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CHARACTERISTIC FINITE DIFFERENCE SCHEME FOR THE 2-DIMENSIONAL RLW EQUATION

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Abstract

We consider characteristic finite difference scheme for two-dimensional RLW equation. We prove the convergence of the format and get H^1-mode optimal error estimates. This format can overcome numerical concussion and is more easy to operate.

1. Introduction

Two-dimensional RLW (regularised long wave) equation is a form of non-linear long wave, it can be used to describe many important physical

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phenomena, such as shallow water waves and plasma waves, etc.. In recent decades, there are many people have studied the numerical solution of the RLW equation. The paper [3] had studied the exact solutions of the two-dimensional RLW equation. After that there were many people have studied it, but the solution stability and accuracy needs further improvement. This paper adopts the characteristic finite difference scheme, and make convergence analysis and error analysis.

Considering the following RLW equation:

$$u_t + \alpha u_x + \beta u_y - \lambda u_{xxt} - \theta u_{yyt} = f(x, y, t), \quad (x, y, t) \in \Omega \times [0, T], \tag{1}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \tag{2}$$

$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times [0, T], \tag{3}$$

where α, β, λ, θ are constants, and $\lambda > 0$, $\theta > 0$, $\Omega = (0, 1) \times (0, 1)$.

Let

$$x_i = ih, \quad y_j = jh, \quad (0 \le i, j \le N),$$

$$h = \frac{1}{N}, \quad t^n = n \cdot \Delta t, \quad (0 \le n \le M), \quad \Delta t = [\frac{T}{M}].$$

Let S denote the collections function in the grids

$$S = \{u_{i,j} | u_{0,j} = u_{N,j} = u_{i,0} = u_{i,N} = 0, \quad i, j = 1, 2, \cdots, N\}.$$

$$f_{i,j}^n = f(x_i, y_j, t^n), \quad f_{x,i,j} = \frac{f_{i+1,j} - f_{i,j}}{h}, \quad f_{y,i,j} = \frac{f_{i,j+1} - f_{i,j}}{h},$$

$$f_{\bar{x},i,j} = \frac{f_{i,j} - f_{i-1,j}}{h}, \quad f_{\bar{y},i,j} = \frac{f_{i,j} - f_{i,j-1}}{h},$$

$$f_{x\bar{x},i,j} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2}, \quad f_{y\bar{y},i,j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{h^2},$$

$$(f, g) = \sum_i \sum_j h_i^2 f_{i,j} g_{i,j},$$

$$\|f\|^2 = (f, f), \quad \|f\|_1^2 = \|f\|_0^2 + \|f_x\|^2 + \|f_y\|^2,$$

$$\|f\|_{l^2} = \sum_i \sum_j h_i^2 \max \{|f(x, y)|^2; |x - x_i| \le R_1 h, |y - y_j| \le R_1 h\},$$

where $R_1 = \max \{\alpha, \beta\}$.

2. Characteristic Finite Difference Scheme

Let $m = \sqrt{1 + \alpha^2 + \beta^2}$, τ is the characteristic direction of operator $u_t + \alpha u_x + \beta u_y$, then

$$\frac{\partial}{\partial \tau} = \frac{1}{m} \frac{\partial}{\partial t} + \frac{\alpha}{m} \frac{\partial}{\partial x} + \frac{\beta}{m} \frac{\partial}{\partial y};$$

Equations (1)-(3) are equivalent to

$$m \frac{\partial u}{\partial \tau} - \lambda u_{xxt} - \theta u_{yyt} = f(x, y, t), \quad (x, y, t) \in \Omega \times [0, T], \tag{4}$$

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \tag{5}$$

$$u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times [0, T], \tag{6}$$

the characteristic finite difference scheme of (4)-(6), define $U_{i,j}^n \in S$ $(n = 1, 2, \cdots, N)$, satisfying

$$\frac{U_{i,j}^n - \overline{U}_{i,j}^{n-1}}{\Delta t} - \lambda \frac{U_{x,\bar{x},i,j}^n - U_{x,\bar{x},i,j}^{n-1}}{\Delta t} - \theta \frac{U_{y,\bar{y},i,j}^n - U_{y,\bar{y},i,j}^{n-1}}{\Delta t} = f_{i,j}^n, \tag{7}$$

$$U_{i,j}^0 = u_0(x_i, y_j), \tag{8}$$

where $\overline{U}_{i,j}^{n-1} = U^{n-1}(\bar{x}_i, \bar{y}_j)$, $\bar{x}_i = x_i - \alpha\Delta t$, $\bar{y}_j = y_j - \beta\Delta t$, $U^{n-1}(x, y)$ is linear interpolation of $U_{i,j}^{n-1}$.

3. Convergence Analysis of the Characteristic Finite Difference Scheme

At (x_i, y_j, t^n), Equation (4) may be rewritten as

$$\frac{u_{i,j}^n - \overline{u}_{i,j}^{n-1}}{\Delta t} - \lambda \frac{u_{x\bar{x},i,j}^n - u_{x\bar{x},i,j}^{n-1}}{\Delta t} - \theta \frac{u_{y\bar{y},i,j}^n - u_{y\bar{y},i,j}^{n-1}}{\Delta t}$$

$$= \frac{u_{i,j}^n - \overline{u}_{i,j}^{n-1}}{\Delta t} - m \frac{\partial u}{\partial \tau}(x_i, y_j, t^n) - \lambda\left(\frac{u_{x\bar{x},i,j}^n - u_{x\bar{x},i,j}^{n-1}}{\Delta t} - u_{xxt}(x_i, y_j, t^n)\right)$$

$$- \left(\theta \frac{u_{y\bar{y},i,j}^n - u_{y\bar{y},i,j}^{n-1}}{\Delta t} - u_{yyt}(x_i, y_j, t^n)\right) + f_{i,j}^n. \tag{9}$$

Define $\eta = u - U$, (9) subtracting (7), we get

$$\frac{\eta_{i,j}^n - \overline{\eta}_{i,j}^{n-1}}{\Delta t} - \lambda \frac{\eta_{x\bar{x},i,j}^n - \eta_{x\bar{x},i,j}^{n-1}}{\Delta t} - \theta \frac{\eta_{y\bar{y},i,j}^n - \eta_{y\bar{y},i,j}^{n-1}}{\Delta t} = \rho_{i,j}^n, \tag{10}$$

where $\rho_{i,j}^n = \frac{u_{i,j}^n - \overline{u}_{i,j}^{n-1}}{\Delta t} - m \frac{\partial u}{\partial \tau}(x_i, y_j, t^n) - \lambda\left(\frac{u_{x\bar{x},i,j}^n - u_{x\bar{x},i,j}^{n-1}}{\Delta t}\right.$

$$\left. - u_{xxt}(x_i, y_j, t^n)\right) - \left(\theta \frac{u_{y\bar{y},i,j}^n - u_{y\bar{y},i,j}^{n-1}}{\Delta t} - u_{yyt}(x_i, y_j, t^n)\right).$$

Multiplying the above by $h_i^2 \eta_{i,j}^n$, and summing on i and j, we get

$$\left(\frac{\eta^n - \overline{\eta}^{n-1}}{\Delta t}, \eta^n\right) - \left(\lambda \frac{\eta_{x\bar{x}}^n - \eta_{x\bar{x}}^{n-1}}{\Delta t}, \eta^n\right) - \left(\theta \frac{\eta_{y\bar{y}}^n - \eta_{y\bar{y}}^{n-1}}{\Delta t}, \eta^n\right) = (\rho^n, \eta^n). \tag{11}$$

Now we estimate the above

$$\left(\frac{\eta^n - \overline{\eta}^{n-1}}{\Delta t}, \eta^n\right) \geq \frac{1}{2\Delta t}\left(\|\eta^n\|^2 - \|\overline{\eta}^{n-1}\|^2\right). \tag{12}$$

From [4], we get

$$\|\overline{\eta}^{n-1}\|^2 = \|\eta^{n-1}\|^2 + o(\Delta t\|\eta^{n-1}\|^2 + h^4\Delta t\|u^{n-1}\|_3^2);$$

$$(\lambda\frac{\eta_{x\bar{x}}^n - \eta_{x\bar{x}}^{n-1}}{\Delta t}, \eta^n) \le \frac{\lambda}{2\Delta t}\{\|\eta_x^{n-1}\|^2 - \|\eta_x^n\|^2\}, \tag{13}$$

$$(\theta\frac{\eta_{y\bar{y}}^n - \eta_{y\bar{y}}^{n-1}}{\Delta t}, \eta^n) \le \frac{\theta}{2\Delta t}\{\|\eta_y^{n-1}\|^2 - \|\eta_y^n\|^2\}, \tag{14}$$

$$(\rho^n, \eta^n) \le \frac{1}{2}(\|\rho^n\|^2 + \|\eta^n\|^2), \tag{15}$$

$$\|\rho^n\|^2 \le \|\frac{u^n - \overline{u}^{n-1}}{\Delta t} - m\frac{\partial u}{\partial\tau}(t^n)\|^2 + \|\lambda\frac{u_{x\bar{x}}^n - u_{x\bar{x}}^{n-1}}{\Delta t} - u_{xxt}(t^n)\|^2$$

$$+ \|\theta\frac{u_{y\bar{y}}^n - u_{y\bar{y}}^{n-1}}{\Delta t} - u_{yyt}(t^n)\|^2, \tag{16}$$

$$\|\frac{u^n - \overline{u}^{n-1}}{\Delta t} - m\frac{\partial u}{\partial\tau}(t^n)\|^2 \le c\Delta t\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;l^2)}, \tag{17}$$

$$\|\frac{u_{x,\bar{x}}^n - u_{x,\bar{x}}^{n-1}}{\Delta t} - u_{xxt}(t^n)\|^2$$

$$\le c\{\|\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}(\frac{\partial^3 u}{\partial x^2\partial t} - (\frac{\partial u}{\partial t})_{x\bar{x}})(\xi)d\xi\|^2$$

$$+ \|\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_{\xi}^{t^n}\frac{\partial^4 u}{\partial x^2\partial t^2}\sigma d\sigma d\xi\|^2\}$$

$$\le c\{\frac{h^4}{\Delta t}\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)} + \Delta t\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)}\}, \tag{18}$$

$$\|\frac{u_{y,\bar{y}}^n - u_{y,\bar{y}}^{n-1}}{\Delta t} - u_{xxt}(t^n)\|^2$$

$$\le c\{\frac{h^4}{\Delta t}\|\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)} + \Delta t\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)}\}. \tag{19}$$

From the above estimations, we get

$$\frac{1}{\Delta t}\{\|\eta^n\|^2-\|\overline{\eta}^{n-1}\|^2+\lambda\|\eta_x^n\|^2-\lambda\|\eta_x^{n-1}\|^2+\theta\|\eta_y^n\|^2-\theta\|\eta_x^{n-1}\|^2\}$$

$$\le c\{\|\eta^n\|^2+\Delta t\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;l^2)}+\frac{\lambda h^4}{\Delta t}\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\lambda\Delta t\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)}+\theta(\frac{h^4}{\Delta t}\|\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\Delta t\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)})\}. \tag{20}$$

So

$$\frac{1}{\Delta t}\{\|\eta^n\|^2-\|\eta^{n-1}\|^2+\lambda\|\eta_x^n\|^2-\lambda\|\eta_x^{n-1}\|^2+\theta\|\eta_y^n\|^2-\theta\|\eta_x^{n-1}\|^2\}$$

$$\le c\{\|\eta^n\|^2+\|\eta^{n-1}\|^2+h^4\|u^{n-1}\|_3^2+\Delta t\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\frac{\lambda h^4}{\Delta t}\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}+\lambda\Delta t\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\theta(\frac{h^4}{\Delta t}\|\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}+\Delta t\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)})\}. \tag{21}$$

Both sides of the above multiplied by Δt, then

$$\|\eta^n\|^2-\|\eta^{n-1}\|^2+\lambda\|\eta_x^n\|^2-\lambda\|\eta_x^{n-1}\|^2+\theta\|\eta_y^n\|^2-\theta\|\eta_x^{n-1}\|^2$$

$$\le c\{\Delta t(\|\eta^n\|^2+\|\eta^{n-1}\|^2+h^4\|u^{n-1}\|_3^2)+\Delta t^2\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\lambda h^4\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}+\lambda\Delta t^2\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)}$$

$$+\theta(h^4\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(t^{n-1},t^n;l^2)}+\Delta t^2\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(t^{n-1},t^n;l^2)})\}. \tag{22}$$

Sum n from 1 to M, we get

$$\|\eta^M\|^2 - \|\eta^0\|^2 + \lambda\|\eta_x^M\|^2 - \lambda\|\eta_x^0\|^2 + \theta\|\eta_y^M\|^2 - \theta\|\eta_x^0\|^2$$

$$\leq c\{\sum_{n=1}^{M}\Delta t(\|\eta^n\|^2 + \|\eta^{n-1}\|^2) + h^4\|u\|^2_{L^2(0,T;H^3)} + \Delta t^2\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(0,T;l^2)}$$

$$+ \lambda h^4\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(0,T;l^2)} + \lambda\Delta t^2\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(0,T;l^2)}$$

$$+ \theta(h^4\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(0,T;l^2)} + \Delta t^2\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(0,T;l^2)})\}. \tag{23}$$

Choose suitable small $\Delta t(\text{s.t. } c\Delta t < \frac{1}{2})$. According to its initial value and Gronwall inequality, we get

$$\|\eta^M\|^2 + \lambda\|\eta_x^M\|^2 + \theta\|\eta_y^M\|^2$$

$$\leq c\{h^4\|u\|^2_{L^2(0,T;H^3)} + \Delta t^2\|\frac{\partial^2 u}{\partial\tau^2}\|^2_{L^2(0,T;l^2)}$$

$$+ \lambda h^4\|\frac{\partial^5 u}{\partial x^4\partial t}\|^2_{L^2(0,T;l^2)} + \lambda\Delta t^2\|\frac{\partial^4 u}{\partial x^2\partial t^2}\|^2_{L^2(0,T;l^2)}$$

$$+ \theta(h^4\frac{\partial^5 u}{\partial y^4\partial t}\|^2_{L^2(0,T;l^2)} + \Delta t^2\|\frac{\partial^4 u}{\partial y^2\partial t^2}\|^2_{L^2(0,T;l^2)})\}. \tag{24}$$

From the above estimations, we obtain:

Theorem. Assume u be the solution of (4)-(6), and $U_{i,j}^n$ be the characteristic difference solution of (7)-(8), then for the suitable small Δt and h, the following estimation founds:

$$\|U^n - u(t_n)\|_1 \leq c(h^2 + \Delta t).$$

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