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MODIFIED PROXIMAL ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITIES

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Abstract

In this paper, some modified decomposition schemes for solving structured variational inequality problems were proposed. The proposed approaches combine the decomposition algorithm introduced in [9] with other proximal-like regularization. Under certain appropriate assumptions, we show that the method generates convergent sequences.

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1. Introduction

Variational inequality theory, was emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in taxation, unemployment etc. The ideas and techniques of variational inequality are being applied in a variety of diverse areas of sciences and proved to be productive and innovative. It has been shown that this theory provides that most natural, direct, simple, unified, and efficient framework for general treatment of a wide class of unrelated linear and nonlinear problems.

Variational inequality has been extended and generalized in several directions by using novel and new techniques, and has been solved by a number of numerical methods including projection method, auxiliary principle, proximal point method, and decomposition. Historically, in 1964, the variational inequality problem was introduced by Stampacchia [28].

In 1980, the books by Kinderlehrer and Stampacchia [14], and Baiocchi and Capelo [6], provide a thorough introduction to the application of inequalities in infinite-dimensional metric spaces.

The projection method provides important tools for finding the approximate solution of variational inequalities. This method is due to Lions and Stampacchia [16]. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem by using the concept of projection. This alternative formulation has played a significant part in developing various projection-type of the variational inequalities.

It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately, these strict conditions rule out many applications of this method.

This fact motivates the modification of the projection method and the development of other methods. To implement the projection-splitting

method, one has to evaluate the projection, which is itself a difficult problem. These facts motivate the use of the auxiliary principle technique.

This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem.

A popular method for solving variational inequality problems is the proximal point method.

This method is in fact the implicit-type method, which arises in the context of discretization of the initial value problems. Martinet [18] considered this method as a regularization technique for the convex optimization. Rockafellar [27] applied this method for finding a zero of the maximal monotone operators.

2. Bregman Functions

First recall the notion of Bregman functions called also ***D***-functions introduced by Bregman ([7], 1967), developed and used in the proximal theory by ([8, 13, 15, 25, 26] and references therein). Let S be an open subset of $\Re^n$ and $h : \overline{S} \to \Re$ be a finite-valued continuously differentiable function on S and define $\mathbf{D_h}$ by

$$\mathbf{D_h}(\mathbf{x}, \mathbf{y}) := \mathbf{h}(\mathbf{x}) - \mathbf{h}(\mathbf{y}) - \langle \nabla \mathbf{h}(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle, \quad \mathbf{x} \in \overline{\mathbf{S}}, \mathbf{y} \in \mathbf{S}. \tag{1}$$

Definition 1. The function h is called a *Bregman function with zone S* or a *D-function*, if

(a) h is continuously differentiable on S and continuous on $\overline{S}$;

(b) h is strictly convex on $\overline{S}$;

(c) for every $\lambda \in \Re$, the partial level sets $L_1(y, \lambda) = \left\{x \in \overline{S} : \mathbf{D_h}(\mathbf{x}, \mathbf{y}) \le \lambda\right\}$ and $L_2(x, \lambda) = \{y \in S : \mathbf{D_h}(\mathbf{x}, \mathbf{y}) \le \lambda\}$ are bounded for every $y \in S$ and $x \in \overline{S}$, respectively;

(d) if $\{y_k\} \in S$ is a convergent sequence with limit y^*, then

$$\mathbf{D_h}(\mathbf{y}^*, \mathbf{y_k}) \to \mathbf{0};$$

(e) if $\{x^k\}$ and $\{y_k\}$ are sequences such that $y_k \to y^* \in \overline{S}$, $\{x^k\}$ is bounded and $\mathbf{D_h}(\mathbf{x^k}, \mathbf{y_k}) \to \mathbf{0}$, then $x^k \to y^*$.

From the above definition, we have the following:

Lemma 1. *Let h be a Bregman function with zone S. Then*

(i) $\mathbf{D_h}(\mathbf{x}, \mathbf{x}) = \mathbf{0}$ *and* $\mathbf{D_h}(\mathbf{x}, \mathbf{y}) \geq \mathbf{0}$ *for* $x \in \overline{S}$ *and* $y \in S$;

(ii) $\forall a, b \in S$ *and* $c \in \overline{S}$,

$$\mathbf{D_h}(\mathbf{c}, \mathbf{a}) + \mathbf{D_h}(\mathbf{a}, \mathbf{b}) - \mathbf{D_h}(\mathbf{c}, \mathbf{b}) = \langle \nabla \mathbf{h}(\mathbf{b}) - \nabla \mathbf{h}(\mathbf{a}), \mathbf{c} - \mathbf{a} \rangle; \tag{2}$$

(iii) $\forall a, b \in S$,

$$\mathbf{D_h}(\mathbf{a}, \mathbf{b}) + \mathbf{D_h}(\mathbf{b}, \mathbf{a}) \leq \|\nabla \mathbf{h}(\mathbf{a}) - \nabla \mathbf{h}(\mathbf{b})\| \times \|\mathbf{a} - \mathbf{b}\|; \tag{3}$$

(iv) $\forall x \in S$,

$$h^*(\nabla h(x)) = \langle x, \nabla h(x) \rangle - h(x);$$

(v) *let* $\{x^k\} \in S$ *such that* $x^k \to x^* \in S$, *then* $\mathbf{D_h}(\mathbf{x}^*, \mathbf{x^k}) \to \mathbf{0}$ *and* $\mathbf{D_h}(\mathbf{x^k}, \mathbf{x}^*) \to \mathbf{0}$.

Remark 1. Some examples

(1) $S = \Re^n$, $h_1(x) = \frac{1}{2}\|x\|^2$, then $D_{h_1}(x, y) = \frac{1}{2}\|x - y\|^2$.

(2) $S = \Re^n_{++}$, $h_2(x) = \sum_{i=1}^{n} x_i \log(x_i) - x_i$, then for $(x, y) \in \overline{\Re}^n \times \Re^n$ and with $0 \log 0 = 0$,

$$D_{h_2}(x, y) = \sum_{i=1}^{n} x_i \log(x_i / y_i) + y_i - x_i,$$

D_{h_2} is called the *Kullback-Leibler relative entropy distance.*

(3) $S =]-1, 1[^n, \quad h_3(x) = -\sum_{i=1}^{n}\sqrt{1-x_i^2}$,

then

$$D_{h_3}(x, y) = \sum_{i=1}^{n}\frac{1-x_iy_i}{\sqrt{1-y_i^2}} - \sum_{i=1}^{n}\sqrt{1-x_i^2}.$$

(4) $S = \Re^n_{++}$, $h_4(x) = \sum_{i=1}^{n} x_i - \sqrt{x_i}$,

then

$$D_{h_4}(x, y) = \sum_{i=1}^{n}\frac{1}{2\sqrt{y_i}}(\sqrt{x_i} - \sqrt{y_i})^2.$$

(5) $S = \Re^n$, $h_5(x) = \frac{1}{2}\langle x, Ax\rangle$,

then

$$D_{h_5}(x, y) = \frac{1}{2}\|x-y\|_A^2 = \frac{1}{2}\langle x-y, A(x-y)\rangle,$$

where $A \in \Re^{n\times n}$ is symmetric positive definite.

(6) Let $g : \Re^n \to \Re$ be a strictly convex function such that

$$g \in C^2(\Re^n) \text{ and } \lim_{\|x\|\to\infty}\frac{g(x)}{\|x\|} = +\infty,$$

then g is a Bregman function.

(7) If h is a Bregman function, then $h(x) + c^\top x + d$ for any $c \in \Re^n$, $d \in \Re$, also is a Bregman function.

Remark 2. $\mathbf{D_h}(.,.)$ cannot be considered as a distance because of the lack of the triangle inequality and the symmetry property. $\mathbf{D_h}(.,.)$ is usually called an *entropy distance*.

Definition 2. A Bregman function h with zone S^o is said to be zone coercive, if the image of S^o under ∇h is equal to $\mathbb{R}^n$.

We close this section, by recalling some basic results about Bregman functions and distances.

Proposition 1. *For any* $a, b, c \in X$ *and* $c \in \overline{X}$,

$$D_\psi(c, a) - D_\psi(c, b) - D_\psi(b, a) = \langle \nabla\psi(b) - \nabla\psi(a), c - b \rangle. \tag{4}$$

Proof. Using the definition of Bregman distances, we get

$$\begin{aligned} D_\psi(c, a) &- D_\psi(c, b) - D_\psi(b, a) \\ &= \psi(c) - \psi(a) - \langle \nabla\psi(a), c - a \rangle - \psi(c) + \psi(b) \\ &\quad + \langle \nabla\psi(b), c - b \rangle - \psi(b) + \psi(a) + \langle \nabla\psi(a), b - a \rangle \\ &= \langle \nabla\psi(b) - \nabla\psi(a), c - b \rangle. \end{aligned}$$

□

Proposition 2. *If the function* h *is strongly convex of coefficient* α, *then*

$$\nabla^2 h \succeq \alpha I, \tag{5}$$

where I *is the identity matrix, and the notation* $\nabla^2 h \succeq \alpha I$ *means that* $(\nabla^2 h - \alpha I)$ *is positive semi-definite.*

We use this proposition to prove the following:

Proposition 3. *If* $h : X \to \Re$ *is a strongly convex of coefficient* α *Bregman function, and* D_h *is the Bregman distance, then*

$$D_h(x, y) \geq \frac{\alpha}{2}\|x - y\|^2, \quad \forall x, y \in X \subset \Re^n.$$

Proof. Since

$$D_h(x,\ y) = h(x) - h(y) - \langle \nabla h(y),\ x - y \rangle.$$

Using Taylor development,

$$h(x) = h(y) + \langle \nabla h(y),\ x - y \rangle + \langle \nabla^2 h(y)(x - y),\ x - y \rangle + \ldots.$$

Using Proposition 2, we get

$$h(x) - h(y) + \langle \nabla h(y),\ x - y \rangle \geq \langle \nabla^2 h(y)(x - y),\ x - y \rangle \geq \frac{\alpha}{2} \|x - y\|^2,$$

which implies that

$$D_h(x,\ y) = h(x) - h(y) + \langle \nabla h(y),\ x - y \rangle \geq \frac{\alpha}{2} \|x - y\|^2. \qquad \square$$

3. Modified Proximal Techniques for VIPs

In this paper, we consider the following monotone variational inequality problem denoted by (VIP):

Find $x^* \in \Re^n$, such that

$$\langle x - x^*,\ f(x^*) \rangle \geq 0, \quad \forall x \in C, \tag{6}$$

where

$$C = \{x \in \Re^n \mid\ g(x) \geq 0\},$$

is a given nonempty closed convex set, g is a concave function, and $f : C \to \Re^n$, $g : \Re^n \to \Re^m$ are given monotone functions.

It is straightforward to see that the above VIP model can be rephrased as:

Find $u^* \in \Omega$, such that

$$\langle u - u^*,\ F(u^*) \rangle \geq 0, \quad \forall u \in \Omega, \tag{7}$$

where

$$u = \begin{pmatrix} x \\ \\ y \end{pmatrix}, \quad F(u) = \begin{pmatrix} f(x) - \nabla g(x)y \\ \\ g(x) \end{pmatrix}, \tag{8}$$

and $\Omega = C \times \Re^m_+$. From now on, Equation (7)-(8) will be denoted by $VIP(\Omega)$.

To develop our new approach, we will use a version of the auxiliary principle (see Marcotte and Zhu [17], Noor [21], and He et al. in [10, 11, 12] and references therein). The method generates the new pair $\tilde{u} = (\tilde{x}, \tilde{y})$ from $u = (x, y)$ as follows:

$$\langle x' - \tilde{x}, r(\tilde{x} - x) + f(\tilde{x}) - \nabla g(\tilde{x})y \rangle \geq 0, \quad \forall x' \in C, \tag{9}$$

and

$$\langle y' - \tilde{y}, g(\tilde{x}) + \nabla h(\tilde{y}) - \nabla h(y) \rangle \geq 0, \quad \forall y' \in \Re^m_+, \tag{10}$$

where h is a Bregman function, and the first algorithm can be summed up as follows:

Algorithm 1

Step 0. Choose a zone coercive Bregman function h_0, with zone C^0, select a constant $\mu > 0$ and define $h : \Re^n \to \Re$,

$$h(x) = h_0(x) + \frac{\mu}{2}\|x\|^2, \tag{11}$$

and choose an initial point $(x^0, y^0) \in \Omega$, $\alpha > 0$, $s > 0$, $r > 0$ and let $k = 0$.

Step 1. Find $x^{k+1} \in \Re^n$, such that

$$\langle x' - x^{k+1}, r(x^{k+1} - x^k) + f(x^{k+1}) - \nabla g(x^{k+1})y^k \rangle \geq 0, \quad \forall x' \in C. \tag{12}$$

Step 2. Find $y^{k+1} \in Y = \Re_+^m$, such that

$$\langle y' - y^{k+1}, \alpha g(x^{k+1}) + s\big(\nabla h(y^{k+1}) - \nabla h(y^k)\big)\rangle \geq 0, \quad \forall y' \in \Re_+^m. \quad (13)$$

Remark 3. The above algorithm is a classical Usawa method for solving VIPs, but we just mix it with the quadratic proximal point approach with respect to the variable x and a non-quadratic proximal point approach, known as Bregman prox with respect to the variable y. Of course, one may modify again this algorithmic scheme by changing the added proximal terms. We mean that we can present (and may be in the literature it was already done) the following schemes, after assuming that all the initializations are done:

Algorithm 2

Step 1. Find $x^{k+1} \in \Re^n$, such that

$$\langle x' - x^{k+1}, r(\nabla h_1(x^{k+1}) - \nabla h_1(x^k)) + f(x^{k+1}) - \nabla g(x^{k+1})y^k\rangle \geq 0, \quad \forall x' \in C. \quad (14)$$

Step 2. Find $y^{k+1} \in Y = \Re_+^m$, such that

$$\langle y' - y^{k+1}, \alpha g(x^{k+1}) + s(\nabla h_2(y^{k+1}) - \nabla h_2(y^k))\rangle \geq 0, \quad \forall y' \in \Re_+^m. \quad (15)$$

Algorithm 3

Step 1. Find $x^{k+1} \in \Re^n$, such that

$$\langle x' - x^{k+1}, r(\nabla h_1(x^{k+1}) - \nabla h_1(x^k)) + f(x^{k+1}) - \nabla g(x^{k+1})y^k\rangle \geq 0, \quad \forall x' \in C. \quad (16)$$

Step 2. Find $y^{k+1} \in Y = \Re_+^m$, such that

$$\langle y' - y^{k+1}, \alpha g(x^{k+1}) + s(y^{k+1} - y^k)\rangle \geq 0, \quad \forall y' \in \Re_+^m. \quad (17)$$

Algorithm 4

Step 1. Find $x^{k+1} \in \Re^n$, such that

$$\langle x' - x^{k+1}, r(x^{k+1} - x^k) + f(x^{k+1}) - \nabla g(x^{k+1}) y^k \rangle \geq 0, \quad \forall x' \in C. \quad (18)$$

Step 2. Find $y^{k+1} \in Y = \Re^m_+$, such that

$$\langle y' - y^{k+1}, \alpha g(x^{k+1}) + s(y^{k+1} - y^k) \rangle \geq 0, \quad \forall y' \in \Re^m_+ k. \quad (19)$$

Also, we can propose to replace the proximal term (the Bregman one or the classical quadratic one) by the ϕ-divergent prox-like term as it was done by Auslender et al. [5] and recently by other authors, for instance, see the work of Han and Sun [9] and the reader may see the references therein. In the next section, we give the convergence analysis of Algorithm 1. To avoid the repetition of almost similar approaches, in this work, we present only the convergence of Algorithm 1 in detail. Minor modifications are needed to prove the convergence of Algorithms 2, 3, and 4.

4. Convergence Analysis

Definition 3. $F = (F_1, F_2)$ is partially co-coercive on Ω, if there exists $\sigma > 0$ such that

$$\langle F(u) - F(v), u - v \rangle \geq \sigma \|F_2(u) - F_2(v)\|^2, \quad \forall u, v \in \Omega.$$

Lemma 2. *For any* $w, v \in \Re^n$, *and* a, b *are positive constants*, *we have*

$$ab\|v\|^2 + a\langle v, w \rangle \geq -\frac{a}{4b}\|w\|^2. \quad (20)$$

Proof. Easy and can be omitted.

We make the following assumptions:

(A1) C is a nonempty closed convex set, $F = (F_1, F_2)$ is partially co-coercive on Ω.

(A2) The optimal solution set of $VIP(\Omega, F)$ denotes Ω^*, is nonempty.

Lemma 3. *For any* $u^* = (x^*, y^*) \in \Omega^*$, *and* $\{(x^k, y^k)\}_k$ *be the sequence generated Algorithm* 1, *it holds*

$$W \geq \langle x^{k+1} - x^*, f(x^{k+1}) - \nabla g(x^{k+1})y^k - (f(x^*) - \nabla g(x^*)y^*)\rangle, \tag{21}$$

where

$$W = \langle f(x^{k+1}) - \nabla g(x^{k+1})y^k, x^{k+1} - x^k\rangle.$$

Proof. Since x^* is a solution of $VIP(\Omega, F)$ and $x^{k+1} \in C$, from (7), we have

$$\langle x^{k+1} - x^*, f(x^*) - \nabla g(x^*)y^*\rangle \geq 0,$$

implies

$$-\langle x^{k+1} - x^*, f(x^*) - \nabla g(x^*)y^*\rangle \leq 0.$$

Set

$$W = \langle f(x^{k+1}) - \nabla g(x^{k+1})y^k, x^{k+1} - x^k\rangle,$$

hence

$$\begin{aligned} W &\geq W - \langle x^{k+1} - x^*, f(x^*) - \nabla g(x^*)y^*\rangle \\ &\geq \langle x^{k+1} - x^*, f(x^{k+1}) - \nabla g(x^{k+1})y^k - (f(x^*) - \nabla g(x^*)y^*)\rangle, \end{aligned}$$

and then the lemma follows:

Lemma 4. *For any* $u^* \in \Omega^*$, *let* $(x^k, y^k)_k$ *be the sequence generated in Algorithm* 1

$$G \geq \langle y^{k+1} - y^*, g(x^{k+1}) - g(x^*)\rangle, \tag{22}$$

where

$$G = \langle y^{k+1} - y^*, g(x^{k+1})\rangle.$$

Proof. Since $y^* \in \Re_+^m$, then

$$\langle y^{k+1} - y^*, g(x^*)\rangle \geq 0,$$

implies that

$$-\langle y^{k+1} - y^*, g(x^*)\rangle \leq 0,$$

set

$$G = \langle y^{k+1} - y^*, g(x^{k+1})\rangle.$$

Therefore,

$$G \geq G - \langle y^{k+1} - y^*, g(x^*)\rangle \geq \langle y^{k+1} - y^*, g(x^{k+1}) - g(x^*)\rangle,$$

and then the lemma follows.

The following theorem will be used to prove the fundamental result in the convergence analysis of Algorithm 1, it is the main theorem of this section.

Theorem 1. *Let* $u^* = (x^*, y^*) \in \Omega^*$ *be a solution point of* $VIP(\Omega, F)$. *Then*

$$\begin{aligned} d \geq \|x^{k+1} - x^k\|^2 + \frac{s\xi}{r\alpha}\|y^{k+1} - y^k\|^2) \\ + \frac{2}{r}[\langle y^{k+1} - y^*, g(x^{k+1}) - g(x^*)\rangle \\ + \langle x^{k+1} - x^*, f(x^{k+1}) - \nabla g(x^{k+1})y^k - (f(x^*) - \nabla g(x^*)y^*)\rangle, \end{aligned} \tag{23}$$

where

$$d = \frac{2}{rs\alpha}\left\{D_h(y^*, y^k) - D_h(y^*, y^{k+1})\right\} + \left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\right).$$

Proof. Let $G = \|x^{k+1} - x^k\|^2 + \frac{2}{rs\alpha} D_h(y^{k+1}, y^k)$, and let

$$d = \frac{2}{rs\alpha} D_h(y^*, y^k) + \|x^k - x^*\|^2 - \frac{2}{rs\alpha} D_h(y^*, y^{k+1}) - \|x^{k+1} - x^*\|^2.$$

So d can be written as follows:

$$d = \frac{2}{rs\alpha}\left(D_h(y^*, y^k) - D_h(y^*, y^{k+1})\right) + \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

Applying Proposition 1 with $a = y^k$, $b = y^{k+1}$, and $c = y^*$, using

$$\|x^* - x^k\|^2 = \|x^* - x^{k+1}\|^2 + 2\langle x^* - x^{k+1}, x^{k+1} - x^k\rangle + \|x^k - x^{k+1}\|^2,$$

one obtains

$$d = \frac{2}{rs\alpha}\left\{D_h(y^{k+1}, y^k) + \langle \nabla h(y^{k+1}) - \nabla h(y^k), y^* - y^{k+1}\rangle\right\}$$

$$+ 2\langle x^* - x^{k+1}, x^{k+1} - x^k\rangle + \|x^k - x^{k+1}\|^2.$$

Now, using the fact that $D_h(a, b) \geq 0.5\xi\|a - b\|^2$, where ξ is the strong convexity coefficient for the Bregman function h, and denote

$$T_1 = \|x^k - x^{k+1}\|^2 + \frac{\xi}{rs\alpha}\|y^k - y^{k+1}\|^2,$$

then one has

$$d \geq T_1 + \frac{2}{rs\alpha}\left\{\langle \nabla h(y^{k+1}) - \nabla h(y^k), y^* - y^{k+1}\rangle\right\} + 2\langle x^* - x^{k+1}, x^{k+1} - x^k\rangle.$$

Now, if we use $x' = x^*$ in Step 1 of Algorithm 1, we can write

$$\langle x^* - x^{k+1}, f(x^{k+1}) - \nabla g(x^{k+1})y^k + r(x^{k+1} - x^k)\rangle \geq 0,$$

which implies

$$\langle x^* - x^{k+1}, r(x^{k+1} - x^k)\rangle \geq \langle x^{k+1} - x^*, f(x^{k+1}) - \nabla g(x^{k+1})y^k\rangle,$$

multiplying both sides by $\frac{2}{r}$, we get

$$2\langle x^* - x^{k+1}, (x^{k+1} - x^k)\rangle \geq \frac{2}{r}\langle x^{k+1} - x^*, f(x^{k+1}) - \nabla g(x^{k+1})y^k\rangle,$$

then by Lemma 1, we get

$$2\langle x^* - x^{k+1},\, x^{k+1} - x^k \rangle \geq \frac{2}{r}\langle x^{k+1} - x^*,\, f(x^{k+1}) - \nabla g(x^{k+1})y^k - (f(x^*) - \nabla g(x^*)y^*)\rangle, \tag{24}$$

and using $y' = y^*$ in Step 2 of Algorithm 1, we obtain

$$\langle y^* - y^{k+1},\, s\alpha g(x^{k+1}) + s(\nabla h(y^{k+1}) - \nabla h(y^k))\rangle \geq 0,$$

and according to Lemma 2, we get

$$\frac{2}{rs\alpha}\langle y^* - y^{k+1},\, \nabla h(y^{k+1}) - \nabla h(y^k)\rangle \geq \frac{2}{rs}\langle y^{k+1} - y^*,\, g(x^{k+1}) - g(x^*)\rangle, \tag{25}$$

which completes the proof.

Proposition 4. *For any $u^* \in \Omega^*$, and let $(x^k, y^k)_k$ be the sequence generated in Algorithm* 1. *Then, we have*

$$d \geq \|x^{k+1} - x^*\|^2 + \frac{\xi}{r\alpha}\|y^{k+1} - y^k\|^2 + \frac{2\sigma}{r}\|g(x^{k+1}) - g(x^*)\|^2 + \frac{2}{rs}\langle y^{k+1} - y^*,\, g(x^{k+1}) - g(x^*)\rangle, \tag{26}$$

where

$$d = \frac{2}{rs\alpha}\left\{D_h(y^*,\, y^k) - D_h(y^*,\, y^{k+1})\right\} + \left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\right).$$

Proof. Let us denote $u = (x^{k+1},\, y^k)$, $u^* = (x^*,\, y^*)$, thus, we have

$$F(u) = (f(x^{k+1}) - \nabla g(x^{k+1})y^k,\, g(x^{k+1})),$$

and $F(u^*) = (f(x^*) - \nabla g(x^*)y^*,\, g(x^*))$. Since F is partially co-coercive with coefficient α, then

$$\langle F(u) - F(u^*),\, u - u^*\rangle \geq \alpha\|g(x^{k+1}) - g(x^*)\|^2,$$

and using $y^{k+1} - y^* = y^{k+1} - y^k + y^k - y^*$, then the proof follows.

To finalize the convergence analysis, we need the following proposition:

Proposition 5. *Let* $u^* = (x^*, y^*) \in \Omega^*$ *be a solution point of* $VIP(\Omega, F)$. *Then*

$$\frac{2s}{r\alpha} D_h(y^*, y^k) + \|x^k - x^*\|^2 \geq \frac{2s}{r\alpha} D_h(y^*, y^{k+1}) + \|x^{k+1} - x^*\|^2 + \|x^{k+1} - x^k\|^2 + \left(\frac{s\xi}{r\alpha} - \frac{1}{2r\sigma}\right)\|y^{k+1} - y^k\|^2. \quad (27)$$

Proof. Using Lemma 4 with $a = \frac{2}{r}$, $b = \sigma$, $v = g(x^{k+1}) - g(x^*)$, and $w = y^{k+1} - y^*$, then $\frac{a}{4b} = \frac{1}{2r\sigma}$, and $ab = \frac{2\sigma}{r}$. Thus, we directly obtain

$$\frac{2\sigma}{r}\|g(x^{k+1}) - g(x^*)\|^2 + \frac{2}{r}\langle y^{k+1} - y^*, g(x^{k+1}) - g(x^*)\rangle \geq -\frac{1}{2r\sigma}\|y^{k+1} - y^*\|^2. \quad (28)$$

From Theorem 1, we get

$$d \geq \|x^{k+1} - x^k\|^2 + \left(\frac{s\xi}{r\alpha} - \frac{1}{2r\sigma}\right)\|y^{k+1} - y^k\|^2,$$

where

$$d = \|x^* - x^k\|^2 + \frac{2}{r\alpha} D_h(y^*, y^k) - \left(\|x^* - x^{k+1}\|^2 + \frac{2}{r\alpha} D_h(y^*, y^{k+1})\right). \quad (29)$$

Thus, the proposition is established.

Before we prove the following theorem, we assume that $\left(\frac{s\xi}{r\alpha} - \frac{1}{2r\alpha\sigma}\right) > 0$, or alternatively $\alpha > 4s\xi\sigma$ to ensure that $d \geq 0$.

Theorem 2. *Let* $u^* = (x^*, y^*) \in \Omega^*$ *be a solution of VIP*(Ω, F) *and* $\{u^k\}$ *be the sequence generated Algorithm* 1, *then*

$$\lim_{k\to\infty} \|u^k - u^*\| = 0.$$

Proof. Since $\|.\| \geq 0$ and $\alpha > 4s\xi\sigma$, then $d \geq 0$ and by taking the sum of d in (29), we obtain

$$\sum_{t=0}^{k} [\|x^* - x^t\|^2 + \frac{2s}{r\alpha} D_h(y^*, y^t) - (\|x^* - x^{t+1}\|^2 + \frac{2s}{r\alpha} D_h(y^*, y^{t+1})) \geq 0,$$

then

$$\begin{aligned} &\frac{2s}{r\alpha} D_h(y^*, y^0) + \|x^0 - x^*\|^2 - \frac{2s}{r\alpha} D_h(y^*, y^1) - \|x^1 - x^*\|^2 \\ &\quad + \frac{2}{r\alpha} D_h(y^*, y^1) + \|x^1 - x^*\|^2 - \frac{2s}{r\alpha} D_h(y^*, y^2) + \|x^2 - x^*\|^2 \\ &\quad + \ldots - \frac{2s}{r\alpha} D_h(y^*, y^k) - \|x^k - x^*\|^2 \geq 0, \end{aligned}$$

implies

$$D_h(y^*, y^0) + \|x^0 - x^*\|^2 \geq D_h(y^*, y^k) + \|x^k - x^*\|^2.$$

Thus, the sequence $\{(x^k, y^k)\}_k$ is bounded.

Now, by taking the sum in (27), we get

$$\begin{aligned} &\sum_{t=0}^{k} [\frac{2s}{r\alpha} D_h(y^*, y^t) + \|x^t - x^*\|^2 - (\frac{2s}{r\alpha} D_h(y^*, y^{t+1}) + \|x^{t+1} - x^*\|^2)] \\ &\geq \sum_{t=0}^{k} (\|x^{t+1} - x^k\|^2 + (\frac{s\zeta}{r\alpha} - \frac{1}{2r\sigma}) \|y^{t+1} - y^t\|^2), \end{aligned}$$

implies that

$$\begin{aligned} &D_h(y^*, y^0) + \|x^0 - x^*\|^2 - D_h(y^*, y^k) - \|x^k - x^*\|^2 \\ &\geq \sum_{t=0}^{k} (\|x^{t+1} - x^t\|^2 + (\frac{s\zeta}{r\alpha} - \frac{1}{2r\sigma}) \|y^{t+1} - y^t\|^2), \end{aligned}$$

then we have

$$\infty \geq \sum_{t=0}^{k} (\|x^{t+1} - x^t\|^2 + (\frac{s\zeta}{r\alpha} - \frac{1}{2r\sigma})\|y^{t+1} - y^t\|^2).$$

If $k \to \infty$, we have

$$\lim_{k\to\infty} \|x^{k+1} - x^k\|^2 = 0,$$

and

$$\lim_{k\to\infty} \|y^{k+1} - y^k\|^2 = 0.$$

And since $(x^k, y^k)_k = (u^k)_k$ is bounded, then any accumulation point $\overline{u} = (\overline{x}, \overline{y}) \in \Omega$ is a solution to $VIP(F, \Omega)$, and by taking u^*, we get the convergence of the whole sequence to the solution u^* of (7)-(8).

As we mentioned prior the convergence analysis section, the above study can be applied under certain modifications to Algorithms 2, 3, and 4.

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