Journal of Pure and Applied Mathematics: Advances and Applications
Volume 9, Number 1, 2013, Pages 1-15

FIXED-POINTS AND UNIQUENESS OF MEROMORPHIC FUNCTIONS

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Abstract

Let $f(z)$, $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n \geq 3k + 7$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM; $f(z)$ and $g(z)$ share ∞ IM, then (1) $f(z) = tg(z)$ for $k \geq 2$; (2) either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$ or $f(z) = tg(z)$ for $k = 1$, where c_1, c_2, and c are three nonzero constants satisfying $4n^2(c_1c_2)^n c^2 = -1$ and t is a constant such that $t^n = 1$.

1. Introduction and Main Results

In this paper, a meromorphic function will mean meromorphic in the whole complex plane. Let $f(z)$ be a nonconstant meromorphic function. We use the following standard notation of value distribution theory:

2010 Mathematics Subject Classification: 30D35.

Keywords and phrases: meromorphic function, sharing value, unicity.

This work is supported by the Tianyuan Foundation of the NNSF of China (Grant No. 11126267).

Received June 1, 2012

$$T(r, f),\ m(r, f),\ N(r, f),\ \overline{N}(r, f),\ N\left(r, \frac{1}{f}\right), \cdots$$

(see Hayman [4] and Yang [6]). We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\{T(r, f)\},$$

as $r \longrightarrow \infty$, possibly outside of a set with finite measure.

Let a be a finite complex number and k be a positive integer. We denote by $N_{(k}(r, \frac{1}{f-a})$ the counting function for zeros of $f(z) - a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted. Set $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \ldots + \overline{N}_{(k}(r, \frac{1}{f-a})$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a be a complex number. If $f(z) - a$ and $g(z) - a$ assume the same zeros, then we say that $f(z)$ and $g(z)$ share the value a IM. If $f(z) - a$ and $g(z) - a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value a CM. We denote by $\overline{N}_L(r, \frac{1}{f-a})$ the counting function for a-point of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity not be counted. Similarly, we have the notation $\overline{N}_L(r, \frac{1}{g-a})$.

Fang and Hua [1] and Yang and Hua [7] obtained the following unicity theorem:

Theorem A. *Let* $f(z), g(z)$ *be two nonconstant meromorphic* (*entire*) *functions, and let* $n \geq 11 (\geq 6)$ *be a positive integer. If* $f^n f'$ *and* $g^n g'$

share 1 *CM*, *then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2, *and* c *are three nonzero constants satisfying* $(c_1 c_2)^{n+1} c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant* t *such that* $t^{n+1} = 1$.

In 2000, Fang and Qiu [2] proved the following result:

Theorem B. *Let* $f(z)$, $g(z)$ *be two nonconstant meromorphic functions, and let* $n \geq 11$ *be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share* z *CM*, *then either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2, *and* c *are three nonzero constants satisfying* $4(c_1 c_2)^{n+1} c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant* t *such that* $t^{n+1} = 1$.

In 2002, Fang [3] proved the following result:

Theorem C. *Let* $f(z)$, $g(z)$ *be two nonconstant entire functions, and let* n, k *be two positive integers with* $n > 2k + 4$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* 1 *CM*, *then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2, *and* c *are three nonzero constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ *or* $f(z) = tg(z)$ *for a constant* t *such that* $t^n = 1$.

Recently, Xu et al. [5] proved the following theorem:

Theorem D. *Let* $f(z)$, $g(z)$ *be two nonconstant meromorphic functions, and let* n, k *be two positive integers with* $n \geq 3k + 11$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* z *CM*; $f(z)$ *and* $g(z)$ *share* ∞ *IM*, *then either* $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, *where* c_1, c_2, *and* c *are three nonzero constants satisfying* $4n^2 (c_1 c_2)^n c^2 = -1$ *or* $f(z) = tg(z)$ *for a constant* t *such that* $t^n = 1$.

In this paper, we improve Theorem D as follows:

Theorem 1. *Let* $f(z)$, $g(z)$ *be two nonconstant meromorphic functions, and let* n, k *be two positive integers with* $n \geq 3k + 7$. *If* $(f^n)^{(k)}$ *and*

$(g^n)^{(k)}$ share z CM; $f(z)$ and $g(z)$ share ∞ IM, then (1) *$f(z) = tg(z)$ for $k \geq 2$;* (2) *either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$ or $f(z) = tg(z)$ for $k = 1$, where c_1, c_2, and c are three nonzero constants satisfying $4n^2(c_1c_2)^n c^2 = -1$ and t is a constant such that $t^n = 1$.*

2. Some Lemmas

In order to prove our results, we need the following lemmas:

Lemma 1 ([8]). *Let $f(z)$ be a nonconstant meromorphic function satisfying $f^{(k)}(z) \not\equiv 0$, and let k be a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2. *Let $f(z)$, $g(z)$ be two nonconstant meromorphic functions. If $f(z)$ and $g(z)$ share 1 CM; $f(z)$ and $g(z)$ share ∞ IM, then one of the following cases must occur*:

(i) $T(r, f) + T(r, g) \leq 2\{N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g})\} + 4\overline{N}(r, f) + 2\{\overline{N}_L(r, f)$ $+ \overline{N}_L(r, g)\} + S(r, f) + S(r, g)$;

(ii) *either $f(z)g(z) \equiv 1$ or $f(z) \equiv g(z)$.*

Proof. Set

$$H = \frac{f''}{f'} - 2\frac{f'}{f-1} - \frac{g''}{g'} + 2\frac{g'}{g-1}. \tag{1}$$

Now, we consider two cases.

Case 1. $H(z) \not\equiv 0$. It follows from the logarithmic derivative lemma that

$$m(r, H) = S(r, f) + S(r, g).$$

Let z_0 be a common simple 1-point of $f(z)$ and $g(z)$. By simple computing, we see that z_0 is a zero of $H(z)$. Thus, we get

$$\begin{aligned}
N_{1)}\left(r, \frac{1}{f-1}\right) = N_{1)}\left(r, \frac{1}{g-1}\right) &\leq N(r, \frac{1}{H}) \\
&\leq T(r, H) + O(1) \leq N(r, H) + S(r, f) + S(r, g) \\
&\leq \overline{N}_{(2}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{g}) + \overline{N}_L(r, f) + \overline{N}_L(r, g) \\
&+ N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r, f) + S(r, g), \qquad (2)
\end{aligned}$$

where $N_0(r, \frac{1}{f'})$ is the counting function, which only counts those points such that $f' = 0$, but $f(f-1) \neq 0$. By the second fundamental theorem, we have

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f-1}) - N_0(r, \frac{1}{f'}) + S(r, f), \qquad (3)$$

and

$$T(r, g) \leq \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) + \overline{N}(r, \frac{1}{g-1}) - N_0(r, \frac{1}{g'}) + S(r, g). \qquad (4)$$

Obviously,

$$\begin{aligned}
\overline{N}(r, \frac{1}{f-1}) + \overline{N}(r, \frac{1}{g-1}) = 2\overline{N}(r, \frac{1}{g-1}) \\
&\leq N_{1)}\left(r, \frac{1}{f-1}\right) + N(r, \frac{1}{g-1}) \\
&\leq N_{1)}\left(r, \frac{1}{f-1}\right) + T(r, g) + S(r, f) + S(r, g). \qquad (5)
\end{aligned}$$

By Equations (2)-(5), we have

$$T(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + 2\overline{N}(r, f) + \overline{N}_L(r, f)$$

$$+ \overline{N}_L(r,\ g) + S(r,\ f) + S(r,\ g). \tag{6}$$

Similarly, we have

$$T(r,\ g) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + 2\overline{N}(r,\ g) + \overline{N}_L(r,\ f)$$

$$+ \overline{N}_L(r,\ g) + S(r,\ f) + S(r,\ g). \tag{7}$$

From Equations (6) and (7), we get (i) of Lemma 2.

Case 2. $H(z) \equiv 0.$ It follows from (1) that

$$\frac{1}{f-1} = \frac{A}{g-1} + B, \tag{8}$$

where $A \neq 0$ and B are constants. By rewriting (8), we get

$$f = \frac{(B+1)g + A - B - 1}{Bg + A - B}. \tag{9}$$

Next, we consider three subcases.

Case 2.1. $B = 0.$ From Equation (9), we obtain that

$$f = \frac{g}{A} + \frac{A-1}{A}.$$

If $A = 1,$ then $f = g.$ If $A \neq 1,$ by the second fundamental theorem, we have

$$T(r,\ f) \leq \overline{N}(r,\ f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f - \frac{A-1}{A}}) + S(r,\ f)$$

$$\leq \overline{N}(r,\ f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + S(r,\ f),$$

which reveals (i) of Lemma 2.

Case 2.2. $B = -1.$ From Equation (9), we obtain that

$$f = \frac{A}{-g + A + 1}.$$

If $A = -1$, then $f(z)g(z) = 1$. If $A \neq -1$, by the second fundamental theorem, we have

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{(A+1)f - A}) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + S(r, f),$$

which reveals (i) of Lemma 2.

Case 2.3. $B \neq 0, -1$. From Equation (9), we obtain that

$$(Bf(z) - B - 1)(Bg(z) + A - B) = -A.$$

By the second fundamental theorem, we have

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{Bf - B - 1}) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, g) + S(r, f),$$

which reveals (i) of Lemma 2. Hence the proof of the Lemma 2 is complete.

Lemma 3 ([6]). *Let* f, g *be two nonconstant meromorphic functions. If* $(f^n)^{(k)} \equiv (g^n)^{(k)}$ *and* $n \geq 2k + 2$, *then* $f(z) = tg(z)$ *for a constant* t *such that* $t^n = 1$.

3. Proof of Theorem 1

Proof of Theorem 1. Set $F = f^n$, $G = g^n$. Thus $\frac{F^{(k)}}{z}$ and $\frac{G^{(k)}}{z}$ share 1 CM; $\frac{F^{(k)}}{z}$ and $\frac{G^{(k)}}{z}$ share ∞ IM. We suppose that

$$T(r, \frac{F^{(k)}}{z}) + T(r, \frac{G^{(k)}}{z}) \leq 2\left\{N_2(r, \frac{z}{F^{(k)}}) + N_2(r, \frac{z}{G^{(k)}})\right\} + 4\overline{N}(r, \frac{F^{(k)}}{z})$$

$$+\ 2\left\{\overline{N}_L(r, \frac{F^{(k)}}{z}) + \overline{N}_L(r, \frac{G^{(k)}}{z})\right\}$$

$$+\ S(r, f) + S(r, g). \tag{10}$$

We note that

$$N_2(r, \frac{1}{F^{(k)}}) + N_2(r, \frac{1}{G^{(k)}}) \le N(r, \frac{1}{F^{(k)}}) - \left(N_{(3}(r, \frac{1}{F^{(k)}}) - 2\overline{N}_{(3}(r, \frac{1}{F^{(k)}})\right)$$

$$+ N(r, \frac{1}{G^{(k)}}) - \left(N_{(3}(r, \frac{1}{G^{(k)}}) - 2\overline{N}_{(3}(r, \frac{1}{G^{(k)}})\right). \tag{11}$$

If z_0 is a zero of $f(z)$ with multiplicity p, then z_0 is a zero of $(f^n)^{(k)}$ with multiplicity $np - k$, we have

$$N_{(3}(r, \frac{1}{F^{(k)}}) - 2\overline{N}_{(3}(r, \frac{1}{F^{(k)}}) \ge (n - k - 2)N(r, \frac{1}{f}). \tag{12}$$

Similarly, we have

$$N_{(3}(r, \frac{1}{G^{(k)}}) - 2\overline{N}_{(3}(r, \frac{1}{G^{(k)}}) \ge (n - k - 2)N(r, \frac{1}{g}). \tag{13}$$

By Equation (10)-(13), we have

$$T(r, \frac{1}{F^{(k)}}) + T(r, \frac{1}{G^{(k)}}) \le T(r, \frac{z}{F^{(k)}}) + T(r, \frac{z}{G^{(k)}}) + 2\log r$$

$$\le T(r, \frac{F^{(k)}}{z}) + T(r, \frac{G^{(k)}}{z}) + 2\log r + O(1)$$

$$\le 2\left(N(r, \frac{1}{F^{(k)}}) + N(r, \frac{1}{G^{(k)}})\right) + 4\overline{N}(r, f)$$

$$+ 2(k + 2 - n)\left(N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right) + 6\log r$$

$$+ 2\left\{\overline{N}_L(r, \frac{F^{(k)}}{z}) + \overline{N}_L(r, \frac{G^{(k)}}{z})\right\}$$

$$+ S(r, f) + S(r, g). \tag{14}$$

Noting that

$$nm(r, \frac{1}{f}) = m(r, \frac{1}{F}) \leq m(r, \frac{1}{F^{(k)}}) + S(r, f)$$

$$\leq T(r, \frac{1}{F^{(k)}}) - N(r, \frac{1}{F^{(k)}}) + S(r, f). \tag{15}$$

Similarly, we have

$$nm(r, \frac{1}{g}) \leq T(r, \frac{1}{G^{(k)}}) - N(r, \frac{1}{G^{(k)}}) + S(r, g). \tag{16}$$

By Equations (14), (15), (16), and Lemma 1, we have

$$n(T(r, f) + T(r, g)) \leq N(r, \frac{1}{F^{(k)}}) + N(r, \frac{1}{G^{(k)}})$$

$$+ (2k + 4 - n)\left(N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right) + 6\log r + 4\overline{N}(r, f)$$

$$+ 2\left(\overline{N}_L(r, \frac{F^{(k)}}{z}) + \overline{N}_L(r, \frac{G^{(k)}}{z})\right) + S(r, f) + S(r, g)$$

$$\leq 2(k + 2)\left(N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right) + (2k + 4)\overline{N}(r, f)$$

$$+ 6\log r + 2\left(\overline{N}_L(r, \frac{F^{(k)}}{z}) + \overline{N}_L(r, \frac{G^{(k)}}{z})\right)$$

$$+ S(r, f) + S(r, g). \tag{17}$$

Noting that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM; $f(z)$ and $g(z)$ share ∞ IM, we get

$$2\left(\overline{N}_L(r, \frac{F^{(k)}}{z}) + \overline{N}_L(r, \frac{G^{(k)}}{z})\right) \leq \overline{N}(r, f) + \overline{N}(r, g).$$

From Equation (17), we have

$$n(T(r, f) + T(r, g)) \leq 2(k+2)\left(N(r, \frac{1}{f}) + N(r, \frac{1}{g})\right) + 6\log r$$
$$+ (k+3)\left(\overline{N}(r, f) + \overline{N}(r, g)\right) + S(r, f) + S(r, g). \quad (18)$$

Next, we consider two cases.

Case 1. Either $f(z)$ or $g(z)$ is a transcendental meromorphic function. If $n \geq 3k+8$, then it follows from Equation (18) that

$$T(r, f) + T(r, g) \leq 6\log r + S(r, f) + S(r, g),$$

a contradiction. If $n = 3k+7$, by Equation (18), we get that

$$N_{(2}(r, f) = S(r, f), \quad N_{(2}(r, g) = S(r, g).$$

Thus

$$\overline{N}_L(r, \frac{F^{(k)}}{z}) = S(r, f), \quad \overline{N}_L(r, \frac{G^{(k)}}{z}) = S(r, g).$$

It follows from Equation (17) that

$$T(r, f) + T(r, g) \leq 6\log r + S(r, f) + S(r, g),$$

a contradiction.

Case 2. Both $f(z)$ and $g(z)$ are two nonconstant rational functions. If $f(z)$ is a polynomial, then $g(z)$ is a polynomial. Thus, it follows from Equation (17) that

$$8\log r \leq (k+3)(T(r, f) + T(r, g)) \leq 6\log r + O(1),$$

a contradiction. Thus, both $f(z)$ and $g(z)$ are non-polynomial rational functions. By Equation (18), we have

$$2(k+2)\left(m(r,\frac{1}{f})+m(r,\frac{1}{g})\right)+(k+3)\left(N_{(2}(r,f)+N_{(2}(r,g)-\overline{N}_{(2}(r,f)-\overline{N}_{(2}(r,g)\right)$$

$$+\ (k+3)(m(r,\ f)+m(r,\ g))\leq 6\log r+O(1). \tag{19}$$

Set

$$f(z)=\frac{p_2(z)}{p_1(z)},\quad g(z)=\frac{q_2(z)}{q_1(z)},$$

where both $p_1(z), p_2(z)$ and $q_1(z), q_2(z)$ are coprime polynomials. If $\deg p_2 > \deg p_1$, then $m(r,\ f)=(\deg p_2-\deg p_1)\log r$. It follows from Equation (19) that

$$N_{(2}(r,\ f)=0,\quad N_{(2}(r,\ g)=0.$$

Thus,

$$\overline{N}_L(r,\ \frac{F^{(k)}}{z})=0,\quad \overline{N}_L(r,\ \frac{G^{(k)}}{z})=0.$$

It follows from Equation (17) that

$$6\log r\leq T(r,\ f)+T(r,\ g)+(2k+4)m(r,\ \frac{1}{g})+(k+2)(m(r,\ f)+m(r,\ g))$$

$$\leq 6\log r+O(1).$$

Hence, we have

$$f(z)=\frac{a_2z^2+a_1z+a_0}{z-z_1},\quad g(z)=\frac{b_1z+b_0}{z-z_1}, \tag{20}$$

where $a_2, a_1, a_0, b_1,$ and b_0 are constants with $a_2b_1\neq 0$. From Equation (20), we have

$$(f^n)^{(k)}=\frac{P(z)}{(z-z_1)^{n+k}},\qquad (g^n)^{(k)}=\frac{Q(z)}{(z-z_1)^{n+k}},$$

where $P(z), Q(z)$ are polynomials with $\deg P=2n$ and $\deg Q=n-1$. Thus $(f^n)^{(k)}-z$ has $2n$ zeros (computing multiplicity), but $(g^n)^{(k)}-z$ has only $n+k+1$ zeros (computing multiplicity). This contradicts $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM.

Thus $\deg p_2 \leq \deg p_1$. If $\deg p_2 < \deg p_1$, then $m(r, \frac{1}{f}) = (\deg p_1 - \deg p_2) \log r$. It follows from Equation (19) that

$$2(k+2)m(r, \frac{1}{f}) \leq 6 \log r + O(1),$$

and

$$N_{(2}(r, f) = 0, \qquad N_{(2}(r, g) = 0.$$

Thus,

$$\overline{N}_L(r, \frac{F^{(k)}}{z}) = 0, \quad \overline{N}_L(r, \frac{G^{(k)}}{z}) = 0.$$

It follows from Equation (17) that

$$8 \log r \leq T(r, f) + T(r, g) + 2(k+2)m(r, \frac{1}{f}) \leq 6 \log r + O(1),$$

a contradiction. Thus $\deg p_2 \geq \deg p_1$. Hence, $\deg p_2 = \deg p_1$. Similarly, we can get that $\deg q_2 = \deg q_1$. Thus by Equation (19), we have

$$(k+3)\left(N_{(2}(r, f) + N_{(2}(r, g) - \overline{N}_{(2}(r, f) - \overline{N}_{(2}(r, g)\right) \leq 6 \log r + O(1). \quad (21)$$

If $f(z)$ has a pole with multiplicity at least 3, then by Equation (21), we have

$$8 \log r \leq 2(k+3) \log r \leq 6 \log r + O(1),$$

a contradiction. If $f(z)$ has two multiple poles, then by (21), we have

$$8 \log r \leq 2(k+3) \log r \leq 6 \log r + O(1),$$

a contradiction. Thus $f(z)$ has at most one multiple pole and its multiplicity is 2. Similarly, we can get that $g(z)$ has at most one multiple pole and its multiplicity is 2. If both $f(z)$ and $g(z)$ have one multiple pole, then by Equation (21), we have

$$8\log r \le 2(k+3)\log r \le 6\log r + O(1),$$

a contradiction. If $f(z)$ has single multiple pole and $g(z)$ has only simple poles, then we have

$$f(z) = \frac{a_t z^t + a_{t-1} z^{t-1} + \ldots + a_0}{(z - z_1)^2 (z - z_2) \ldots (z - z_{t-1})},$$

$$g(z) = \frac{b_{t-1} z^{t-1} + \ldots + b_0}{(z - z_1)(z - z_2) \ldots (z - z_{t-1})}, \tag{22}$$

where $z_l\,(l = 1, 2, \ldots, t-1)$ are distinct complex numbers and $a_i\,(i = 0, 1, \ldots, t)$, $b_j\,(j = 0, 1, \ldots, t-1)$ are constants with $a_t b_{t-1} \neq 0$. From Equation (22), we have

$$(f^n)^{(k)} = \frac{P_1(z)}{(z - z_1)^{2n+k} (z - z_2)^{n+k} \ldots (z - z_{t-1})^{n+k}},$$

and

$$(g^n)^{(k)} = \frac{Q_1(z)}{(z - z_1)^{n+k} (z - z_2)^{n+k} \ldots (z - z_{t-1})^{n+k}},$$

where $P_1(z), Q_1(z)$ are polynomials with $\deg P_1 \le nt + kt - 2k - 1$ and $\deg Q_1 \le nt + kt - n - 2k - 1$. Thus $(f^n)^{(k)} - z$ has $nt + kt - k + 1$ zeros (computing multiplicity), but $(g^n)^{(k)} - z$ has only $nt + kt - n - k + 1$ zeros (computing multiplicity). This contradicts $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM. Similarly, if $g(z)$ has single pole and $f(z)$ has only simple poles, we can get a contradiction. Therefore, both $f(z)$ and $g(z)$ have only simple poles, then we have

$$(f^n)^{(k)} = \frac{h_1(z)}{P_2(z)}, \quad (g^n)^{(k)} = \frac{h_2(z)}{P_2(z)},$$

where both $h_1(z)$, $P_2(z)$ and $h_2(z)$, $P_2(z)$ are coprime polynomials with $\max\{\deg h_1, \deg h_2\} < \deg P_2$. Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, $h_1(z) \equiv h_2(z)$. Thus, $(f^n)^{(k)} \equiv (g^n)^{(k)}$.

Therefore, by Lemma 2, we get that either $(f^n)^{(k)} \equiv (g^n)^{(k)}$ or $(f^n)^{(k)}(g^n)^{(k)} \equiv z^2$. If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, by Lemma 3, we get $f(z) = tg(z)$, where t is a constant such that $t^n = 1$. If $(f^n)^{(k)}(g^n)^{(k)} \equiv z^2$, then $f(z)$ and $g(z)$ are two entire functions and have no zeros. Set $f(z) = e^{\alpha}$, $g(z) = e^{\beta}$, where α, β are entire functions, we have

$$(f^n)^{(k)} = A[(\alpha')^k + P_{k-1}(\alpha')]e^{n\alpha}, \quad (g^n)^{(k)} = B[(\beta')^k + Q_{k-1}(\beta')]e^{n\beta},$$

where A, B are nonzero constants, $P_{k-1}(\alpha')$, $Q_{k-1}(\beta')$ are differential polynomials in α' and β' of degree at most $k-1$, respectively. Thus, we get

$$AB[(\alpha')^k + P_{k-1}(\alpha')][(\beta')^k + Q_{k-1}(\beta')]e^{n(\alpha+\beta)} = z^2. \tag{23}$$

We deduce $\alpha + \beta = C$, where C is a constant. Hence from Equation (23), we get

$$A_1(\alpha')^{2k} = z^2 + \widetilde{P}_{2k-1}(\alpha'), \tag{24}$$

where A_1 is a nonzero constant and $\widetilde{P}_{2k-1}(\alpha')$ is a differential polynomials in α' of degree at most $2k-1$. Thus, we obtain

$$T(r, \alpha') \le 2\log r + S(r, \alpha').$$

Therefore, α' is a polynomial. If $k \ge 2$, then from Equation (24), we obtain a contradiction. If $k = 1$, from Equation (24), we get that $\alpha' = \frac{c}{2}z + b$. Thus,

$$f(z) = c_1 e^{cz^2 + bz}, \quad g(z) = c_2 e^{-cz^2 - bz},$$

and

$$-c_1^n n(2cz+b)c_2^n n(2cz+b) = z^2.$$

We deduce $b = 0$ and $4n^2(c_1c_2)^n c^2 = -1,$ thus

$$f(z) = c_1 e^{cz^2}, \quad g(z) = c_2 e^{-cz^2}.$$

The proof of Theorem 1 is complete.

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