Journal of Mathematical Sciences: Advances and Applications
Volume 21, 2013, Pages 39-48

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS FOR MILDLY DEGENERATE QUASILINEAR HYPERBOLIC EQUATIONS

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Abstract

We consider the initial-boundary value problem for mildly degenerate quasilinear hyperbolic equations of Kirchhoff type: $u_{tt} - \|u_x(t)\|^{2\gamma} u_{xx} + u_t = |u|^p u$ with the Dirichlet boundary value condition. When $\gamma \geq 1$ and $p > 4\gamma$, we obtain the optimal decay rate of the norms $\|u_x(t)\|$ and $\|u_{xx}(t)\|$.

1. Introduction

In this paper, we study on the asymptotic behaviour of solutions to the initial-boundary value problem for the following degenerate quasilinear hyperbolic equations of Kirchhoff type:

$$u_{tt} - \|u_x(t)\|^{2\gamma} u_{xx} + u_t = f(u) \quad \text{in} \quad (0, 1) \times (0, \infty), \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{and} \quad u(0, t) = u(1, t) = 0,$$

2010 Mathematics Subject Classification: 35L15.

Keywords and phrases: mildly degenerate, Kirchhoff strings, decay estimates.

Received May 10, 2013

where $u = u(x, t)$ is an unknown real value function, $u_t = \partial u / \partial t$, $u_x = \partial u / \partial x$, the symbol $\| \cdot \| = \| \cdot \|_{L^2}$ is the usual norm of $L^2 = L^2(\Omega)$ with $\Omega = (0, 1)$, $\gamma \geq 1$, and $f(u)$ is a C^1-function on $[0, \infty)$ satisfying

$$f(u) = |u|^p u \quad \text{with} \quad p > 0. \tag{1.2}$$

The Equation (1.1) describes small amplitude vibrations of an elastic string (see Kirchhoff [8] for the original equation, and also [11]).

In the case of initial data belonging to usual Sobolev spaces, Arosio and Garavaldi [1] have carried out detailed analysis about the existence of local solutions for the Kirchhoff type equations (also see [2], [3], [4], [14], and the references cited therein).

In the case of non-degenerate type, Hosoya and Yamada [6] have proved the exponential decay and the global existence of solutions under the small data condition (also see [15]). On the other hand, in the case of degenerate type and $f(u) \equiv 0$, many authors have studied the global existence and the decay properties of solutions under the small data condition (see [5], [12], [13], [17], [20], and the references cited therein).

We define the energies associated with the degenerate equation (1.1) by

$$E(u, u_t) \equiv \frac{1}{2}\|u_t\|^2 + J(u),$$

$$E_1(u, u_t) \equiv \frac{1}{2}\|u_t\|^2 + \frac{1}{2(\gamma+1)}\|u_x\|^{2(\gamma+1)},$$

$$H(u, u_t) \equiv \frac{\|u_{xt}\|^2}{\|u_x\|^{2\gamma}} + \|u_{xx}\|^2,$$

where we set the potential by

$$J(u) \equiv \frac{1}{2(\gamma+1)}\|u_x\|^{2(\gamma+1)} - \frac{1}{p+2}\|u\|_{p+2}^{p+2},$$

and $\| \cdot \|_q$ is the norm of $L^q = L^q(\Omega)$ with $\Omega = (0, 1)$.

By standard calculation, we find that the energy $E(u(t),\, u_t(t))$ has the so-called energy identity

$$\frac{d}{dt}\, E(u(t),\, u_t(t)) + \|u_t(t)\|^2 \,=\, 0,$$

or

$$E(u(t),\, u_t(t)) + \int_0^t \|u_t(s)\|^2 ds \,=\, E(u_0,\, u_1).$$

In order to get the global existence theorem (see Theorem 1.1) for the nonlinear equation (1.1), we introduce the K-positive set (i.e., the modified potential well) as

$$\mathcal{W}_* \,\equiv\, \{u \,\in\, H^2 \,\cap\, H_0^1 \mid K(u) > 0\} \bigcup \{\, 0 \,\},$$

where we set $K(u) \equiv \|u_x\|^{2(\gamma+1)} - \|u\|_{p+2}^{p+2}$ (see [10], [15], and also [7], [18], [19]).

It is easy to see that if $u \,\in\, \mathcal{W}_*$ and $p > 2\gamma,$ then

$$J(u) \geq \frac{p - 2\gamma}{2(\gamma + 1)(p + 2)} \|u_x\|^{2(\gamma+1)},$$

and hence

$$\frac{p - 2\gamma}{2(\gamma + 1)(p + 2)}\, E_1(u,\, u_t) \leq E(u,\, u_t) \leq E_1(u,\, u_t).$$

We notice that the K-negative set $\mathcal{V}_* \,\equiv\, \{u \,\in\, H^2 \,\cap\, H_0^1 \mid K(u) < 0\}$ plays an important role in the global non-existence theorem (e.g., see [15] and the references cited therein).

In the previous paper [15], we proved the following global existence theorem and the upper decay estimates of the first order derivatives (i.e., the first energy) for (1.1) when $p > 2\gamma.$

Theorem 1.1. *Suppose that the initial data* $\{u_0, u_1\} \in \mathcal{W}_* \times H_0^1$ *are suitably small and* $u_0 \neq 0$ *and* $p > 2\gamma$. *Then, there exists a unique global solution* $u(t)$ *of* (1.1) *such that*

$$u(t) \in C^0([0, \infty); \mathcal{W}_*) \cap C^1([0, \infty); H_0^1) \cap C^2([0, \infty); L^2), \tag{1.3}$$

and

$$0 < \|u_x(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}} \quad \text{and} \quad \|u_t(t)\|^2 \leq C(1+t)^{-1-\frac{1}{\gamma}}, \tag{1.4}$$

for $t \geq 0$ *with some positive constant* C *and*

$$\frac{|M'(t)|}{M(t)} < \frac{2}{\gamma + 2} \quad \text{for} \quad t \geq 0, \tag{1.5}$$

where we set $M(t) \equiv \|u_x(t)\|^2$.

Moreover, when $\gamma = 1$, in the previous paper [16], we improved the decay rate of (1.4) like as (1.7) with $\gamma = 1$, and derived the lower decay estimate like as (1.6). However, we can not apply in the case of $\gamma > 1$ directly.

Our aim in this paper is to expand the results in [16] to the case $\gamma > 1$ (and $p > 4\gamma$). In particular, we derive the uniformly boundedness of $\|u_{xx}(t)\|^2 / \|u_x(t)\|^2$ for $t \geq 0$, and we obtain the decay estimate of $\|u_{xx}(t)\|^2$, which plays an important role in the argument of the lower decay.

Our main result is as follows:

Theorem 1.2. *In addition to the assumption of Theorem* 1.1, *suppose that* $p > 4\gamma$. *Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$C^{-1}(1+t)^{-\frac{1}{\gamma}} \leq \|u_x(t)\|^2, \quad \|u_{xx}(t)\|^2 \leq C(1+t)^{-\frac{1}{\gamma}}, \tag{1.6}$$

$$\|u_t(t)\|^2 \leq C(1+t)^{-2-\frac{1}{\gamma}} \quad \text{for} \quad t \geq 0, \tag{1.7}$$

with some positive constant C.

Theorem 1.2 follows from Propositions 2.1-2.3 in the following section:

The notations we use in this paper are standard. The symbol $(\cdot,\cdot)$ means the inner product in L^2 or sometimes duality between the space X and its dual X'. Positive constants will be denoted by C and will change from line to line.

2. Decay Properties

In this section, we carry out the proof of Theorem 1.2. We put $M(t) \equiv \|u_x(t)\|^2$ for simplicity of the notation.

Proposition 2.1. *In addition to the assumption of Theorem* 1.1, *suppose that* $p > 4\gamma$. *Then, the solution* $u(t)$ *of* (1.1) *satisfies*

$$\frac{\|u_{xx}(t)\|^2}{M(t)} \le C \quad \text{and } \|u(t)\|_{H^2}^2 \le C(1+t)^{-\frac{1}{\gamma}}. \tag{2.1}$$

Proof. Using the Equation (1.1), we observe

$$\begin{aligned}\frac{d}{dt}\frac{\|u_{xx}(t)\|^2}{M(t)} &= \frac{1}{M(t)^{\gamma+2}}\left(2(M(t)^{\gamma},\, u_{xxt})M(t) - (M(t)^{\gamma}u_{xx},\, u_{xx})M'(t)\right)\\ &= -2Q(t) - R(t) + S(t),\end{aligned} \tag{2.2}$$

where

$$Q(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(M(t)\|u_{xt}(t)\|^2 - \left(\frac{1}{2}M'(t)\right)^2\right) \ge 0,$$

$$R(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(2M(t)(u_{xtt},\, u_{xt}) + M'(t)\left(\|u_{xt}(t)\|^2 - \frac{1}{2}M''(t)\right)\right),$$

$$S(t) \equiv \frac{1}{M(t)^{\gamma+2}}\left(-2M(t)(f(u),\, u_{xxt}) + M'(t)(f(u),\, u_{xx})\right).$$

On the other hand, we have

$$\frac{d}{dt}Q(t) = -(\gamma+2)\frac{M'(t)}{M(t)}Q(t)$$

$$+\frac{1}{M(t)^{\gamma+2}}\left(2M(t)(u_{xtt},\,u_{xt}) + M'(t)\|u_{xt}(t)\|^2 - \frac{1}{2}M'(t)M''(t)\right)$$

$$= -(\gamma+2)\frac{M'(t)}{M(t)}Q(t) + R(t). \tag{2.3}$$

Adding (2.2) and (2.3), we observe

$$\frac{d}{dt}\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t)\right) = -\left(2 + (\gamma+2)\frac{M'(t)}{M(t)}\right)Q(t) + S(t).$$

Since it follows from (1.5) that

$$2 + (\gamma+2)\frac{M'(t)}{M(t)} \geq 0,$$

and

$$|S(t)| \leq C\frac{1}{M(t)^{\gamma+2}}\|u(t)\|_\infty^p\|u_x(t)\|^3\|u_{xt}(t)\|$$

$$\leq CM(t)^{\frac{p}{2}-\gamma}H(t)^{\frac{1}{2}} \leq CM(t)^{\frac{p}{2}-\gamma},$$

we have

$$\frac{d}{dt}\left(\frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t)\right) \leq CM(t)^{\frac{p}{2}-\gamma},$$

and hence, if $p > 4\gamma$, from (1.4), we obtain that

$$\frac{\|u_{xx}(t)\|^2}{M(t)} + Q(t) \leq C,$$

which implies the desired estimate (2.1). □

Proposition 2.2. *Under the assumption of Proposition* 2.1, *the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|u_x(t)\|^2 \geq C'(1+t)^{-\frac{1}{\gamma}} \quad \text{for} \quad t \geq 0, \tag{2.4}$$

with some positive constant C'.

Proof. Multiplying (1.1) by $2u_t(t)M(t)^{-\gamma-2}$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} + \frac{1}{M(t)}\right) + \left(2 + (\gamma+2)\frac{M'(t)}{M(t)}\right)\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} \\
&\quad = -2\frac{M'(t)}{M(t)} + \frac{2}{M(t)^{\gamma+2}}(f(u),\, u_t) \\
&\quad \leq C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+2}{2}}}\frac{\|u_{xx}\|}{M(t)^{\frac{1}{2}}}M(t)^{\frac{\gamma-1}{2}} + C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+2}{2}}}M(t)^{\frac{p-2\gamma}{2}}M(t)^{\frac{\gamma-1}{2}} \\
&\quad \leq C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+2}{2}}}M(t)^{\frac{\gamma-1}{2}},
\end{aligned}$$

where we used (2.1) at the last inequality. Since it follows from (1.5) that

$$2 + (\gamma+2)\frac{M'(t)}{M(t)} \geq 0,$$

the Young inequality yields

$$\frac{d}{dt}\left(\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} + \frac{1}{M(t)}\right) \leq CM(t)^{\gamma-1}\left(\leq C(1+t)^{-1+\frac{1}{\gamma}}\right),$$

and

$$\frac{\|u_t(t)\|^2}{M(t)^{\gamma+2}} + \frac{1}{M(t)} \leq C(1+t)^{\frac{1}{\gamma}},$$

which implies the desired estimate (2.4). □

Proposition 2.3. *Under the assumption of Proposition* 2.1, *the solution* $u(t)$ *of* (1.1) *satisfies*

$$\|u_t(t)\|^2 \le C(1+t)^{-2-\frac{1}{\gamma}} \quad \text{for} \quad t \ge 0. \tag{2.5}$$

Proof. Multiplying (1.1) by $2u_t(t)M(t)^{-\gamma-1}$ and integrating it over Ω, we have

$$\begin{aligned}
&\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} + \left(2+(\gamma+1)\frac{M'(t)}{M(t)}\right)\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \\
&\quad = -\frac{M'(t)}{M(t)} + \frac{2}{M(t)^{\gamma+1}}(f(u),\, u_t) \\
&\quad \le C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}\frac{\|u_{xx}(t)\|}{M(t)^{\frac{1}{2}}}M(t)^{\frac{\gamma}{2}} + C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}M(t)^{\frac{p-2\gamma}{2}}M(t)^{\frac{\gamma}{2}} \\
&\quad \le C\frac{\|u_t(t)\|}{M(t)^{\frac{\gamma+1}{2}}}M(t)^{\frac{\gamma}{2}},
\end{aligned}$$

where we used (2.1) at the last inequality. Since it follows from (1.5) that

$$2+(\gamma+1)\frac{M'(t)}{M(t)} \ge \frac{2}{\gamma+2},$$

the Young inequality yields

$$\frac{d}{dt}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} + \frac{1}{\gamma+2}\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \le CM(t)^{\gamma} \left(\le C(1+t)^{-1}\right),$$

and

$$\frac{\|u_t(t)\|^2}{M(t)^{\gamma+1}} \le C(1+t)^{-1},$$

which implies the desired estimate (2.5). □

Proof of Theorem 1.2. The estimate (1.6) follows (2.1) and (2.4) and the estimate (1.7) follows (2.5). □

Acknowledgement

This work was in part supported by Grant-in-Aid for Science Research (C) 21540186 of JSPS (Japan Society for the Promotion of Science).

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