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SEQUENCES OF RIGHT AND LEFT NEIGHBOURS OF SIX TYPE INDEFINITE BINARY QUADRATIC FORMS AND THEIR PROPER AUTOMORPHISMS

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Abstract

In this work, we consider the sequences of right and left neighbours of indefinite binary quadratic forms $F = F(x, y) = ax^2 + bxy + cy^2$ of discriminant $\Delta = b^2 - 4ac$. In the first section, we give some definitions and notations from binary quadratic forms and their neighbours. In the second section, we try to determine the sequences of right and left neighbours of some specific indefinite quadratic forms and in the last section, we formulate the set of proper automorphisms of them.

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1. Introduction

A real binary quadratic form F is a polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2, \tag{1.1}$$

with real coefficients a, b, c. We denote F shortly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. F is an integral form if and only if $a, b, c \in \mathbb{Z}$, and is indefinite if and only if $\Delta(F) > 0$. An indefinite quadratic form $F = (a, b, c)$ of discriminant Δ is said to be reduced, if

$$\left|\sqrt{\Delta} - 2|a|\right| < b < \sqrt{\Delta}. \tag{1.2}$$

There is a strong connection between binary quadratic forms and the extended modular group $\overline{\Gamma}$ (see [7]). Gauss (1777-1855) defined the group action of $\overline{\Gamma}$ on the set of forms as follows:

$$gF(x, y) = (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy$$
$$+ (at^2 + btu + cu^2)y^2,$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \overline{\Gamma}$, that is, $gF = F(rx + ty, sx + uy)$. Let F_1 and F_2 be two forms. If there exists a $g \in \overline{\Gamma}$ such that $gF_1 = F_2$, then F_1 and F_2 are called *equivalent*. If $\det g = 1$, then F_1 and F_2 are called *properly equivalent* and if $\det g = -1$, then F_1 and F_2 are called *improperly equivalent*. An element $g \in \overline{\Gamma}$ is said to be an automorphism of F, if $gF = F$. If $\det g = 1$, then g is called *proper automorphism* and if $\det g = -1$, then g is called *improper automorphism*. The set of proper automorphisms of F is denoted by $Aut^+(F)$ and the set of improper automorphisms of F is denoted by $Aut^-(F)$. A quadratic form F is said to be ambiguous, if it is improperly equivalent to itself.

Let $\rho(F)$ denotes the normalization (it means that replacing F by its normalization, for further details, see [1]) of $(c, -b, a)$. To be more explicit, we set

$$\rho(F) = (c, -b + 2cs, cs^2 - bs + a), \tag{1.3}$$

where

$$s = s(F) = \begin{cases} \operatorname{sign}(c)\left\lfloor \dfrac{b}{2|c|} \right\rfloor, & \text{for } |c| \geq \sqrt{\Delta}, \\ \operatorname{sign}(c)\left\lfloor \dfrac{b + \sqrt{\Delta}}{2|c|} \right\rfloor, & \text{for } |c| < \sqrt{\Delta}. \end{cases} \tag{1.4}$$

Note that, if F is reduced, then $\rho(F)$ is also reduced. In fact, ρ is a permutation of the set of all reduced indefinite forms. Set

$$\chi(F) = (-c, b - a),\ \tau(F) = (-a, b, -c), \text{ and } \mu(F) = (a, -b, c). \tag{1.5}$$

Then

$$\tau(\rho(F)) = \rho(\tau)F = (-c, -b + 2cs, -a + bs - cs^2).$$

If F is reduced, then $\tau(F)$ is also reduced. If $\chi(F) = F$, i.e., $F = (a, b, -a)$, then F is called *symmetric*. We assume that $F = (a, b, c)$ is indefinite and integral throughout the paper.

The cycle of F is the sequence $((\tau\rho)^i(G))$ for $i \in \mathbb{Z}$, where $G = (k, l, m)$ is a reduced form with $k > 0$, which is equivalent to F. Similarly, the proper cycle of F is the sequence $(\rho^i(G))$ for $i \in \mathbb{Z}$, where G is a reduced form, which is properly equivalent to F. The cycle and the proper cycle of F are invariants of the equivalence class of F. We represent the cycle or proper cycle of F by its period $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ of length l. The cycle and proper cycle of F is given below:

Lemma 1. *Let* $F = (a, b, c)$ *be a reduced indefinite binary quadratic form of discriminant* Δ. *Let* $F_0 = F = (a_0, b_0, c_0)$,

$$s_i = |s(F_i)| = \left\lfloor \frac{b_i + \sqrt{\Delta}}{2|c_i|} \right\rfloor, \tag{1.6}$$

and let

$$F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = (|c_i|, -b_i + 2s_i|c_i|, -(a_i + b_i s_i + c_i s_i^2)), \tag{1.7}$$

for $1 \le i \le l-2$. *Then the cycle of* F *is* $F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-1}$ *of length* l. *Moreover*,

(1) *If* l *is odd, then the proper cycle of* F *is*

$$F_0 \sim \tau(F_1) \sim F_2 \sim \cdots \sim \tau(F_{l-2}) \sim F_{l-1} \sim \tau(F_0) \sim F_1 \sim \cdots \sim \tau(F_{l-1}),$$

of length $2l$. *In this case, the equivalence class of* F *is equal to the proper equivalence class of* F.

(2) *If* l *is even, then the proper cycle of* F *is*

$$F_0 \sim \tau(F_1) \sim F_2 \sim \cdots \sim F_{l-2} \sim \tau(F_{l-1}),$$

of length l. *In this case, the equivalence class of* F *is the disjoint union of the proper equivalence class of* F *and the proper equivalence class of* $\tau(F)$ [1].

The right neighbour of $F = (a, b, c)$ is denoted by $R(F)$ is the form (A, B, C) determined by the following three conditions:

(i) $A = c$,

(ii) $b + B \equiv 0 \pmod{2A}$ and $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$,

(iii) $B^2 - 4AC = \Delta$.

It is clear from definition that

$$R(F) = (A, B, C) = \begin{pmatrix} 0 & -1 \\ 1 & -\delta \end{pmatrix} (a, b, c), \tag{1.8}$$

where

$$\delta = \frac{b + B}{2c}. \tag{1.9}$$

For some forms F, we have $R^n(F) = F$ for some integer $n \geq 1$, but for some forms F, we get $R^n(F) = R^m(F) \neq F$ for integers $m < n$. The left neighbour $L(F)$ of F is defined as

$$L(F) = L(a, b, c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(c, b, a). \tag{1.10}$$

So F is properly equivalent to its right and left neighbour (for further details on binary quadratic forms and their neighbours, see also [2, 3, 4, 5, 6]).

2. Sequence of Right and Left Neighbours of Some Specific Quadratic Forms

In [8, 9], the first author considered the proper cycles of indefinite quadratic forms and their right neighbours. He proved the following theorem:

Theorem 1 ([8], Theorem 2.1)**.** *Let* $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ *be the cycle of* F *of length* l *and* $R^i(F_0)$ *be the consecutive right neighbours of* F_0 *for* $i \geq 0$. *Then*

(1) *If* l *is odd, then the proper cycle of* F *is* $F_0 \sim R^1(F_0) \sim \cdots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$ *of length* $2l$.

(2) *If* l *is even, then the proper cycle of* F *is* $F_0 \sim R^1(F_0) \sim \cdots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)$ *of length* l.

Later, Tekcan et al. able to prove the following two theorems on the left neighbours of indefinite forms:

Theorem 2 ([10], Theorem 4)**.** *Let* $F_0 \sim F_1 \sim \cdots \sim F_{l-1}$ *be the cycle of* F *of length* l *and* $L^i(F_0)$ *be the consecutive left neighbours of* F_0 *for* $i \geq 0$. *Then*

(1) *If* l *is odd, then the proper cycle of* F *is* $F_0 \sim L^{2l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$ *of length* $2l$.

(2) *If* l *is even, then the proper cycle of* F *is* $F_0 \sim L^{l-1}(F_0) \sim \cdots \sim L^2(F_0) \sim L^1(F_0)$ *of length* l.

Theorem 3 ([10], Theorem 7)**.** *Let* l *be the length of the cycle of* F.

(1) *If* l *is odd, then* $L^i(F_0) = R^{2l-i}(F_0)$ *for* $1 \leq i \leq 2l-1$.

(2) *If* l *is even, then* $L^i(F_0) = R^{l-i}(F_0)$ *for* $1 \leq i \leq l-1$.

In the present paper, we will consider the sequences of right and left neighbours of some specific indefinite binary quadratic forms F. For this reason, let

$$R_F = \{R^1(F), R^2(F), \cdots, R^n(F)\} \text{ and } L_F = \{L^1(F), L^2(F), \cdots, L^m(F)\},$$

denote the sequences of right and left neighbours of F, respectively, for some positive integers n and m. Here our main problems are:

(1) We want to determine which indefinite quadratic forms F have one right and one left neighbour and they are same, that is, $R^1(F) = L^1(F)$.

(2) Which indefinite quadratic forms F have infinitely many right and left neighbours for which $R^i(F) = R^j(F)$ for $j > i$ and $L^k(F) = L^m(F)$ for $m > k$ and again they are same.

(3) Is there any connection between indefinite quadratic forms F for which $R^1(F) = L^1(F)$ and $R^i(F) = R^j(F)$ for $j > i$ and $L^k(F) = L^m(F)$ for $m > k$?

(4) We want to determine the proper automorphisms of indefinite forms F for which $R^1(F) = L^1(F)$ and $R^i(F) = R^j(F)$ for $j > i$ and $L^k(F) = L^m(F)$ for $m > k$.

In this section, we try to answer these problems. The following three forms and the next three forms (which are the application to μ the previous ones) are the forms what we are looking for.

Let $a, t \geq 1$ be two integers. Then, we set the indefinite quadratic forms as

$$F_1 = (a, at, -a), \quad F_2 = (a, at, -at), \quad \text{and } F_3 = (at, at, -a). \tag{2.1}$$

Then, we can give the following theorem:

Theorem 4. *$F_1, F_2,$ and F_3 have one right and one left neighbour and they are same, that is,*

$$R^1_{F_1} = L^1_{F_1} = \{(-a, at, a)\},$$

$$R^1_{F_2} = L^1_{F_2} = \{(-at, at, a)\},$$

$$R^1_{F_3} = L^1_{F_3} = \{(-a, at, at)\}.$$

Proof. We start with F_1. For $i = 0$, we get from (1.8), $\delta_0 = -t$ and hence the first right neighbour of F_1 is

$$R^1(F_1) = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix}(a, at, -a) = (-a, at, a).$$

For $i = 1$, we get $\delta_1 = t$ and hence

$$R^2(F_1) = \begin{pmatrix} 0 & -1 \\ 1 & -t \end{pmatrix}(-a, at, a) = (a, at, -a) = F_1.$$

So F_1 has one right neighbour. The left neighbours of F_1 are hence

$$L^1(F_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(-a, at, a) = (-a, at, a);$$

$$L^2(F_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R(a, at, -a) = (a, at, -a) = F_1.$$

So $R^1_{F_1} = L^1_{F_1} = \{(-a, at, a)\}$.

With the same method, we find that

$$R^1(F_2) = (-at, at, a), \quad R^2(F_2) = (a, at, -at) = F_2,$$

$$L^1(F_2) = (-at, at, a), \quad L^2(F_2) = (a, at, -at) = F_2,$$

$$R^1(F_3) = (-a, at, at), \quad R^2(F_3) = (at, at, -a) = F_3,$$

$$L^1(F_3) = (-a, at, at), \quad L^2(F_3) = (at, at, -a) = F_3.$$

Therefore, $R^1_{F_2} = L^1_{F_2} = \{(-at, at, a)\}$ and $R^1_{F_3} = L^1_{F_3} = \{(-a, at, at)\}$. □

For our second and third questions, let $\mu(F)$ be the transformation in (1.5). Note that

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \overline{\Gamma}.$$

So $\mu(a, b, c) = (a, -b, c)$, that is, the form (a, b, c) is improperly equivalent to $(a, -b, c)$. For the forms F_1, F_2, and F_3 in (2.1), we set

$$\begin{aligned} \mu(F_1) &= (a, -at, -a) = F_1^*, \\ \mu(F_2) &= (a, -at, -at) = F_2^*, \\ \mu(F_3) &= (at, -at, -a) = F_3^*. \end{aligned} \tag{2.2}$$

So, the answer of our third question is that the form F_i is improperly equivalent to F_i^* for $i = 1, 2, 3$. In the following theorem, we will see that

$$R^3(F_i^*) = R^1(F_i^*), \quad R^4(F_i^*) = R^2(F_i^*),$$

$$L^3(F_i^*) = L^1(F_i^*), \quad L^4(F_i^*) = L^2(F_i^*),$$

and they are same for $i = 1, 2, 3$.

Theorem 5. *For* F_1^*, F_2^*, *and* F_3^*, *we have*

$$R_{F_1^*} = L_{F_1^*} = \{\overline{(-a, at, a), (a, at, -a)}\},$$

$$R_{F_2^*} = L_{F_2^*} = \{\overline{(-at, at, a), (a, at, -at)}\},$$

$$R_{F_3^*} = L_{F_3^*} = \{\overline{(-a, at, at), (at, at, -a)}\}.$$

Proof. As in Theorem 4, we find that the right and left neighbours of F_1^* are

$$R^1(F_1^*) = (-a, at, a), \quad R^2(F_1^*) = (a, at, -a),$$

$$R^3(F_1^*) = (-a, at, a) = R^1(F_1^*), \quad R^4(F_1^*) = (a, at, -a) = R^2(F_1^*),$$

$$L^1(F_1^*) = (-a, at, a), \quad L^2(F_1^*) = (a, at, -a),$$

$$L^3(F_1^*) = (-a, at, a) = L^1(F_1^*), \quad L^4(F_1^*) = (a, at, -a) = L^2(F_1^*).$$

So $R_{F_1^*} = L_{F_1^*} = \{\overline{(-a, at, a), (a, at, -a)}\}$.

Similarly, we find that

$$R^1(F_2^*) = (-at, at, a), \quad R^2(F_2^*) = (a, at, -at),$$

$$R^3(F_2^*) = (-at, at, a) = R^1(F_2^*), \quad R^4(F_2^*) = (a, at, -at) = R^2(F_2^*),$$

$$L^1(F_2^*) = (-at, at, a), \quad L^2(F_2^*) = (a, at, -at),$$

$$L^3(F_2^*) = (-at, at, a) = L^1(F_2^*), \quad L^4(F_2^*) = (a, at, -at) = L^2(F_2^*),$$

and

$$R^1(F_3^*) = (-a,\ at,\ at), \quad R^2(F_3^*) = (at,\ at,\ -a),$$

$$R^3(F_3^*) = (-a,\ at,\ at) = R^1(F_3^*), \quad R^4(F_3^*) = (at,\ at,\ -a) = R^2(F_3^*),$$

$$L^1(F_3^*) = (-a,\ at,\ at), \quad L^2(F_3^*) = (at,\ at,\ -a),$$

$$L^3(F_3^*) = (-a,\ at,\ at) = L^1(F_3^*), \quad L^4(F_3^*) = (at,\ at,\ -a) = L^2(F_3^*).$$

So

$$R_{F_2^*} = L_{F_2^*} = \{\overline{(-at,at,a),(a,at,-at)}\},\ R_{F_3^*} = L_{F_3^*} = \{\overline{(-a,at,at),(at,at,-a)}\},$$

as we wanted. □

3. Automorphisms

In this section, we will consider the proper automorphism of the forms F_1, F_2, F_3 and F_1^*, F_2^*, F_3^*. Let δ be the integer in (1.9). Set

$$T(\delta) = \begin{pmatrix} -\delta & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.1}$$

Note that $T(\delta) \in \overline{\Gamma}$. Let $R^i(F)$ be the consecutive right neighbours of F for $0 \le i \le N$. Set

$$\tau_{F,N} = T(\delta_0)T(\delta_1)\cdots T(\delta_{N-1}). \tag{3.2}$$

Then we can give the following two theorems:

Theorem 6 ([3], Theorem 9.4)**.** *Let F be an integral form of positive nonsquare discriminant. Let $N > 0$ be the smallest positive integer such that $R^N(F) = F$. Then*

(1) *$\tau_{F,N}$ is an element of $Aut^+(F)$ of infinite order.*

(2) $Aut^+(F) = \{ \pm (\tau_{F,N})^m : m \in \mathbb{Z}\}$.

Theorem 7 ([3], Corollary 9.5)**.** *Let F be an integral form of positive nonsquare discriminant such that $R^N(F) = R^M(F)$ for smallest integers $N > M$. Then*

$$Aut^+(F) = \{ \pm (\tau_{F,N}\tau_{F,M}^{-1})^m : m \in \mathbb{Z}\}.$$

By virtue of above two theorems, we can give the following theorem related to proper automorphisms of the forms F_1, F_2, F_3 and F_1^*, F_2^*, F_3^*.

Theorem 8. *For the forms F_1, F_2, F_3 and F_1^*, F_2^*, F_3^*, we have the followings*:

(1)

$$Aut^+(F_1) = \{ \pm (\tau_{F_1,2})^m : m \in \mathbb{Z}\}, \text{ where } \tau_{F_1,2} = \begin{pmatrix} -t^2-1 & t \\ t & -1 \end{pmatrix},$$

and

$$(\tau_{F_1,2})^m = \begin{pmatrix} (\tau_{F_1,2})_{11} & (\tau_{F_1,2})_{12} \\ (\tau_{F_1,2})_{21} & (\tau_{F_1,2})_{22} \end{pmatrix},$$

where

$$(\tau_{F_1,2})_{11} = (-1)^m \sum_{i=0}^{m} \binom{2m-i}{2m-2i} t^{2m-2i},$$

$$(\tau_{F_1,2})_{12} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{2m-1-2i},$$

$$(\tau_{F_1,2})_{21} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{2m-1-2i},$$

$$(\tau_{F_1,2})_{22} = (-1)^{m} \sum_{i=0}^{m-1} \binom{2m-2-i}{2m-2-2i} t^{2m-2-2i},$$

for $m \geq 1$.

(2) $Aut^{+}(F_2) = \{ \pm (\tau_{F_2,2})^m : m \in \mathbb{Z}\}$, *where* $\tau_{F_2,2} = \begin{pmatrix} -t-1 & 1 \\ t & -1 \end{pmatrix}$,

and

$$(\tau_{F_2,2})^m = \begin{pmatrix} (\tau_{F_2,2})_{11} & (\tau_{F_2,2})_{12} \\ (\tau_{F_2,2})_{21} & (\tau_{F_2,2})_{22} \end{pmatrix},$$

where

$$(\tau_{F_2,2})_{11} = (-1)^m \sum_{i=0}^{m} \binom{2m-i}{2m-2i} t^{m-i},$$

$$(\tau_{F_2,2})_{12} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-1-i},$$

$$(\tau_{F_2,2})_{21} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-i},$$

$$(\tau_{F_2,2})_{22} = (-1)^{m} \sum_{i=0}^{m-1} \binom{2m-2-i}{2m-2-2i} t^{m-1-i},$$

for $m \geq 1$.

(3) $Aut^{+}(F_3) = \{ \pm (\tau_{F_3,2})^m : m \in \mathbb{Z}\}$, *where* $\tau_{F_3,2} = \begin{pmatrix} -t-1 & t \\ 1 & -1 \end{pmatrix}$,

and

$$(\tau_{F_3,2})^m = \begin{pmatrix} (\tau_{F_3,2})_{11} & (\tau_{F_3,2})_{12} \\ (\tau_{F_3,2})_{21} & (\tau_{F_3,2})_{22} \end{pmatrix},$$

where

$$(\tau_{F_3,2})_{11} = (-1)^m \sum_{i=0}^{m} \binom{2m-i}{2m-2i} t^{m-i},$$

$$(\tau_{F_3,2})_{12} = (-1)^{m+1}\sum_{i=0}^{m-1}\binom{2m-1-i}{2m-1-2i}t^{m-i},$$

$$(\tau_{F_3,2})_{21} = (-1)^{m+1}\sum_{i=0}^{m-1}\binom{2m-1-i}{2m-1-2i}t^{m-1-i},$$

$$(\tau_{F_3,2})_{22} = (-1)^{m}\sum_{i=0}^{m-1}\binom{2m-2-i}{2m-2-2i}t^{m-1-i},$$

for $m \geq 1$.

(4) $Aut^{+}(F_1^*) = \left\{\pm(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})^m : m \in \mathbb{Z}\right\}$,

where

$$\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1} = \begin{pmatrix} -1 & t \\ t & -t^2-1 \end{pmatrix},$$

and

$$(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})^m = \begin{pmatrix} (\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{11} & (\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{12} \\ (\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{21} & (\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{22} \end{pmatrix},$$

where

$$(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{11} = (-1)^{m}\sum_{i=0}^{m-1}\binom{2m-2-i}{2m-2-2i}t^{2m-2-2i},$$

$$(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{12} = (-1)^{m+1}\sum_{i=0}^{m-1}\binom{2m-1-i}{2m-1-2i}t^{2m-1-2i},$$

$$(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{21} = (-1)^{m+1}\sum_{i=0}^{m-1}\binom{2m-1-i}{2m-1-2i}t^{2m-1-2i},$$

$$(\tau_{F_1^*,3}\tau_{F_1^*,1}^{-1})_{22} = (-1)^{m}\sum_{i=0}^{m}\binom{2m-i}{2m-2i}t^{2m-2i},$$

for $m \geq 1$.

(5) $Aut^{+}(F_2^*) = \left\{ \pm \left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)^m : m \in \mathbb{Z} \right\}$,

where

$$\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} = \begin{pmatrix} -1 & 1 \\ t & -t-1 \end{pmatrix},$$

and

$$\left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)^m = \begin{pmatrix} \left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{11} & \left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{12} \\ \left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{21} & \left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{22} \end{pmatrix},$$

where

$$\left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{11} = (-1)^m \sum_{i=0}^{m-1} \binom{2m-2-i}{2m-2-2i} t^{m-1-i},$$

$$\left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{12} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-1-i},$$

$$\left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{21} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-i},$$

$$\left(\tau_{F_2^*,3} \tau^{-1}_{F_2^*,1} \right)_{22} = (-1)^m \sum_{i=0}^{m-1} \binom{2m-i}{2m-2i} t^{m-i},$$

for $m \geq 1$.

(6) $Aut^{+}(F_3^*) = \left\{ \pm \left(\tau_{F_3^*,3} \tau^{-1}_{F_3^*,1} \right)^m : m \in \mathbb{Z} \right\}$,

where

$$\tau_{F_3^*,3} \tau^{-1}_{F_3^*,1} = \begin{pmatrix} -1 & t \\ 1 & -t-1 \end{pmatrix},$$

and

$$(\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})^m = \begin{pmatrix} (\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{11} & (\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{12} \\ (\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{21} & (\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{22} \end{pmatrix},$$

where

$$(\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{11} = (-1)^m \sum_{i=0}^{m-1} \binom{2m-2-i}{2m-2-2i} t^{m-1-i},$$

$$(\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{12} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-i},$$

$$(\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{21} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{m-1-i},$$

$$(\tau_{F_3^*,3}\tau_{F_3^*,1}^{-1})_{22} = (-1)^m \sum_{i=0}^{m} \binom{2m-i}{2m-2i} t^{m-i},$$

for $m \geq 1$.

Proof. (1) We see in Theorem 4 that for the form $F_1 = (a, at, -a)$, we get $\delta_0 = -t$ and $\delta_1 = t$. So from (3.2),

$$\tau_{F_1,2} = T(\delta_0)T(\delta_1) = \begin{pmatrix} -t^2-1 & t \\ t & -1 \end{pmatrix}.$$

Let $m = 1$, then we obtain

$$(\tau_{F_1,2}) = \begin{pmatrix} -t^2-1 & t \\ t & -t \end{pmatrix} = \tau_{F_1,2}.$$

So it is true for $m = 1$. Let us assume that this relation is satisfied for $m-1$, that is,

$$(\tau_{F_1,2})^{m-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = (-1)^{m-1}\sum_{i=0}^{m-1}\binom{2m-2-i}{2m-2-2i}t^{2m-2-2i},$$

$$B = (-1)^{m}\sum_{i=0}^{m-2}\binom{2m-3-i}{2m-3-2i}t^{2m-1-2i},$$

$$C = (-1)^{m+1}\sum_{i=0}^{m-1}\binom{2m-3-i}{2m-3-2i}t^{2m-1-2i},$$

$$D = (-1)^{m-1}\sum_{i=0}^{m-2}\binom{2m-4-i}{2m-4-2i}t^{2m-4-2i}.$$

Then

$$\begin{pmatrix} -t^2-1 & t \\ t & -t \end{pmatrix}(\tau_{F_1,2})^{m-1} = \begin{pmatrix} (\tau_{F_1,2})^m_{11} & (\tau_{F_1,2})^m_{12} \\ (\tau_{F_1,2})^m_{21} & (\tau_{F_1,2})^m_{22} \end{pmatrix}.$$

Here

$$(\tau_{F_1,2})^m_{11} = (-t^2-1)\left[(-1)^{m-1}\sum_{i=0}^{m-1}\binom{2m-2-i}{2m-2-2i}t^{2m-2-2i}\right]$$

$$+ t\left[(-1)^{m}\sum_{i=0}^{m-2}\binom{2m-3-i}{2m-3-2i}t^{2m-3-2i}\right]$$

$$= (-1)^{m-1}\left[\begin{array}{c}\binom{2m-2}{2m-2}t^{2m-2}(-t^2-1)+\binom{2m-3}{2m-4}t^{2m-4}(-t^2-1) \\ +\binom{2m-4}{2m-6}t^{2m-6}(-t^2-1)+\cdots+\binom{m}{2}t^2(-t^2-1) \\ +\binom{m-1}{0}(-t^2-1)\end{array}\right]$$

$$+(-1)^m\begin{bmatrix}\binom{2m-3}{2m-3}t^{2m-3}\cdot t+\binom{2m-4}{2m-5}t^{2m-4}\cdot t+\cdots\\ +\binom{m}{3}t^m\cdot t+\binom{m-1}{1}t^{m-1}\cdot t\end{bmatrix}$$

$$=(-1)^m\begin{bmatrix}\binom{2m-2}{2m-2}(t^{2m}+t^{2m-2})+\binom{2m-3}{2m-3}t^{2m-2}\\ +\binom{2m-4}{2m-6}(t^{2m-4}+t^{2m-6})+\binom{2m-5}{2m-7}t^{2m-4}+\cdots\\ +\binom{m}{2}(t^4+t^2)+\binom{m}{3}t^{m+1}+\binom{m-1}{1}t^m\end{bmatrix}$$

$$=(-1)^m\begin{bmatrix}t^{2m}+t^{2m-2}+t^{2m-2}+(2m-3)(t^{2m-2}+t^{2m-4})\\ +(2m-4)t^{2m-3}+\frac{(2m-4)(2m-3)}{2}(t^{2m-4}+t^{2m-6})\\ +\frac{(2m-5)(2m-6)}{2}t^{2m-4}+\cdots+\frac{m(m-1)}{2}(t^4+t^2)\\ +\frac{m(m-1)(m-2)}{2}t^{m+1}+(t^2+1)+(m-1)t^m\end{bmatrix}$$

$$=(-1)^m\begin{bmatrix}\binom{2m}{2m}t^{2m}+\binom{2m-1}{2m-1}t^{2m-2}+\binom{2m-2}{2m-4}t^{2m-4}\\ +\binom{2m-3}{2m-6}t^{2m-6}+\cdots+\binom{m+1}{2}t^2+1\end{bmatrix}$$

$$=(-1)^m\sum_{i=0}^{m}\binom{2m-i}{2m-2i}t^{2m-2i}.$$

The other cases $(\tau_{F_1,2})_{12}^m$, $(\tau_{F_1,2})_{21}^m$, and $(\tau_{F_1,2})_{22}^m$ can be proved similarly.

(2) and (3) can be proved similarly.

(4) For the form $F_1^* = (a, -at, -a)$, we obtain the following table:

i	0	1	2	3
a_i	a	$-a$	a	$-a$
b_i	$-at$	at	at	at
c_i	$-a$	a	$-a$	a
δ_i	0	t	$-t$	

Here $R^3(F_1^*) = R^1(F_1^*)$. So $N = 3$ and $M = 1$. Therefore,

$$\tau_{F_1^*,3} = \begin{pmatrix} -t & -1 \\ t^2+1 & t \end{pmatrix} \text{ and } \tau^{-1}_{F_1^*,1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So

$$\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1} = \begin{pmatrix} -1 & t \\ t & -t^2-1 \end{pmatrix}$$

by Theorem 7. Again, it can be proved by induction on m that

$$(\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})^m = \begin{pmatrix} (\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{11} & (\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{12} \\ (\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{21} & (\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{22} \end{pmatrix},$$

where

$$(\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{11} = (-1)^m \sum_{i=0}^{m-1} \binom{2m-2-i}{2m-2-2i} t^{2m-2-2i},$$

$$(\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{12} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{2m-1-2i},$$

$$(\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1})_{21} = (-1)^{m+1} \sum_{i=0}^{m-1} \binom{2m-1-i}{2m-1-2i} t^{2m-1-2i},$$

$$\left(\tau_{F_1^*,3}\tau^{-1}_{F_1^*,1}\right)_{22} = (-1)^m \sum_{i=0}^{m} \binom{2m-i}{2m-2i} t^{2m-2i},$$

for $m \geq 1$.

(5) and (6) can be proved similarly. □

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