently large, the operator T^n is a contraction operator. Therefore, there exists in $R(h, K)$ a unique solution $u(t) \in C^0_{m-p}[0, h]$ solving the initial -value problem (3.1)-(3.2). The proof of Theorem 4.1 is complete.

Corollary 4.2. *Taking K sufficiently large in* (4.2), *i.e.*,

$$K \geq \frac{a^{m-\sigma}\|f\|_{\sigma}\Gamma(1-\sigma)}{\Gamma(1-\sigma+p)},$$

we have $h = a$ *and Theorem* 4.1 *shows that the uniqueness and global existence of* $u(t) \in C^0_{m-p}[0, a]$ *can be obtained.*

Remark 4.3. As in the proof of Theorem 3.2 and subsequent remarks, Theorem 4.1 actually gives us a means of determining this solution by a Picard-type iteration process, which is very similar to expressions (3.11)-(3.12).

Example 4. Finally, let us concern iterative method for the solution $u(t) \in C^0_{m-p}[0, a]$ of the initial -value problem described in Example 3.

$${}_0D_t^p u(t) = t^{-\mu}u(t), \quad \left[{}_0D_t^{p-j}u(t)\right]_{t=0} = c_j,\ (j = 1, 2),\ c_2 \neq 0,$$

where $1 < p < 2$, $0 < \mu < 1 - (2 - p)$. Obviously, the above initial-value problem satisfies all conditions in Theorem 4.1 and $\sigma = \mu$; so we can determine the solution by a Picard-type iteration process

$$u_0(t) = \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} t^{p-j},$$

$$u_1(t) = \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1}\tau^{-\mu}u_0(\tau)d\tau$$

$$= \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} \frac{\Gamma(1+p-j-\mu)}{\Gamma(1+2p-j-\mu)} t^{2p-j-\mu},$$

$$u_2(t) = \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} t^{p-j} + \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} \frac{\Gamma(1+p-j-\mu)}{\Gamma(1+2p-j-\mu)} t^{2p-j-\mu}$$

$$+ \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} \frac{\Gamma(1+p-j-\mu)}{\Gamma(1+2p-j-\mu)} \frac{\Gamma(1+2p-j-2\mu)}{\Gamma(1+3p-j-2\mu)} t^{3p-j-2\mu},$$

and taking the limit of $u_n(t)$ as $n \to \infty$, we obtain

$$u(t) = \lim_{n\to\infty} u_n(t) = \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} t^{p-j}$$

$$+ \sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{2} \frac{c_j}{\Gamma(p-j+1)} \left[\prod_{k=1}^{n} \frac{\Gamma(1+kp-k\mu-j)}{\Gamma(1+(k+1)p-k\mu-j)} \right] t^{(n+1)p-n\mu-j} \right\}.$$

Figure 2 shows computed solutions $u_0(t), u_1(t), \cdots, u_5(t)$ in the closed interval $[0.01, 3.01]$ with $a = 3.01$, $p = 1.5$, $\mu = 0.3$, $c_1 = 1$, and $c_2 = 1$. Obviously, the solution $u_4(t)$ and $u_5(t)$ are very closed to each other, so we may take $u(t) \approx u_5(t)$.

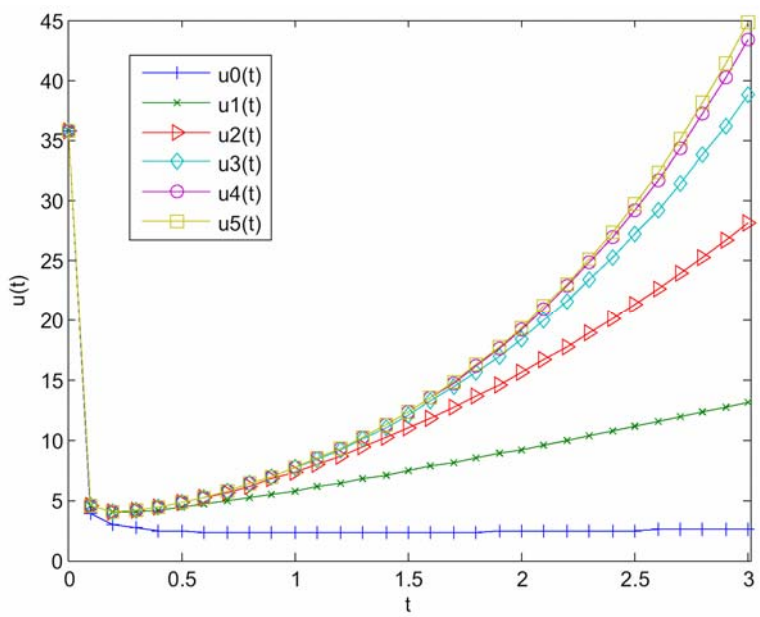

Figure 2. Computed solutions of the initial-value problem described in Example 4 obtained with Picard-type iteration process with $a = 3.01$, $p = 1.5$, $\mu = 0.3$, $c_1 = 1$, and $c_2 = 1$.

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