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SOME PROPERTIES OF FIBONACCI AND LUCAS SYMBOL ELEMENTS

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Abstract

In this paper, we study some properties of the Fibonacci and Lucas numbers. With these relations, we proved some new properties for the Fibonacci quaternions and Fibonacci symbol elements.

1. Introduction

In the paper [2], Flaut and Shpakivskyi found some properties of the generalized Fibonacci quaternions and in the paper [1], the authors gave the definition of the Fibonacci symbol elements. In this paper, we introduce the Lucas symbol elements and we will show that many properties satisfied by the Fibonacci and Lucas numbers are also satisfied by the Fibonacci symbol elements and Lucas symbol elements.

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2. Elementary Properties of the Fibonacci and Lucas Numbers

From [1], the following properties of Fibonacci numbers are known:

Proposition 2.1. *Let* $(f_n)_{n\geq 0}$ *be the Fibonacci sequence* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $\forall n \in \mathbb{N}$. *Then*

(i) $f_n + f_{n+3} = 2f_{n+2}, \quad \forall n \in \mathbb{N}$;

(ii) $f_n + f_{n+4} = 3f_{n+2}, \quad \forall n \in \mathbb{N}$.

In the following, we give other elementary properties of the Fibonacci and Lucas numbers. □

Proposition 2.2. *Let* $(l_n)_{n\geq 0}$ *be the Lucas sequence* $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$, $\forall n \in \mathbb{N}$. *Then*

$$l_n + l_{n+3} = 2l_{n+2}, \quad \forall n \in \mathbb{N}.$$

Proof. By induction for $n \in \mathbb{N}$. □

Proposition 2.3. *Let* $(f_n)_{n\geq 0}, (l_n)_{n\geq 0}$ *be the Fibonacci and Lucas sequences* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $\forall n \in \mathbb{N}$, $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$, $\forall n \in \mathbb{N}$. *Then*

$$f_n + f_{n+6} = 2l_{n+3}, \quad \forall n \in \mathbb{N}.$$

Proof. We use the known relation $f_{n-1} + f_{n+1} = l_n$, $\forall n \in \mathbb{N}^*$, and we use induction for $n \in \mathbb{N}$. □

Proposition 2.4. *Let* $(f_n)_{n\geq 0}, (l_n)_{n\geq 0}$ *be the Fibonacci and Lucas sequences* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $\forall n \in \mathbb{N}$, $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$, $\forall n \in \mathbb{N}$. *Then*

$$l_n + f_n = 2f_{n+1}, \quad \forall n \in \mathbb{N}.$$

Proof. The proof is obviously, using the recurrence relation for the Fibonacci numbers and the known relation $f_{n-1} + f_{n+1} = l_n$, $\forall n \in \mathbb{N}^*$. □

Proposition 2.5. *Let* $(f_n)_{n\geq 0}$, $(l_n)_{n\geq 0}$ *be the Fibonacci and Lucas sequences* $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$, $\forall n \in \mathbb{N}$, $l_0 = 2$, $l_1 = 1$, $l_{n+2} = l_{n+1} + l_n$, $\forall n \in \mathbb{N}$. *Then*

(i) $f_n + l_{n+2} = f_{n+4}$, $\quad \forall n \in \mathbb{N}$;

(ii) $f_{n+2} + l_n = 3f_{n+1}$, $\quad \forall n \in \mathbb{N}$.

Proof. (i) The proof is obviously, using the recurrence relation for the Fibonacci numbers and the known relation $f_{n-1} + f_{n+1} = l_n$, $\forall n \in \mathbb{N}^*$.

(ii) Using Proposition 2.2, and the relation $f_{n-1} + f_{n+1} = l_n$, $\forall n \in \mathbb{N}^*$, we have

$$\begin{aligned} f_{n+2} + l_n &= f_{n+2} + f_{n+1} + f_{n-1} \\ &= f_{n+3} + f_{n-1} \\ &= 3f_{n+1}. \end{aligned}$$

3. Properties of Fibonacci Symbol Elements and Lucas Symbol Elements

Let n be an arbitrary positive integer, and K be a field, whose $char(K)$ does not divide n and contains a primitive n-th root of unity. Let $K^* = K\setminus\{0\}$, $a, b \in K^*$ and let S be the algebra over K generated by elements x and y, where

$$x^n = a, \quad y^n = b, \quad yx = \omega xy.$$

This algebra is called a *symbol algebra* (also known as a *power norm residue algebra*) and it is denoted by $\left(\frac{a, b}{K, \omega}\right)$.

Let ω be a primitive root of the unity of order 3; $K = \mathbb{Q}(\omega)$ be the cyclotomic field; and $S = \left(\frac{a, b}{K, \omega}\right)$ be the symbol algebra of degree 3. We know that S has a K-basis $\{x^i y^j | 0 \leq i, j < 3\}$ such that $x^3 = a \in K^*$, $y^3 = b \in K^*$, $yx = \omega xy$. Let $z \in S$, $z = \sum_{i, j=1}^{n} x^i y^j c_{ij}$.

In [1], we defined n-th Fibonacci symbol element

$$F_n = f_n \cdot 1 + f_{n+1} \cdot x + f_{n+2} \cdot x^2 + f_{n+3} \cdot y + f_{n+4} \cdot xy$$
$$+ f_{n+5} \cdot x^2 y + f_{n+6} \cdot y^2 + f_{n+7} \cdot xy^2 + f_{n+8} \cdot x^2 y^2.$$

Let $(l_n)_{n\geq 0}$ be the Lucas sequence $l_0 = 2, l_1 = 1, l_{n+2} = l_{n+1} + l_n, \forall n \in \mathbb{N}$. We define the n-th Lucas symbol element as in the following:

$$L_n = l_n \cdot 1 + l_{n+1} \cdot x + l_{n+2} \cdot x^2 + l_{n+3} \cdot y + l_{n+4} \cdot xy$$
$$+ l_{n+5} \cdot x^2 y + l_{n+6} \cdot y^2 + l_{n+7} \cdot xy^2 + l_{n+8} \cdot x^2 y^2.$$

We remark immediately that the sequence of Fibonacci symbol elements and the sequence of Lucas symbol elements satisfy the same recurrence as Fibonacci sequence and Lucas sequence, that is, $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$, $\forall n \in \mathbb{N}$, $n \geq 2$.

Proposition 3.1. *Let $(F_n)_{n\geq 0}$ be the sequence of Fibonacci symbol elements. Then*

(i) $F_n + F_{n+3} = 2F_{n+2}$, $\quad \forall n \in \mathbb{N}$;

(ii) $F_n + F_{n+4} = 3F_{n+2}$, $\quad \forall n \in \mathbb{N}$.

Proof. (i) Using Proposition 2.1 (i), we have

$$F_n + F_{n+3} = f_n \cdot 1 + f_{n+1} \cdot x + f_{n+2} \cdot x^2 + f_{n+3} \cdot y + f_{n+4} \cdot xy$$
$$+ f_{n+5} \cdot x^2 y + f_{n+6} \cdot y^2 + f_{n+7} \cdot xy^2 + f_{n+8} \cdot x^2 y^2$$
$$+ f_{n+3} \cdot 1 + f_{n+4} \cdot x + f_{n+5} \cdot x^2 + f_{n+6} \cdot y + f_{n+7} \cdot xy$$

$$+ f_{n+8} \cdot x^2y + f_{n+9} \cdot y^2 + f_{n+10} \cdot xy^2 + f_{n+11} \cdot x^2y^2$$

$$= 2f_{n+2} \cdot 1 + 2f_{n+3} \cdot x + 2f_{n+4} \cdot x^2 + 2f_{n+5} \cdot y$$

$$+ 2f_{n+6} \cdot xy + 2f_{n+7} \cdot x^2y + 2f_{n+8} \cdot y^2$$

$$+ 2f_{n+9} \cdot xy^2 + f_{n+10} \cdot x^2y^2$$

$$= 2F_{n+2}.$$

In the same way, we prove (ii). □

Similarly, we obtain the following results:

Proposition 3.2. *Let* $(F_n)_{n\geq 0}, (L_n)_{n\geq 0}$ *be the sequence of Fibonacci symbol elements and the sequence of Lucas symbol elements. Then*

(i) $L_n + L_{n+3} = 2L_{n+2}, \quad \forall n \in \mathbb{N}$;

(ii) $F_n + F_{n+6} = 2L_{n+3}, \quad \forall n \in \mathbb{N}$;

(iii) $L_n + F_n = 2F_{n+1}, \quad \forall n \in \mathbb{N}$;

(iv) $F_n + L_{n+2} = 2F_{n+4}, \quad \forall n \in \mathbb{N}$;

(v) $F_{n+2} + L_n = 3F_{n+3}, \quad \forall n \in \mathbb{N}$.

□

In 1876, Lucas proved that $\sum_{k=1}^{n} f_k = f_{n+2} - 1, \forall n \in \mathbb{N}^*$. We explore whether a similar result is true for the sequence of Fibonacci symbol elements.

Proposition 3.3. *Let* $(F_n)_{n\geq 0}$ *be the sequence of the Fibonacci symbol elements. Then*

$$\sum_{k=1}^{n} F_k = F_{n+2} - F_2, \quad \forall n \in \mathbb{N}^*.$$

Proof. Using the above relations and Lucas's result, we have

$$\sum_{k=1}^{n} F_k = \sum_{k=1}^{n} f_k + x \cdot \sum_{k=1}^{n} f_{k+1} + x^2 \cdot \sum_{k=1}^{n} f_{k+2} + y \cdot \sum_{k=1}^{n} f_{k+3} + xy \cdot \sum_{k=1}^{n} f_{k+4}$$

$$+ x^2y \cdot \sum_{k=1}^{n} f_{k+5} + y^2 \cdot \sum_{k=1}^{n} f_{k+6} + xy^2 \cdot \sum_{k=1}^{n} f_{k+7} + x^2y^2 \cdot \sum_{k=1}^{n} f_{k+8}$$

$$= f_{n+2} - 1 + x \cdot (f_{n+3} - 1 - f_1) + x^2 \cdot (f_{n+4} - 1 - f_1 - f_2)$$

$$+ y \cdot (f_{n+5} - 1 - f_1 - f_2 - f_3) + xy \cdot (f_{n+6} - 1 - f_1 - f_2 - f_3 - f_4)$$

$$+ x^2y \cdot (f_{n+7} - 1 - f_1 - f_2 - f_3 - f_4 - f_5)$$

$$+ y^2 \cdot (f_{n+8} - 1 - f_1 - f_2 - f_3 - f_4 - f_5 - f_6)$$

$$+ xy^2 \cdot (f_{n+9} - 1 - f_1 - f_2 - f_3 - f_4 - f_5 - f_6 - f_7)$$

$$+ x^2y^2 \cdot (f_{n+10} - 1 - f_1 - f_2 - f_3 - f_4 - f_5 - f_6 - f_7 - f_8)$$

$$= F_{n+2} - f_2 - x \cdot f_3 - x^2 \cdot f_4 - y \cdot f_5 - xy \cdot f_6 - x^2y \cdot f_7$$

$$- y^2 \cdot f_8 - xy^2 \cdot f_9 - x^2y^2 \cdot f_{10}$$

$$= F_{n+2} - F_2.$$

□

Other old result about Fibonacci numbers is

$$f_{2n+i} = \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+i}, \quad \forall\, n,\, i \in \mathbb{N}, \quad n \geq 1.$$

Using this relation, we obtain the following result about Fibonacci symbol elements.

Proposition 3.4. *Let* $(F_n)_{n\geq 0}$ *be the sequence of Fibonacci symbol elements. Then*

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} \cdot F_k, \quad \forall\, n \in \mathbb{N}^*.$$

Proof. Using the above relations and the previous result, we have

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \cdot F_k &= \sum_{k=0}^{n} \binom{n}{k} \cdot f_k + x \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+1} + x^2 \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+2} \\ &\quad + y \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+3} + xy \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+4} \\ &\quad + x^2 y \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+5} + y^2 \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+6} \\ &\quad + xy^2 \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+7} + x^2 y^2 \cdot \sum_{k=0}^{n} \binom{n}{k} \cdot f_{k+8} \\ &= F_{2n}.\end{aligned}$$

□

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