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BARABANOV'S AUXILIARY SYSTEMS FOR A CLASS OF PERTURBED DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we study the stability properties of the perturbed family $\sum_{\Delta}: \dot{x} = [A + B\Delta C]x$, when $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, $l, q \in \mathbb{Z}_{+}^{\star}$, A is a Hurwitz-stable matrix, $B \neq 0$, $C \neq 0$ are given matrices specifying the structure of the perturbation, $\Delta \in \mathbb{R}^{l \times q}$ represents the uncertainty of the perturbation; applying a theorem proved by Barabanov (see [1]). We obtain the explicit expressions for the so called auxiliary Barabanov's systems. We also state a way in order to find the stability radius of the perturbed matrix A.

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1. Introduction

Consider a family of systems of differential equations, namely, let $(A, B, C) \in L_{n,l,q}(\mathbb{R}) := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times l} \times \mathbb{R}^{q\times n}$, where A is a Hurwitz-stable matrix, i.e., the spectrum of A is contained in the open left complex semi-plane. $B \neq \mathbf{0}$, $C \neq \mathbf{0}$ are given matrices specifying the structure of the perturbation, $l, q \in \mathbb{Z}_+^\star$. We consider for each matrix $\Delta \in \mathbb{R}^{l\times q}$, the system

$$\Sigma_\Delta : \dot{x} = [A + B\Delta C]x,$$

and consider for each positive number r, the differential inclusion

$$\Sigma_r : \dot{x} \in F_r(x) := \{[A + B\Delta C]x : \Delta \in \mathbb{R}^{l\times q}, \quad \|\Delta\| \leq r\}, \tag{1.1}$$

where $\|\cdot\|$ is some norm on the matrix space $\mathbb{R}^{l\times q}$.

Following ([6], [7], [4]), we define the real stability radius of the matrix A for linear time-varying perturbations of structure (B, C) as

$$r_{\mathbb{R},t}^-(A, B, C) = \inf\{r > 0 : \Sigma_r \text{ is not asymptotically stable}\}. \tag{1.2}$$

An upper bound for the time-varying stability radius (1.2) is the number

$$r_{\mathbb{R}}^-(A, B, C) = \inf\{\|\Delta\| : \Delta \in \mathbb{R}^{l\times q}, \sigma(A + B\Delta C) \cap \mathbb{C}_+ \neq \emptyset\}, \tag{1.3}$$

where $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \Re e(\lambda) \geq 0\}$ and $\sigma(M)$ denotes the spectrum of the matrix M.

In the paper [8], it is presented a formula for the calculation of the number

$$r_{\mathbb{R}}^-(A, B, C), \quad (A, B, C) \in L_{n,l,q}(\mathbb{R}),$$

in the case when the perturbations Δ are measured with the spectral norm. Interesting results concerning stability radii $r^-_{\mathbb{R},t}(A, B, C)$ and $r^-_{\mathbb{R}}(A, B, C)$ are obtained in the papers [2], [3], [4], [5], [9], [10]. In this work, our final goal is to study the stability properties of the family (1.1) when $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, $l, q \in \mathbb{Z}_+^\star$, via the number $r^-_{\mathbb{R},t}(A, B, C)$, using a theorem obtained by Barabanov (see [1]) about the stability of a differential inclusion.

The present note is organized as follows. In Section 2, the Barabanov's theorem is presented, then in Section 3, this theorem is applied to the differential inclusion (1.1). We assign to each triple (A, B, C) two systems, which are in some sense extremes for the differential inclusion (1.1). These systems will be called auxiliary Barabanov's systems. In Section 4, we give the expressions of the auxiliary Barabanov's systems as explicit function of the data (A, B, C) and the positive parameter r. Section 5 presents some examples for which we calculate the explicit forms of the auxiliary Barabanov's systems.

2. The Barabanov's Theorem

In this section, because of the importance for our work, we present a result obtained by Barabanov and published in [1].

To each vector $z \in \mathbb{R}^2$, we associate a convex, closed, and bounded set $F(z)$ of vectors of $\mathbb{R}^2$, so that $F(z)$ depends continuously of z, $0 \notin F(z)$ for $z \neq 0$, $F(0) = \{0\}$.

If $z \neq 0$, we denote by $P^+(z)$ the closed half plane with boundary line $L = \{\alpha z / \alpha \in \mathbb{R}\}$ and such that it contains the counterclockwise rotations of the radius vector Oz with angle less than π. Also, let be $P^-(z)$ the closure of $\mathbb{R}^2 \setminus P^+(z)$.

We introduce the following notations:

$$F^+(z) = F(z) \cap P^+(z), \quad D^+ = \{z \in \mathbb{R}^2 \,/\, F^+(z) \neq \varnothing\},$$

$$F^-(z) = F(z) \cap P^-(z), \quad D^- = \{z \in \mathbb{R}^2 \,/\, F^-(z) \neq \varnothing\}. \tag{2.1}$$

It is easy to see that

$$D^+ = \{z \in \mathbb{R}^2 \,/\, f_2 z_1 - f_1 z_2 > 0, \;\; f = (f_1, f_2) \in F(z)\},$$

$$D^- = \{z \in \mathbb{R}^2 \,/\, f_2 z_1 - f_1 z_2 < 0, \;\; f = (f_1, f_2) \in F(z)\}. \tag{2.2}$$

Define on D^+ and D^-, respectively, the applications

$$f^+(z) = \arg\max_{f \in F^+(z)} \frac{(f, z)}{\|f\|},$$

$$f^-(z) = \arg\max_{f \in F^-(z)} \frac{(f, z)}{\|f\|}, \tag{2.3}$$

and we consider the differential inclusion

$$\dot{z} \in F(z), \tag{2.4}$$

and the differential equation systems

$$\dot{z} = f^+(z),$$

$$\dot{z} = f^-(z). \tag{2.5}$$

It is assumed that $F(z)$ satisfies the following conditions:

(a) There exists $\delta > 0$ such that, for every $z \neq 0$ and $f \in F(z)$, there exist $\mu_1 \geq 0$, $\mu_2 \geq 0$, such that $\delta < \mu_1 + \mu_2 < \delta^{-1}$ and $f = \mu_1 f^+(z) - \mu_2 z$ or $f = \mu_1 f^+(z) - \mu_2 z$.

(b) There exists $\delta > 0$ such that, for every $z \neq 0$, $\lambda > 0$, and $f \in F(z)$, can be found $\mu \in \mathbb{R}$ and $f_1 \in F(\lambda z)$, such that $\delta < \mu < \delta^{-1}$ and $f_1 = \lambda \mu f$.

The condition (a) means that the set $F(z)$ is contained in the union of two closed cones with border rays $\{\lambda f^+(z)\}$, $\{-\lambda z\}$ and $\{\lambda f^-(z)\}$, $\{-\lambda z\}$; with $\lambda > 0$. The condition (b) is a certain generalization of a positive homogeneity property of the function F. When this is true, the global asymptotic stability of inclusion (2.4) is equivalent to convergence to the origin of coordinates of all solutions.

Under the conditions imposed on $F(z)$ is valid the following result:

Theorem 1. *For global asymptotic stability of inclusion* (2.4) *is necessary and sufficient that the systems* (2.5) *be globally asymptotically stable.*

Proof. For the proof, see [1]. □

3. Application of the Barabanov's Theorem

In the following, we consider for matrices $M \in \mathbb{R}^{l\times q}$ the Frobenius norm, which will be denoted by $\|M\|_F$, and for vectors m the 2-norm, which will be denoted by $\|m\|$. Furthermore, we will use the notation $M_{i\star}$, $M_{\star i}$, m_{ij}, respectively, for the i-th row, the i-th column, and the elements of the matrix M.

In the work [10], it is presented the following formula for the computation of the stability radius $r_{\mathbb{R}}^-(A, B, C)$, when the size of the perturbation is measured by the Frobenius norm, for arbitrary $(A, B, C) \in L_{2,l,q}(\mathbb{R})$,

$$r_{\mathbb{R}}^-(A, B, C) = \min\left\{\frac{-tr\,A}{\|CB\|_F}, \frac{\sqrt{2}\det A}{\sqrt{\|E\|_F^2 + \sqrt{\|E\|_F^4 - 4\mu\det^2 A}}}\right\}, \tag{3.1}$$

where $E = B^T\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}\end{bmatrix}C^T$, $\mu = \det(BB^T)\det(C^TC)$, and so we can calculate this bound without difficulties. For the determination of the

number $r^-_{\mathbb{R},t}(A, B, C)$, we must investigate the asymptotic stability of the differential inclusion $\sum_r$ for $r \in (0, r^-_{\mathbb{R}}(A, B, C))$. From the definition of the set $F_r(x)$ in (1.1), and from the inclusion $r \in (0, r^-_{\mathbb{R}}(A, B, C))$, there are deduced immediately, following the ideas of Barabanov, the properties for $F_r(x)$ (see, for example, [6]):

(1) $F_r(x)$ is a convex, closed, and bounded subset of the plane for all $x \in \mathbb{R}^2$;

(2) $F_r(x)$ depends linearly on x;

(3) $F_r(0) = 0$, $0 \notin F_r(x)$ if $x \neq \mathbf{0}$;

(4) $F_r(\lambda x) = \lambda F_r(x)$, if $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^2$;

(5) $\lambda x \notin F_r(x)$, for all $x \in \mathbb{R}^2$, $x \neq \mathbf{0}$ and $\lambda \geq 0$.

We now define for each $x \in \mathbb{R}^2$, the sets of points of $\mathbb{R}^2$

$$F^+_r(x) = \{f = (f_1, f_2)^T \in F_r(x) : x_2 f_1 - x_1 f_2 < 0\},$$

$$F^-_r(x) = \{f = (f_1, f_2)^T \in F_r(x) : x_2 f_1 - x_1 f_2 > 0\},$$

and consider for each $x \in \mathbb{R}^2$, the optimization problems

$$\begin{cases} \text{maximize} & \dfrac{\langle f, x\rangle}{\|f\|} = \dfrac{f_1 x_1 + f_2 x_2}{\sqrt{f_1^2 + f_2^2}}, \\ \text{subject to:} & f \in F^+_r(x); \end{cases} \tag{3.2}$$

and

$$\begin{cases} \text{maximize} & \dfrac{\langle f, x\rangle}{\|f\|} = \dfrac{f_1 x_1 + f_2 x_2}{\sqrt{f_1^2 + f_2^2}}, \\ \text{subject to:} & f \in F^-_r(x). \end{cases} \tag{3.3}$$

Note that the vector starting in x and directed to the origin of coordinates does not belongs to the sets $F_r^+(x)$ or $F_r^-(x)$.

In the following, we denote by $f_r^+(x)$ and $f_r^-(x)$ the solutions of the problems (3.2) and (3.3), respectively, and consider the differential equation systems

$$\dot{x} = f_r^+(x), \tag{3.4}$$

$$\dot{x} = f_r^-(x), \tag{3.5}$$

which are defined, respectively, on the regions

$$D_r^+ = \{x \in \mathbb{R}^2 : F_r^+ \neq \emptyset\},$$

$$D_r^- = \{x \in \mathbb{R}^2 : F_r^- \neq \emptyset\}.$$

The properties (1)-(5) mentioned above for $F_r(x)$ allow us to apply the theorem of Barabanov. Due to this result, we can guarantee the asymptotic stability for the differential inclusion $\sum_r$ if and only if the corresponding systems (3.4) and (3.5) are asymptotically stable (a.s.). Hence taking into account, the definition of the number $r_{\mathbb{R},t}^-(A, B, C)$, and the fact that $r_{\mathbb{R},t}^-(A, B, C) \leq r_{\mathbb{R}}^-(A, B, C)$, we conclude that

$$r_{\mathbb{R},t}^-(A, B, C) = \inf\{r \in (0, r_{\mathbb{R}}^-(A, B, C)] : \text{at least one of the (3.6)}$$

$$\text{systems (3.4) or (3.5) is not a.s.}\}.$$

In the following, we will refer to the systems (3.4) and (3.5) as Barabanov's auxiliary systems associated with the triple (A, B, C). In the next section, our goal is to obtain the explicit expressions of the Barabanov's auxiliary systems in terms of the data (A, B, C).

4. Expressions for the Barabanov's Auxiliary Systems

In this section, we solve the optimization problem (3.2) in order to obtain the explicit expression of the Barabanov's system (3.4). By analogy, we obtain also the corresponding expression for the other Barabanov's system (3.5).

For a given $x \in \mathbb{R}^2$, let $f = (f_1, f_2)^T \in F_r^+(x)$. Then according to (1.1), we have that

$$\begin{cases} f_1 = A_{1\star}x + B_{1\star}\sum_{i=1}^{q} C_{i\star}x\Delta_{\star i}, \\ f_2 = A_{2\star}x + B_{2\star}\sum_{i=1}^{q} C_{i\star}x\Delta_{\star i}, \end{cases} \tag{4.1}$$

i.e., $f_r^+(x)$ has components given by expressions (4.1), with matrix $\Delta \in \mathbb{R}^{l\times q}$, which is a solution of the extreme problem

$$\begin{cases} \text{maximize} & \dfrac{\langle f, x\rangle}{\|f\|} = \dfrac{f_1x_1 + f_2x_2}{\sqrt{f_1^2 + f_2^2}}, \\ \text{subject to:} & \|\Delta\|_F^2 = \sum_{i=1}^{l}\sum_{j=1}^{q}\delta_{ij}^2 \le r^2, \\ & x_2f_1 - x_1f_2 < 0. \end{cases} \tag{4.2}$$

Lemma 2. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R}) := \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times l} \times \mathbb{R}^{q\times 2}$, *where* A *is a stable matrix,* $B \neq \mathbf{0}$, $C \neq \mathbf{0}$, $r \in (0, r_{\mathbb{R}}^-(A, B, C))$ *and* $x \in D_r^+$. *For the optimization problem* (4.2), *there exists an optimal solution* $\hat{\Delta} \in \mathbb{R}^{l\times q}$. *Furthermore,* $\Delta = 0$ *is not an optimal solution for this problem.*

Proof. Let $\widetilde{f} = (\widetilde{f}_1, \widetilde{f}_2) \in F_r(x)$ such that $x_2\widetilde{f}_1 - x_1\widetilde{f}_2 = 0$. Then, it is easy to see that the angle (positive) from vector x to vector $\widetilde{f}$, that we

denote by $\angle(x, \widetilde{f})$, is π. On the other hand, we can write, for each vector $f \in F_r^+(x)$,

$$\frac{\langle f, x\rangle}{\|f\|} = \|x\| \cos\theta,$$

$$\theta = \angle(x, f), \quad 0 \le \theta \le \pi.$$

From above, we have

$$\frac{\langle f, x\rangle}{\|f\|} \ge \frac{\langle \widetilde{f}, x\rangle}{\|\widetilde{f}\|},$$

for all $f \in F_r^+(x)$. This proves that we can substitute the second restriction of the problem (4.2) by the inequality $x_2 f_1 - x_1 f_2 \le 0$, obtaining no change in the solution of the problem, since we consider $x \ne \mathbf{0}$ with $F_r^+(x) \ne \emptyset$; and by the geometric meaning of the solution of problem (4.2) that vector cannot be directed to the origin. Then by the continuity of the objective function and the compacity of the restriction set, we conclude that for the problem (4.2), there exists an optimal solution $\hat{\Delta} \in \mathbb{R}^{l\times q}$.

Derivating the objective function and evaluating for $\Delta = \mathbf{0}$, it is obtained that

$$sign \frac{\partial}{\partial \delta_{ij}} \frac{\langle f, x\rangle}{\|f\|}\bigg|_{\Delta=\mathbf{0}} = - sign\ [C_{j\star}x(b_{2i}A_{1\star}x - b_{1i}A_{2\star}x)].$$

Thus for $x \in D_r^+$, there exist $i \in \{\overline{1, l}\}$, $j \in \{\overline{1, q}\}$, such that

$$\frac{\partial}{\partial \delta_{ij}} \frac{\langle f, x\rangle}{\|f\|}\bigg|_{\Delta=\mathbf{0}} \ne 0,$$

because the point $x \in D_r^+$ cannot belong to all the lines $C_{j\star}x = 0$, $j = \overline{1, q}$, and $b_{2i}A_{1\star}x - b_{1i}A_{2\star}x = 0$, $i = \overline{1, l}$. Hence the matrix $\Delta = \mathbf{0}$,

which is in the interior of the restriction set, does not satisfy the necessary condition of extremum and so it is not a solution of the problem (4.2). □

The problem (4.2) is a non-linear programming problem with variables δ_{ij}, $i \in \{\overline{1, l}\}$, $j \in \{\overline{1, q}\}$. The corresponding auxiliary Lagrange function is

$$\mathcal{L}(\Delta, \lambda_0, \lambda) = \lambda_0 \frac{f_1 x_1 + f_2 x_2}{\sqrt{f_1^2 + f_2^2}} + \lambda \left(\sum_{i=1}^{l} \sum_{j=1}^{q} \delta_{ij}^2 - r^2 \right).$$

Let $x \in D_r^+$ and let be $\hat{\Delta} = \left(\hat{\delta}_{ij}\right)_{i \in \{\overline{1, l}\},\, j \in \{\overline{1, q}\}}$ a maximum point of the problem (4.2). Then, there exist Lagrange factors λ_0, λ, not both zero, such that

(i) $\lambda_0 \leq 0$, $\lambda \geq 0$;

(ii) $\mathcal{L}_\Delta \left(\hat{\Delta}, \lambda_0, \lambda\right) = 0$;

(iii) $\lambda \left(\sum_{i=1}^{l} \sum_{j=1}^{q} \hat{\delta}_{ij}^2 - r^2 \right) = 0.$

Writing (ii) in an explicit way, we obtain

$$\lambda_0 \frac{C_{j\star} x}{(f_1^2 + f_2^2)^{3/2}} (x_2 f_1 - x_1 f_2)(b_{2i} f_1 - b_{1i} f_2) + 2\lambda \delta_{ij} = 0, \qquad (4.3)$$

$$i = \overline{1, l},\ j = \overline{1, q},$$

so, $\hat{\delta}_{ij}$, $i = \overline{1, l}$, $j = \overline{1, q}$, is a solution of the system (4.3).

Note that if $\lambda_0 = 0$, then $\hat{\delta}_{ij} = 0$, $i = \overline{1, l}$, $j = \overline{1, q}$, is the unique solution of the system (4.3), which according to Lemma 2 is not a solution of the problem (4.2). So in the following, we can take $\lambda_0 = -1$.

Note also that we can consider that the elements of the first row of the matrix C are not all equal to zero. If this row is null, then we can eliminate it of C and also the first column of the disturbance matrix Δ without changes in the expression of f_1 and f_2.

Let $\Delta = (\delta_{ij})_{i\in\{\overline{1,l}\},\, j\in\{\overline{1,q}\}}$ be a solution of the system (4.2), then we have that

$$\frac{\delta_{ij}}{\delta_{i1}} = \frac{C_{j\star}x}{C_{1\star}x},$$

for $i \in \{\overline{1, l}\}$, $j \in \{\overline{1, q}\}$, and x not belonging to the line $C_{1\star}x = 0$. Hence

$$\Delta_{\star j} = \frac{C_{j\star}x}{C_{1\star}x}\Delta_{\star 1}, \quad j = 1, \ldots, q, \tag{4.4}$$

for x out of the line $C_{1\star}x = 0$, and so the solution Δ of the system (4.3) has the form

$$\Delta = \left(\Delta_{\star 1}, \frac{C_{2\star}x}{C_{1\star}x}\Delta_{\star 1}, \ldots, \frac{C_{q\star}x}{C_{1\star}x}\Delta_{\star 1}\right). \tag{4.5}$$

The expression (4.5) shows that in order to calculate the solution Δ of the system (4.3), it is sufficient to solve the subsystem of (4.3) with respect to the components of the first column $\Delta_{\star 1}$, which can be written as

$$b_{2i}f_1 - b_{1i}f_2 + \Lambda\delta_{i1} = 0, \quad i = \overline{1, l}, \tag{4.6}$$

where

$$\Lambda = \frac{-2\lambda(f_1^2 + f_2^2)^{3/2}}{C_{1\star}x(x_2f_1 - x_1f_2)}. \tag{4.7}$$

On the other hand, according to (4.1) and (4.5), the vector $f = (A + B\Delta C)x \in F_r^+(x)$ takes the form

$$f = Ax + \varphi(x)B\Delta_{\star 1}, \tag{4.8}$$

where

$$\varphi(x) = \frac{\|Cx\|^2}{C_{1\star}x}. \tag{4.9}$$

If we put now in the system (4.6) the components f_1 and f_2 of the vector f given by the expression (4.8), we obtain

$$b_{2i}A_{1\star}x - b_{1i}A_{2\star}x + \varphi(x)\left[b_{2i}\sum_{k=1}^{l} b_{1k}\delta_{k1} - b_{1i}\sum_{k=1}^{l} b_{1k}\delta_{k1}\right] + \Lambda\delta_{i1} = 0. \tag{4.10}$$

The following notations:

$$\begin{cases} \gamma_i(x) := b_{2i}A_{1\star}x - b_{1i}A_{2\star}x, \quad i = \overline{1, l}, \\ \gamma(x) = (\gamma_1(x), \ldots, \gamma_l(x))^T := A_{1\star}xB_{2\star}^T - A_{2\star}xB_{1\star}^T, \\ \widetilde{B} = \left(\widetilde{b}_{ij}\right) := (\det[B_{\star i}, B_{\star j}]), \quad i, j = \overline{1, l}, \end{cases} \tag{4.11}$$

allow us to write the system (4.10) in the form

$$\left[\Lambda I_l - \varphi(x)\widetilde{B}\right]\Delta_{\star 1} = -\gamma(x). \tag{4.12}$$

It is clear that $\gamma(x) = \mathbf{0}$ only if $x = \mathbf{0}$ or on the straight line $A_{1\star}xB_{2\star}^T - A_{2\star}xB_{1\star}^T = \mathbf{0}$, and the last case may take place only if $rk(B) = 1$. For the rest of points of the phase plane is $\|\gamma(x)\| > 0$.

Note that Λ given by (4.7) is a real function of the entries δ_{ij} of the matrix Δ and of the Lagrange multiplier $\lambda \geq 0$. So to find the solutions of the system (4.12) for all $\lambda \geq 0$, we can consider Λ as a number (no depending on δ_{ij} and λ) equal to zero or of the same sign than the number $C_{1\star}x$.

From the application of the necessary condition to the optimization problem (4.2) given above, we conclude that the following lemma holds:

Lemma 3. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where* A *is a stable matrix,* $B \neq \mathbf{0}$, $C \neq \mathbf{0}$, $r \in (0, r_{\mathbb{R}}^{-}(A, B, C))$ *and* $x \in D_r^+$. *If* $\hat{\Delta}$ *is a point of maximum of the problem* (4.2), *then it is given by expression* (4.5), *where* $\hat{\Delta}_{\star 1}$ *is a solution of the system* (4.12) *for some number* Λ *such that* $\Lambda = 0$ *or sign* $\Lambda = sign C_{1\star} x$. *If* $\Lambda \neq 0$, *then*

$$\| \hat{\Delta} \|_F^2 = r^2.$$

Furthermore, the vector $f = (A + B\hat{\Delta}C)x \in F_r^+(x)$ *is obtained substituting* $\hat{\Delta}_{\star 1}$ *in expression* (4.8).

In the following, we state some auxiliary lemmas in order to investigate the algebraic linear system (4.12) for $\Lambda = 0$ and Λ such that $\operatorname{sign} \Lambda = \operatorname{sign} C_{1\star} x$.

Lemma 4. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where A is a stable matrix. With the notations* (4.9) *and* (4.11), *it is true that*

(1) $\tilde{B}^2 B^T = -\mu_1 B^T$;

(2) $\tilde{B}^2 \gamma(x) = -\mu_1 \gamma(x)$;

(3) $B\tilde{B}B^T = \mu_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$;

(4) $\det \left[\Lambda I_l - \varphi(x)\tilde{B}\right] = \Lambda^{l-2}[\Lambda^2 + \mu_1 \varphi^2(x)]$,

where

$$\mu_1 = \sum_{\substack{i=1,\,\ldots,\,l \\ i<j}} \sum_{j=1}^{l} \tilde{b}_{ij}^2 = \det(BB^T). \tag{4.13}$$

Proof. The assertions are a direct consequence of the definitions and straightforward calculations, which had been done in [10]. □

Lemma 5. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where* A *is a stable matrix,* $x \in D_r^+$ *and* $r \in (0, r_{\mathbb{R}}^-(A, B, C))$. *The system* (4.12) *with unknown* $\Delta_{\star 1}$ *in the case* $\Lambda = 0$ *is compatible if and only if* $\mu_1 \neq 0$ *and then the solution set is*

$$\mathcal{S} = \left\{ \Delta_{\star 1} = -\frac{1}{\mu_1 \varphi(x)} \widetilde{B}\gamma(x) + \gamma_0 \,:\, \gamma_0 \in \ker \widetilde{B} \right\}. \tag{4.14}$$

Furthermore,

$$\|\Delta\|_F \geq r_{\mathbb{R}}^-(A, B, C), \tag{4.15}$$

for all Δ *given by expression* (4.5) *with* $\Delta_{\star 1} \in \mathcal{S}$.

Proof. In this case, the linear system (4.12) is

$$\varphi(x)\widetilde{B}\Delta_{\star 1} = \gamma(x), \tag{4.16}$$

and matrix $\varphi(x)\widetilde{B}$ has, according to the assertion (4) of Lemma 4, determinant equal zero.

If $\mu_1 = 0$, then by definition (4.13) is $\widetilde{B} = \mathbf{0}$ and the system (4.16) is incompatible because is $\gamma(x) \neq \mathbf{0}$ for $x \in D_r^+$.

If $\mu_1 \neq 0$, then applying the assertion (2) of Lemma 4, we obtain

$$\varphi(x)\widetilde{B}\left[\widetilde{B}\gamma(x)\right] = \varphi(x)\widetilde{B}^2\gamma(x) = -\mu_1\varphi(x)\gamma(x),$$

and so the system (4.16) is compatible because

$$\Delta_{\star 1} = -\frac{1}{\mu_1 \varphi(x)} \widetilde{B}\gamma(x),$$

it is a particular solution, which implies that the solution set of this system is given by (4.14).

On the other hand, if we put $\Delta_{\star 1} \in \mathcal{S}$ in expression (4.8) and taking into account (4.5), we obtain

$$f = Ax + \varphi(x)B\left[-\frac{\widetilde{B}\gamma(x)}{\mu_1\varphi(x)} + \gamma_0\right], \tag{4.17}$$

consequently, when we calculate $\|\Delta\|_F^2$ as

$$\|\Delta\|_F^2 = \sum_{j=1}^{q}\frac{(C_{j\star}x)^2}{(C_{1\star}x)^2}\|\Delta_{\star 1}\|^2 = \frac{\|Cx\|^2}{(C_{1\star}x)^2}\left\{\left\|\frac{\widetilde{B}\gamma(x)}{\mu_1\varphi(x)}\right\|^2 + \|\gamma_0\|^2\right\}. \tag{4.18}$$

But from assertion (1) of Lemma 4, it follows that $B\widetilde{B}^2 = -\mu_1 B$ and so, as $\gamma_0 \in \ker \widetilde{B}$, it is obtained that

$$B\gamma_0 = -\frac{1}{\mu_1}B\widetilde{B}^2\gamma_0 = 0, \tag{4.19}$$

and

$$B\widetilde{B}\gamma(x) = B\widetilde{B}[A_{1\star}xB_{2\star}^T - A_{2\star}xB_{1\star}^T] = A_{1\star}xB\widetilde{B}B_{2\star}^T - A_{2\star}xB\widetilde{B}B_{1\star}^T.$$

Hence, by making use of assertion (3) of the Lemma 4, it is obtained

$$B\widetilde{B}\gamma(x) = \mu_1 A_{1\star}x\begin{bmatrix}1\\ \\0\end{bmatrix} - \mu_1 A_{2\star}x\begin{bmatrix}0\\ \\-1\end{bmatrix} = \mu_1 Ax. \tag{4.20}$$

The substitution of the expressions (4.19) and (4.20) in (4.17) give us

$$f = \mathbf{0}. \tag{4.21}$$

The equality (4.21) implies that for the matrix Δ given by expression (4.5) with

$$\Delta_{\star 1} = -\frac{1}{\mu_1\varphi(x)}\widetilde{B}\gamma(x) + \gamma_0,$$

the corresponding system

$$\dot{y} = (A + B\Delta C)y,$$

is not a.s. and so by the definition (1.3) is $\|\Delta\|_F \geq r_{\mathbb{R}}^-(A, B, C)$, which proves the statement (4.15) of the lemma. □

From the above lemma, the case $\Lambda = 0$ does not give us a possible solution Δ of the extreme problem (4.2). However, we can use it to obtain an inequality, valid in general for each triple $(A, B, C) \in L_{2,l,q}(\mathbb{R})$.

Corollary 6. *For each triple* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, $r \in (0, r_{\mathbb{R}}^{-}(A, B, C))$ *and each* x *such that* $\gamma(x) \neq \mathbf{0}$, *we have that*

$$\|\gamma(x)\|^2 - r^2 \mu_1 \|Cx\|^2 > 0.$$

Proof. If $rk(B) = 1$, then $\mu_1 = 0$ and the inequality to prove is evident. If $rk(B) = 2$ is $\mu_1 \neq 0$. Let Δ be the solution of the system (4.16). If we use the expression (4.18) for $\gamma_0 = 0$ and the inequality (4.15), taking into account the assertion (1) of Lemma 4, the definition (4.9) of $\varphi(x)$, and the expression (4.11) for matrix $\widetilde{B}$, we obtain

$$\|\Delta\|_F^2 = \frac{\|Cx\|^2}{(C_{1\star}x)^2} \frac{\gamma^T(x)\widetilde{B}^2\gamma(x)}{\mu_1^2\varphi^2(x)} = \frac{\gamma^T(x)\gamma(x)}{\mu_1\|Cx\|^2}$$

$$\geq [r_{\mathbb{R}}^{-}(A, B, C)]^2 > r^2,$$

which proves the statement of the corollary. □

Lemma 7. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where* A *is a stable matrix,* $B \neq \mathbf{0}$, $C \neq \mathbf{0}$, $x \in D_r^+$, $r \in (0, r_{\mathbb{R}}^{-}(A, B, C))$. *For each number* Λ, *such that* $sign\,\Lambda = sign C_{1\star}x$, *the system* (4.12) *has a unique solution*

$$\Delta_{\star 1} = -\frac{\left[\Lambda I_l + \varphi(x)\widetilde{B}\right]\gamma(x)}{\Lambda^2 + \mu_1\varphi^2(x)}, \tag{4.22}$$

which gives according to the expression (4.5), *a matrix* Δ *that satisfies the stationary condition of the optimization problem* (4.2). *For such matrix* Δ, *it holds that*

$$\|\Delta\|_F^2 = r^2, \tag{4.23}$$

only in the case when

$$\Lambda = \frac{\|Cx\|}{rC_{1\star}x}\sqrt{\|\gamma(x)\|^2 - r^2\mu_1\|Cx\|^2}. \tag{4.24}$$

Proof. As $\Lambda \neq 0$, the assertion (4) of Lemma 4 shows that

$$\det\left[\Lambda I_l - \varphi(x)\widetilde{B}\right] \neq 0,$$

and so the solution of the system (4.12) is unique.

On the other hand, we have

$$\left[\Lambda I_l - \varphi(x)\widetilde{B}\right]\left[\Lambda I_l + \varphi(x)\widetilde{B}\right]\gamma(x) = \left[\Lambda^2 I_l - \varphi^2(x)\widetilde{B}^2\right]\gamma(x)$$

$$= \left[\Lambda^2 + \mu_1\varphi^2(x)\right]\gamma(x),$$

equality that shows that the solution $\Delta_{\star 1}$ of the system (4.12) is given by expression (4.22).

If we put this expression of $\Delta_{\star 1}$ in (4.5), we obtain a matrix Δ such that

$$\|\Delta\|_F^2 = \sum_{j=1}^{q}\frac{(C_{j\star}x)^2}{(C_{1\star}x)^2}\|\Delta_{\star 1}\|^2$$

$$= \frac{\|Cx\|^2}{(C_{1\star}x)^2}\frac{\gamma^T(x)\left[\Lambda I_l - \varphi(x)\widetilde{B}\right]\left[\Lambda I_l + \varphi(x)\widetilde{B}\right]\gamma(x)}{\left[\Lambda^2 + \mu_1\varphi^2(x)\right]^2}$$

$$= \frac{\|Cx\|^2}{(C_{1\star}x)^2}\frac{\gamma^T(x)\gamma(x)}{\Lambda^2 + \mu_1\varphi^2(x)},$$

from what, taking into account the statements of the Corollary 6, it follows that the matrix Δ satisfies (4.23) if and only if Λ is given by the expression (4.24). □

Theorem 8. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where* A *is a stable matrix,* $B \neq \mathbf{0}$, $C \neq \mathbf{0}$, $r \in (0, r_{\mathbb{R}}^{-}(A, B, C))$. *For each point* $x \in D_r^+$ *such that* $\gamma(x) \neq \mathbf{0}$, *the vector* $f_r^+(x)$, *which corresponds to the maximum point of the problem* (4.2) *has the expression*

$$f_r^+(x) = \frac{\Lambda^2}{\Lambda^2 + \mu_1 \varphi^2(x)} [Ax - \alpha(x, r) B\gamma(x)], \tag{4.25}$$

where

$$\alpha(x, r) = \frac{\varphi(x)}{\Lambda} = \frac{r\|Cx\|}{\sqrt{\|\gamma(x)\|^2 - r^2 \mu_1 \|Cx\|^2}}, \tag{4.26}$$

and Λ *is given by* (4.24).

Proof. Lemmas 5 and 7 show that there is a unique value of Λ given by (4.24) such that the corresponding system (4.12) has a solution that satisfies the restrictions of the problem (4.2). For such Λ, the solution $\Delta_{\star 1}$ of the system (4.12) is unique (it is given by (4.22)) and so according to Lemma 3, there is a unique possible point of maximum for the problem (4.2). But Lemma 2 states that the problem (4.2) has a maximum point and so the unique maximum point $\hat{\Delta}$ is obtained putting in (4.5) the solution $\Delta_{\star 1}$ of (4.12). Only rest to put $\Delta_{\star 1}$ in the expression of f given by (4.8) to obtain the sought vector $f_r^+(x)$. □

By analogy, we obtain the expression of the function $f_r^-(x)$. It is given in the theorem below:

Theorem 9. *Let* $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, *where* A *is a stable matrix,* $B \neq \mathbf{0}$, $C \neq \mathbf{0}$, $r \in (0, r_{\mathbb{R}}^{-}(A, B, C))$. *For each point* $x \in D_r^-$ *such that* $\gamma(x) \neq \mathbf{0}$, *the vector* $f_r^-(x)$, *which gives the solution of the problem* (4.2), *in the case when* $f = (f_1, f_2)^T \in F_r^-(x)$, *has the expression*

$$f_r^-(x) = \frac{\Lambda^2}{\Lambda^2 + \mu_1 \varphi^2(x)} [Ax + \alpha(x, r) B\gamma(x)], \tag{4.27}$$

where $\alpha(x, r)$ *is given by* (4.26) *and* Λ *by* (4.24).

The expressions (4.25) and (4.27) have sense for all $r \in (0, r_{\mathbb{R}}^-(A, B, C))$ in each point of the phase plane, although $x \in D_r^+$ or $x \in D_r^-$.

Now, we define the vector functions

$$g_r^+(x) = Ax - \alpha(x, r) B\gamma(x), \tag{4.28}$$

$$g_r^-(x) = Ax + \alpha(x, r) B\gamma(x), \tag{4.29}$$

and consider the systems

$$\dot{x} = g_r^+(x), \tag{4.30}$$

$$\dot{x} = g_r^-(x). \tag{4.31}$$

From the expressions (3.6), (4.25), and (4.27) and the definitions (4.28), (4.29) of the functions $g_r^+(x)$ and $g_r^-(x)$, taking into account that the systems (3.4) and (3.5) are a.s. if and only if the systems (4.30) and (4.31) have this property, we conclude that

$$r_{\mathbb{R},t}^-(A, B, C) = \inf\{r \in (0, r_{\mathbb{R}}^-(A, B, C)]: \text{ at least one of the systems (4.30) or (4.31) is not a.s.}\}. \tag{4.32}$$

Remark. In order to study the stability properties of the family of systems of differential equations (1.1), via the number $r_{\mathbb{R},t}^-(A, B, C)$ defined by (4.32), it is sufficient to study the stability properties of the systems (4.30) and (4.31), and we can obtain the analytic expressions for these systems: the Barabanov's auxiliary systems.

5. Examples

In this section, we give some triples $(A, B, C) \in L_{2,l,q}(\mathbb{R})$, and for them, we obtain the corresponding Barabanov's auxiliary systems.

Example 10.

$$A = \begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \end{bmatrix};\ C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For this triple $\mu_1 = 5$; the vector $(Cx)^T = [2x_1 + x_2 \quad x_1 - x_2]$; the vector function $\gamma^T(x) = [-5x_1 - 2x_2 \ \ 2x_1 - 4x_1]$.

The vector $(B\gamma(x))^T = [-20x_1 - 5x_1 - 2x_2]$. So, the corresponding Barabanov's auxiliary systems (4.30) and (4.31), respectively, are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 10\,\alpha(x, r) - 5 & -2 \\ 5\,\alpha(x, r) + 2 & 2\alpha(x, r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10\,\alpha(x, r) - 5 & -2 \\ -5\,\alpha(x, r) + 2 & -2\alpha(x, r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

with $\alpha(x, r) = r\sqrt{\dfrac{5x_1^2 + 4x_1x_2 + 2x_2^2}{20x_1^2 + 20x_1x_2 + 4x_2^2 - 5r^2(5x_1^2 + 4x_1x_2 + 2x_2^2)}}$.

Example 11.

$$A = \begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix};\ B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix};\ C = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

For this triple $\mu_1 = 9$; the vector $(Cx)^T = [x_1 + x_2 - x_1 - x_2 \ \ x_1]$; the vector function $\gamma^T(x) = [3x_1 + 2x_2 - 6x_1]$; the vector $(B\gamma(x))^T = [3x_1 + 2x_2 - 18x_1]$. So, the corresponding Barabanov's auxiliary systems (4.30) and (4.31), respectively, are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3\alpha(x, r) - 2 & -3\alpha(x, r) \\ 18\alpha(x, r) - 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3\alpha(x, r) - 2 & 3\alpha(x, r) \\ -18\alpha(x, r) - 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

with $\alpha(x, r) = r\sqrt{\dfrac{3x_1^2 + 4x_1x_2 + 2x_2^2}{45x_1^2 + 12x_1x_2 + 4x_2^2 - 9r^2(3x_1^2 + 4x_1x_2 + 2x_2^2)}}$.

Example 12.

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix};\ B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

For this triple $\mu_1 = 3$; the vector $(Cx)^T = [x_1\ \ 2x_2]$; the vector function $\gamma^T(x) = [\,-x_1 + x_2 - x_2\ \ x_1\,]$; the vector $(B\gamma(x))^T = [\,-x_1 + 2x_2 - x_1 - x_2\,]$. So, the corresponding Barabanov's auxiliary systems (4.30) and (4.31), respectively, are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \alpha(x, r) - 1 & -2\alpha(x, r) \\ \alpha(x, r) + 1 & \alpha(x, r) - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha(x, r) - 1 & 2\alpha(x, r) \\ -\alpha(x, r) + 1 & -\alpha(x, r) - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};$$

with $\alpha(x, r) = r\sqrt{\dfrac{x_1^2 + 4x_2^2}{2x_1^2 - 2x_1x_2 + 2x_2^2 - 3r^2(x_1^2 + 4x_2^2)}}$.

Conclusion: In each of the above examples, once we have found the expressions for the Barabanov's auxiliary systems, we note that those systems are not linear and that we must study the stability properties of the solution of each in order to obtain the number $r^-_{\mathbb{R},t}(A, B, C)$ and with it a full characterization of the family of systems (1.1) in the case when $n = 2$ and $l, q \in \mathbb{Z}^\star_+$ are arbitrary.

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