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ANNOTATIONS OF TWO EXAMPLES ABOUT MARKOV PROCESS

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Abstract

In this paper, we discuss two examples that Markov process does not satisfy the strong Markov property. We point out their mistake and analyze the reason of mistake. In the end, we prove the two Markov processes of the examples are both strong Markov processes.

1. The Examples and their Proofs

People think some Markov processes, which have not the strong Markov property, exists since someone has given these examples. By our analysis, we think them might be error. In this paper, we firstly give the two examples and their proofs, afterwards analyze the reason of mistake, and prove that they both are strong Markov processes given in examples.

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Example 1.1. Let $\{B_t(\omega);\ t \geq 0\}$ be a Brownian motion. Put

$$A_t(\omega) = \begin{cases} B_t(\omega) & \text{if } B_0(\omega) \neq 0, \\ 0 & \text{otherwise,} \end{cases} = B_t(\omega)\mathcal{X}_{\{B_0 \neq 0\}}(\omega);$$

$$P(t;\ x,\ B) = \begin{cases} \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy, & \text{if } x \neq 0, \\ \mathcal{X}_B(0), & \text{if } x = 0, \end{cases} \qquad B \in \mathcal{B} \triangleq \mathcal{B}((-\infty, \infty)).$$

Then $\{A_t(\omega);\ t \geq 0\}$ is a Markov process with transition probability function $P(t;\ x,\ B)$, but it has not the strong Markov property, where $\mathcal{X}_{\{B_0 \neq 0\}}(\omega)$ is an indicator function relative to set $\{B_0 \neq 0\}$.

Proof. Let $\mathcal{F}_t \triangleq \mathcal{F}(B_u;\ u \leq t)$ be the σ-algebra generated by the process prior to t of B_u. Obviously, A_t is $\mathcal{F}_t$-measurable. For any $B \in \mathcal{B}$ and $t,\ s > 0$,

$$E(\mathcal{X}_B(A_{t+s})|\mathcal{F}_s) = E(\mathcal{X}_B(B_{t+s})\mathcal{X}_{\{B_0 \neq 0\}}(\omega)|\mathcal{F}_s) + E(\mathcal{X}_B(0)\mathcal{X}_{\{B_0 = 0\}}(\omega)|\mathcal{F}_s)$$

$$= \mathcal{X}_{\{B_0 \neq 0\}}(\omega)\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(A_s(\omega)-y)^2}{2t}} dy + \mathcal{X}_{\{B_0 = 0\}}(\omega)\mathcal{X}_B(0)$$

$$= \mathcal{X}_{\{A_s \neq 0\}}(\omega)\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(A_s(\omega)-y)^2}{2t}} dy + \mathcal{X}_{\{A_s = 0\}}(\omega)\mathcal{X}_B(A_s)$$

$$+ \mathcal{X}_{\{A_s = 0,\, B_0 \neq 0,\, B_s = 0\}}(\omega)\left[\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(A_s(\omega)-y)^2}{2t}} dy - \mathcal{X}_B(0)\right]$$

$$= P(t;\ A_s(\omega),\ B), \quad P\text{ - a.e.,} \tag{1.1}$$

where the last equality follows from that the last term of (1.1) is equal to 0 almost everywhere. So A_t is a Markov process with transition probability function $P(t;\ x,\ B)$, but A_t does not satisfy the strong Markov property. For if does, suppose that $\tau(\omega)$ is first hitting time of 0 of $A_t(\omega)$, that is, let $\tau(\omega) = \inf\{t > 0;\ A_t(\omega) = 0\}$. Then,

$$P_x(A_1 \neq 0,\ \tau \leq 1) = E_x[P(A_1 \neq 0,\ \tau \leq 1|\mathcal{F}_t)]$$

$$\leq E_x[P(1-\tau;\ A_\tau,\ \{0\}^c)]$$

$$= E_x[P(1-\tau;\ 0,\ \{0\}^c)] = 0, \tag{1.2}$$

where $\{0\}^c \triangleq R - \{0\}$. On the other hand, since $P_x(A_1 = 0) = 0$ as $x \neq 0$, it follows:

$$P_x(A_1 \neq 0,\ \tau \leq 1) = P_x(\tau \leq 1)$$

$$= P_x(\omega : \text{there exists } t \leq 1 \text{ such that } B_t(\omega) = 0) > 0, \tag{1.3}$$

which contradicts (1.2). □

Example 1.2. For each prime $p \geq 2$, let r_p denote a variable point of the sequence $\{kp^{-n}|k = 0, 1, \cdots, p^n - 1;\ n = 1, 2, \cdots\}$. For each p, we define a space $\Omega^{(p)}$ of paths $\{\omega = x(\tau, \omega)|\tau \geq 0\}$ each of which is the inverse of a Cantor type function, proceeding monotonically from 0 to 1, moving only by jumps and pausing at each point r_p for a time $T(r_p, \omega)$. $\Omega^{(p)}$ can be represented as the space of the random variables $T(r_p, \omega)$, a countably dimensioned product space. A Borel measure is determined on $\Omega^{(p)}$ by requiring that the waiting times be independent, and that, if the irreducible form of r_p is kp^{-n}, $T(r_p, \omega)$ have an exponential distribution with lifetime $\mu(r_p) = p^{-2n}$: $P_r^{(p)}\{T(r_p, \omega) \leq t\} = 1 - \exp(-t/\mu(r_p))$. If $r_p = 0$, we set $\mu(r_p) = 1/p$. With probability one $\sum T(r_p, \omega)$ converges, and the function

$$\tau(x, \omega) = \sum_{r_p \leq x} T(r_p, \omega),$$

is bounded and strictly increasing on the interval $0 \le x \le 1$. The corresponding path $x(\tau, \omega)$ in $\Omega^{(p)}$ is the inverse of this function:

$$x(\tau, \omega) = x, \quad \sum_{r_p < x} T(r_p, \omega) \le \tau \le \sum_{r_p \le x} T(r_p, \omega), \tag{1.4}$$

$$x(\tau, \omega) = x, \quad \tau(1, \omega) = \sum_{r_p < x} T(r_p, \omega) \le \tau. \tag{1.5}$$

It is clear that almost every path thus constructed is continuous for $\tau \ge 0$, and that with the measure $P_r^{(p)}\{\cdot\}$ determined by the waiting times, we have a stationary Markov process on $[0, 1]$.

Consider now the space of paths

$$\Omega = \bigcup_p \Omega^{(p)}, \tag{1.6}$$

with the measure

$$P_r\{\cdot\} = \sum_p C_p P_r^{(p)}\{\cdot \bigcap \Omega^{(p)}\}, \tag{1.7}$$

where $C_p = 1/p' - 1/p.\ p'$ being the largest prime less than p, or if $p = 2$, then $p' = 1$. We state that these form a stationary Markov process on $[0, 1]$: the conditional probability of $x(t + s, \omega)$ being in a Borel set E, given $x(\tau, \omega)$ for $\tau \le t$, is a function only of E, s, and $x(t, \omega)$ for almost every ω. Indeed, for $t \ge 0$, the value of $x(t, \omega)$ is with probability one in the sequence $\{r_p\}$, whenever ω is in $\Omega^{(p)}$; and since these sequences for different primes p have only 0 in common, if $x(t, \omega) = r_p > 0$, then ω is in $\Omega^{(p)}$ with probability one, and the distribution of ω is given by $P_r^{(p)}$. In this case, the Markov property is a

consequence of the independence of waiting times used to construct this measure. If on the other hand, $x(t, \omega) = 0$, the conditional probability in question is simply $P_r\{x(s, \omega) \in E\}$.

Note 1. The process $\{\Omega, P_r\{\cdot\}\}$ does not have the strong Markov property. In particular, let I be the open interval $[0, \frac{1}{2})$ of $[0, 1]$. If $\omega \in \Omega^{(p)}$, then $\mathfrak{J}_I(\omega) = \sum_{r_p < \frac{1}{2}} T(r_p, \omega)$, while $x(t + \mathfrak{J}_I(\omega), \omega)$ depends only on $T(r_p, \omega)$ for $r_p \geq \frac{1}{2}$. Hence for ω in $\Omega^{(p)}$, $\mathfrak{J}_I(\omega)$ is independent of $x(t + \mathfrak{J}_I(\omega), \omega)$. But then clearly $\mathfrak{J}_I(\omega)$ is independent of $x(t + \mathfrak{J}_I(\omega), \omega)$ when ω ranges over all of Ω, if and only if $\mathfrak{J}_I(\omega)$ is independent of the choice of component $\Omega^{(p)}$; and this last statement is certainly not true. □

2. Annotations of the Two Examples

2.1. Annotations of the first example

Remark 2.1. (1) The condition $P(A_0(\omega) = 0) \neq 0$ is not used in the proof, that is, it is holds if $P(A_0(\omega) = 0) = 0$ even if $\{A_0(\omega) = 0\} = \emptyset$. So we may suppose $P(A_0(\omega) = 0) = 0$. Hence $A_s(\omega)$ and $B_s(\omega)$ are two equivalent, and even the same process since $P(A_s(\omega) = B_s(\omega)) = 1$ for every $s \geq 0$, further, $A_s(\omega)$ and $B_s(\omega)$ have the same sample path except the null measurable set $\{A_0(\omega) = 0\}$. Therefore, we think that $A_s(\omega)$ also satisfies the strong Markov property since $B_s(\omega)$ is a strong Markov process (see [1, p. 343] and [7, Theorem 3.10]).

(2) The following theorems indicate that: If we select different transition probability function for the same process, then it has different conclusions whether the strong Markov property holds.

For convenience, suppose without loss of generality that $P(A_0 = 0) = 0$. So, another transition probability function of $A_t(\omega)$ is

$$p(s, t; x, B) = \int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy. \tag{2.1}$$

The following Lemma 2.2 is quoted from [11, Theorem 2.3].

Lemma 2.2. *Let* $\alpha(\omega)$ *be an arbitrary non negative random variable*; $X(t, \omega) \triangleq \{x(t, \omega), t \geq 0\}$ *be an arbitrary random process valuing in measurable space* $(E, \mathcal{E})$. *Put*

$$B_{t_1 \cdots t_n s}(A_1, \cdots, A_n) = \{x(t_1, \omega) \in A_1, \cdots, x(t_n, \omega) \in A_n, s \leq \alpha(\omega)\};$$

$$\prod = \{B_{t_1 \cdots t_n s}(A_1, \cdots, A_n) : t_1 \leq \cdots \leq t_n \leq s, A_1, \cdots, A_n \in \mathcal{E}, n \geq 1\}.$$

Then,

$$\mathcal{N}_\alpha^+ \triangleq \mathcal{F}(x(t, \omega); t \leq \alpha(\omega)) = \mathcal{F}(\prod),$$

where $\mathcal{F}(\cdot)$ *denotes the* σ*-algebra generated by all sets of the bracket*.

Theorem 2.3. (1) $A_s(\omega)$ *and* $B_s(\omega)$ *are two equivalent processes*, *if* $P(A_0(\omega) = 0) = 0$.

(2) $A_s(\omega)$ *is a strong Markov process if* $P(A_0 = 0) = 0$ *and the strong Markov property is defined by transition probability function* $p(s, t; x, B)$.

Proof. (1) Since $P(A_0(\omega) = 0) = 0$ and $A_s(\omega) = B_s(\omega)$ for every $\omega \notin \{A_0(\omega) = 0\}$, and the two processes are said to be equivalent, if and only if $P(A_s(\omega) = B_s(\omega)) = 1$. From which it follows (1) holds.

(2) Let $\alpha(\omega)$ be an arbitrary stopping time. Set

$$A_{t_1 \cdots t_n s}(C_1, \cdots, C_n) = \{A_{t_1} \in C_1, \cdots, A_{t_n} \in C_n, s \leq \alpha(\omega)\};$$

$$B_{t_1 \cdots t_n s}(C_1, \cdots, C_n) = \{B_{t_1} \in C_1, \cdots, B_{t_n} \in C_n, s \leq \alpha(\omega)\};$$

$$\prod_{A_t} = \{A_{t_1 \cdots t_n s}(C_1, \cdots, C_n) : t_1 \le \cdots \le t_n \le s,\ C_1, \cdots, C_n \in \mathcal{B}(R),\ n \ge 1\},$$

$$\prod_{B_t} = \{B_{t_1 \cdots t_n s}(C_1, \cdots, C_n) : t_1 \le \cdots \le t_n \le s,\ C_1, \cdots, C_n \in \mathcal{B}(R),\ n \ge 1\},$$

where $\mathcal{B}(R)$ is the Borel σ-algebra generated by R. Since $\prod_{A_t} \subseteq \prod_{B_t}$, by Lemma 2.2, it follows

$$\mathcal{F}(A_u;\ u \le \alpha) \subseteq \mathcal{F}(B_u;\ u \le \alpha). \tag{2.2}$$

For every $A \in \mathcal{F}(A_u;\ u \le \alpha)$,

$$\int_A P(A_{t+\alpha} \in B | \mathcal{F}(A_u;\ u \le \alpha)) P(d\omega)$$

$$\overset{1}{=} \int_A \mathcal{X}_{\{A_{t+\alpha} \in B\}} P(d\omega)$$

$$\overset{2}{=} \int_{A\{A_0 \ne 0\}} \mathcal{X}_{\{B_{t+\alpha} \in B\}} P(d\omega)$$

$$\overset{3}{=} \int_A E[\mathcal{X}_{\{B_{t+\alpha} \in B\}} | \mathcal{F}(B_u;\ u \le \alpha)] P(d\omega)$$

$$\overset{4}{=} \int_A \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(B_{\alpha(\omega)}(\omega) - y)^2}{2t}} dy P(d\omega)$$

$$\overset{5}{=} \int_A p(\alpha(\omega),\ t + \alpha(\omega);\ A_{\alpha(\omega)}(\omega),\ B) P(d\omega),$$

where the first equality follows from the definition of conditional expectation; the second equality follows from the definition of $A_s(\omega)$, and $P(\{A_0 = 0\}) = 0$; the third equality follows from $A \in \mathcal{F}(A_u;\ u \le \alpha) \subseteq \mathcal{F}(B_u;\ u \le \alpha)$; the fourth equality follows from the strong Markov property of $B_s(\omega)$; and the last equality follows from $A_s(\omega) = B_s(\omega)$ for every $\omega \in \{A_0(\omega) \ne 0\}$ and $\{A_0 \ne 0\} = \Omega$, P-a.e.. Again, $p(\alpha(\omega),\ t + \alpha(\omega);\ A_{\alpha(\omega)}(\omega),\ B)$ is $\mathcal{F}(A_\alpha)$-measurable, and $\mathcal{F}(A_\alpha) \subseteq \mathcal{F}(A_u;\ u \le \alpha)$. So by Radon-Nikodym's theorem, we have

$$P(A_{t+\alpha} \in B|\mathcal{F}(A_u;\ u \le \alpha) = p(\alpha(\omega),\ t + \alpha(\omega);\ A_{\alpha(\omega)}(\omega),\ B),\ P_{\mathcal{N}_\alpha^+}\text{ - a.e.,}$$

(2.3)

which is the strong Markov property if the strong Markov property is defined by transition probability function $p(s,\ t;\ x,\ B)$. □

By [1, p. 343] and [7, Theorem 3.10], we know Brownian motion $B_s(\omega)$ is a strong Markov process. But we will prove that there always exists its a version $\overline{p}(s,\ t;\ x,\ B)$ of transition probability function such that the strong Markov property, defined by $\overline{p}(s,\ t;\ x,\ B)$, is not valid.

Theorem 2.4. *Let $B_s(\omega)$ be the Brownian motion, then there exists a version $\overline{p}(s,\ t;\ x,\ B)$ of transition probability function of $B_s(\omega)$ such that the strong Markov property, defined by $\overline{p}(s,\ t;\ x,\ B)$, is not valid.*

Proof. Suppose without loss of generality that $P(B_0(\omega) = 0) = 0$, we take, for any $0 \le s < t$,

$$\overline{p}(s,\ t;\ x,\ B) = \begin{cases} \mathcal{X}_B(0), & \text{if } x = 0, \\ \displaystyle\int_B \frac{1}{\sqrt{2\pi(t-s)}}\, e^{-\frac{(x-y)^2}{2(t-s)}}\, dy, & \text{if } x \ne 0. \end{cases}$$

From Example 1.1, we know $\overline{p}(s,\ t;\ x,\ B)$ is a version of transition probability function of $B_s(\omega)$. Let $\overline{\tau} = \inf(t > 0;\ B_t = 0)$. If we use the strong Markov property, we have

$$\int_B \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}}\, dy \overset{1}{=} p(0,\ t;\ x,\ B) \overset{2}{=} P_x(B_t \in B) \overset{3}{=} E_x[E_x(\mathcal{X}_{\{B_t \in B\}}|\mathcal{F}(B_u;\ u \le \overline{\tau}))]$$

$$\overset{4}{=} E_x[E_x(\mathcal{X}_{\{\overline{\tau} \le t\}}\mathcal{X}_{\{B_t \in B\}}|\mathcal{F}(B_u;\ u \le \overline{\tau})) + E_x(\mathcal{X}_{\{\overline{\tau} > t\}}\mathcal{X}_{\{B_t \in B\}}|\mathcal{F}(B_u;\ u \le \overline{\tau}))]$$

$$\overset{5}{=} E_x[\mathcal{X}_{\{\overline{\tau} \le t\}}\overline{p}(\overline{\tau},\ t;\ B_{\overline{\tau}},\ B) + \mathcal{X}_{\{\overline{\tau} > t\}}E_x(\mathcal{X}_{\{B_t \in B\}}|\mathcal{F}(B_u;\ u \le \overline{\tau}))]$$

$\geq E_x[\mathcal{X}_{\{\bar{\tau}\leq t\}}\mathcal{X}_B(0)] = P_x(\bar{\tau} \leq t)\mathcal{X}_B(0),$

contradiction. So B_s is not a strong Markov process. Here, the first term of the fifth equality follows from the strong Markov property. □

If the strong Markov property is defined by the transition probability function, from Theorems 2.3 and 2.4, we know whether the strong Markov property holds depends on the transition probability function selecting. So, we should further perfect the definition defined by transition probability function.

2.2. Annotations of the second example

The following we comment on Example 1.2.

Remark 2.5. (1) Although we analyze the example on basis of (1.4), we think it is more rational that (1.4) is replaced by $\sum_{r_p<x} T(r_p, \omega) \leq \tau < \sum_{r_p\leq x} T(r_p, \omega)$.

If $x \notin E_p \triangleq \{kp^{-n} | k = 0, 1, \cdots, p^n - 1;\ n = 1, 2, \cdots\}$ for fixed p, then $\sum_{r_p\leq x} T(r_p, \omega) = \sum_{r_p<x} T(r_p, \omega)$. By the central limit theorem, we know $\sum_{r_p<x} T(r_p, \omega)$ has a normal distribution. From which and (1.4), it follows, for every $\tau \geq 0$,

$$\{x\ (\tau, \omega) = x\} = \{\sum_{r_p<x} T(r_p, \omega) = \tau\} = \emptyset, \quad P_r \text{ - a.e..}$$

Thus, if $x \notin \bigcup_p E_p$, x may be seen the non-state of $x(\tau, \omega)$ since $P(x(\tau, \omega) = x) = 0$ for any τ. Another, if $x \in E_p$, then by (1.4) and the total probability formula, we have, for every $\tau \geq 0$,

$$P(x(\tau, \omega) = x) = P(\sum_{r_p<x} T(r_p, \omega) \leq \tau \leq \sum_{r_p\leq x} T(r_p, \omega))$$

$$= \int_0^{\tau} P(\tau - s \leq \sum_{r_p=x} T(r_p, \omega))dF(s) > 0,$$

where $F(s)$ is the distribution function of $\sum_{r_p<x} T(r_p, \omega)$. So, the minimal state space of $x(\tau, \omega)$ is $E \triangleq \bigcup_p E_p$. Obviously, E is countable. If $i \in E$, then there exists only p_0, such that $i \in E_{p_0}$ since E_p have only 0 in common (it is quoted from the example), in this case, by (1.4), we have

$$\{\omega : x(\tau, \omega) = i\} = \{\omega^{(p_0)} : x(\tau, \omega^{(p_0)}) = i, \omega^{(p_0)} \in \Omega^{(p_0)}\}, \quad P_r \text{ - a.e..} \tag{2.4}$$

(2) For every $x \in E$, suppose might as well that $x \in E_p$, By (1.4) and (2.4), it follows

$$\begin{aligned}
&\{\omega : x(t + \mathfrak{J}_I(\omega), \omega) = x\} \\
&\quad = \{\omega : \sum_{r_p<x} T(r_p, \omega) \le t + \mathfrak{J}_I(\omega) \le \sum_{r_p\le x} T(r_p, \omega)\} \\
&\quad = \begin{cases} \{\omega : \sum_{\frac{1}{2}\le r_p<x} T(r_p, \omega) \le t \le \sum_{\frac{1}{2}\le r_p\le x} T(r_p, \omega)\}, & \text{if } x \ge \dfrac{1}{2}, \\ \emptyset, & \text{if } x < \dfrac{1}{2}, \end{cases}
\end{aligned} \tag{2.5}$$

almost everywhere.

No matter whether the statements of the last paragraph of Example 1.2 is true or false, it does not show that the Markov property or the strong Markov property is not valid, because the conditions of the Markov property and strong Markov property are weaker than independence. In fact, the conditions of Markov property and strong Markov property require only the conditional independence instead of independence, that is, the "future" is independent of the "past" given the "present". In the same time, strong Markov property require that "time" is a stopping time instead of any a nonnegative random variable. But it does not prove that $\mathfrak{J}_I(\omega)$ is a stopping time in the example.

(3) In the end, we shall prove $x(t, \omega)$ is a strong Markov process if $x(t, \omega)$ is a Markov process.

Lemma 2.6. *Let* $p(s, t; x, A)$ *be the transition probability function of* $x(t, \omega)$. *Set*

$$\alpha^{(n)}(\omega) = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathcal{X}_{\{\frac{k-1}{2^n} < \alpha \le \frac{k}{2^n}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha > n\}}(\omega).$$

Then we have, for almost every ω_0 *with* $\omega_0 \in \{\alpha < t\}$,

$$\lim_{n\to\infty} p(\alpha^{(n)}(\omega_0), t; x(\alpha^{(n)}(\omega_0), \omega_0), A) = p(\alpha(\omega_0), t; x(\alpha(\omega_0), \omega_0), A).$$

Proof. For an arbitrary sequence of number $\{s_n; n \ge 1\}$ with $s_n \downarrow s_0$ as $n \uparrow \infty$, let

$$A_n = \{s_0 \le \sum_{r_p \le x} T(r_p, \omega)\} - \{s_n \le \sum_{r_p \le x} T(r_p, \omega)\}.$$

Then $A_n \downarrow \{s_0 = \sum_{r_p \le x} T(r_p, \omega)\} = \emptyset$ almost everywhere as $n \uparrow \infty$. Again,

$$\begin{aligned}
\{x(s_n, \omega) = x\} &= \{\sum_{r_p < x} T(r_p, \omega) \le s_n \le \sum_{r_p \le x} T(r_p, \omega)\} \\
&= \{\sum_{r_p < x} T(r_p, \omega) \le s_n\} \cap \{s_n \le \sum_{r_p \le x} T(r_p, \omega)\} \\
&= \{\sum_{r_p < x} T(r_p, \omega) \le s_n\} \cap [\{s_0 \le \sum_{r_p \le x} T(r_p, \omega)\} - A_n] \\
&= \{\sum_{r_p < x} T(r_p, \omega) \le s_n\} \cap \{s_0 \le \sum_{r_p \le x} T(r_p, \omega)\} \\
&\quad - \{\sum_{r_p < x} T(r_p, \omega) \le s_n\} \cap A_n.
\end{aligned}$$

Noting $\bigcap_{n=1}^{\infty}\{\sum_{r_p<x}T(r_p,\ \omega)\le s_n\}\cap A_n\ =\emptyset,\quad P_r$ - a.e., we have

$$\bigcap_{n=1}^{\infty}\{x(s_n,\ \omega)=x\}=\{x(s_0,\ \omega)=x\},\quad P_r \text{ - a.e..}$$

By the arbitrariness of $\{s_n;\ n\ge 1\}$, we have

$$\lim_{s\downarrow s_0}\{x(s,\ \omega)=x\}=\{x(s_0,\ \omega)=x\},\quad P_r \text{ - a.e..} \tag{2.6}$$

Again, since almost every path of $x(\tau,\ \omega)$ is continuous (it is quoted from the example), and the state space E is countable, there exists enough large N such that $x(\alpha^{(n)}(\omega_0),\ \omega_0)=x(\alpha(\omega_0),\ \omega_0)$ for every $n\ge N$, P_r - a.e.. So by $\alpha^{(n)}(\omega_0)\downarrow\alpha(\omega_0)$ as $n\uparrow\infty$, we have, for almost every $\omega_0\in\{\alpha<\infty\}$,

$$\lim_{n\to\infty}\{\omega:x(\alpha^{(n)}(\omega_0),\ \omega)=x(\alpha^{(n)}(\omega_0),\ \omega_0)\}$$

$$=\{\omega:x(\alpha(\omega_0),\ \omega)=x(\alpha(\omega_0),\ \omega_0)\},\quad P_r \text{ - a.e..}$$

By the definition of $x(\tau,\ \omega)$ and (2.4), we know

$$P_r(x(s,\ \omega)=x)=C_pP_r^{(p)}(\sum_{r_p<x}T(r_p,\ \omega^{(p)})\le s\le\sum_{r_p\le x}T(r_p,\ \omega^{(p)}))>0,$$

for every $s\ge 0$ and $x\in E$, where $x\in E_p$. Then, for almost every $\omega_0\in\{\alpha<t\}$,

$$p(\alpha(\omega_0),\ t;\ x(\alpha(\omega_0),\ \omega_0),\ A)$$

$$=\frac{P_r(x(t,\ \omega)\in A,\ x(\alpha(\omega_0),\ \omega)=x(\alpha(\omega_0),\ \omega_0))}{P_r(x(\alpha(\omega_0),\ \omega)=x(\alpha(\omega_0),\ \omega_0))}$$

$$=\frac{P_r(x(t,\ \omega)\in A,\ \lim_{n\to\infty}\{x(\alpha^{(n)}(\omega_0),\ \omega)=x(\alpha^{(n)}(\omega_0),\ \omega_0)\})}{P_r(\lim_{n\to\infty}\{x(\alpha^{(n)}(\omega_0),\ \omega)=x(\alpha^{(n)}(\omega_0),\ \omega_0)\})}$$

$$= \frac{\lim_{n\to\infty} P_r(x(t,\omega)\in A,\ \{x(\alpha^{(n)}(\omega_0),\omega) = x(\alpha^{(n)}(\omega_0),\omega_0)\})}{\lim_{n\to\infty} P_r(\{x(\alpha^{(n)}(\omega_0),\omega) = x(\alpha^{(n)}(\omega_0),\omega_0)\})}$$

$$= \lim_{n\to\infty} p(\alpha^{(n)}(\omega_0), t;\ x(\alpha^{(n)}(\omega_0),\omega_0),\ A). \qquad \square$$

Lemma 2.7. *Let* $\alpha(\omega)$ *be an arbitrary non negative random variable. Then*

$$p(\alpha(\omega), t;\ x(\alpha(\omega),\omega),\ A)\mathcal{X}_{\{\alpha(\omega)<t\}},$$

is measurable relative to $\mathcal{N}_\alpha^+$ *and* $\mathcal{F}(x(\alpha))$, *respectively*,

where $\mathcal{F}(x(\alpha)) \triangleq \{\{\omega : (x(\alpha(\omega),\omega),\alpha(\omega)) \in A\} : A \in \mathcal{E}\times\mathcal{B}([0,\infty])\};\ \mathcal{E} = \mathcal{B}(E)$.

Proof. Note

$$p(\alpha(\omega), t;\ x(\alpha(\omega),\omega),\ A)\mathcal{X}_{\{\alpha<t\}}(\omega) = \sum_{s<t}\sum_{x\in E} p(s, t;\ x,\ A)\mathcal{X}_{\{x(s,\omega)=x,\,\alpha=s\}}(\omega). \tag{2.7}$$

Since $\{u \le \alpha(\omega)\} = \{x(u,\omega)\in E,\ u\le\alpha(\omega)\}$, by Lemma 2.2, we obtain $\{u\le\alpha(\omega)\}\in\mathcal{N}_\alpha^+$ for every $u\ge 0$. Hence, $\{\alpha(\omega)<u\}\in\mathcal{N}_\alpha^+$ for every $u\ge 0$. Therefore,

$$\{x(s,\omega)=x,\ \alpha(\omega)=s\} = \lim_{u\downarrow s}\{x(s,\omega)=x,\ s\le\alpha(\omega)\}\bigcap\{\alpha(\omega)<u\}\in\mathcal{N}_\alpha^+.$$

From which and that E is at most countable yield

$$\sum_{x\in E} p(s, t;\ x,\ A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega),$$

is $\mathcal{N}_\alpha^+$-measurable for every $s<t$. The following we will prove

$$\sum_{s<t}\sum_{x\in E} p(s, t;\ x,\ A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega),$$

is $\mathcal{N}_\alpha^+$-measurable by means of transfinite induction.

Suppose that $\preceq$ is well ordering on $[0, t)$ with the first element a_0. Obviously,

$$\sum_{s\preceq a_0}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha(\omega)=s\}}(\omega) = \sum_{x\in E} p(a_0, t; x, A)\mathcal{X}_{\{x(a_0)=x,\,\alpha=a_0\}}(\omega),$$

which is $\mathcal{N}_\alpha^+$-measurable. Suppose that $\sum_{s\preceq a}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega)$ is $\mathcal{N}_\alpha^+$- measurable for every a with $a \prec T$. We take an increasing sequence $\{a_n; n \geq 1, a \prec T\}$ satisfying $\lim_{n\to\infty} a_n = T$, that is, for any given a number $a \prec T$, there exists one a_n such that $a \preceq a_n$. So, we have

$$\sum_{s\prec T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega) = \lim_{n\to\infty}\sum_{s\preceq a_n}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega).$$

So $\sum_{s\prec T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega)$ is $\mathcal{N}_\alpha^+$-measurable. Noting

$$\sum_{s\preceq T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega) = \sum_{s\prec T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega) + p(T, t; x, A)\mathcal{X}_{\{x(T)=x,\,\alpha=T\}}(\omega),$$

it follows $\sum_{s\preceq T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega)$ is $\mathcal{N}_\alpha^+$-measurable. Hence, by transfinite induction, we have, for every $T \in [0, t)$,

$$\sum_{s\preceq T}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega),$$

is $\mathcal{N}_\alpha^+$-measurable. Again, take an increasing sequence $\{T_n; n \geq 1\}$, which satisfies that for any given number $s \in [0, t)$, there exists a T_n such that $s \preceq T_n$. So,

$$\sum_{s<t}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega) = \lim_{n\to\infty}\sum_{s\preceq T_n}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega).$$

From which, it follows $\sum_{s<t}\sum_{x\in E} p(s, t; x, A)\mathcal{X}_{\{x(s)=x,\,\alpha=s\}}(\omega)$ is $\mathcal{N}_\alpha^+$-measurable.

Next, noting $\{x(s, \omega) = x, \alpha(\omega) = s\} \in \mathcal{F}(x(\alpha))$ for every $s < t$ from the definition of $\mathcal{F}(x(\alpha))$, by transfinite induction and same proving manner above, we know

$$p(\alpha(\omega), t; x(\alpha(\omega), \omega), A)\mathcal{X}_{\{\alpha<t\}}(\omega),$$

is also $\mathcal{F}(x(\alpha))$-measurable. □

Theorem 2.8. *Let* $x(t, \omega)$ *be the process constructed by Example* 1.2. *If* $x(t, \omega)$ *is a Markov process, then* $x(t, \omega)$ *must be strong Markov process, that is, for an arbitrary stopping time* $\alpha(\omega)$ *and* $A \subseteq E$, $t > 0$, *we have*

$$\begin{aligned} P_r(x(t, \omega) \in A|\mathcal{N}_\alpha^+) &= p(\alpha(\omega), t; x(\alpha(\omega), \omega), A) \\ &= P_r(x(t, \omega) \in A|\mathcal{F}(x(\alpha))), \quad (P_r)_{\{\alpha<t\}}\text{ - a.e.}, \end{aligned} \tag{2.8}$$

where $(P_r)_{\{\alpha<t\}}$ *means the equality holds for every* $\omega \in \{\alpha < t\}$ *except one null set of* $\{\alpha < t\}$.

Proof. Since $\alpha(\omega)$ is a stopping time, $\{\alpha(\omega) \le t\} \in \mathcal{N}_t^+ \triangleq \mathcal{F}(x(u); u \le t)$ for every $t \ge 0$. Set

$$\alpha^{(n)}(\omega) = \sum_{k=0}^{n2^n} \frac{k}{2^n}\mathcal{X}_{\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}}(\omega) + (n+1)\mathcal{X}_{\{\alpha>n\}}(\omega).$$

By the constructing method of measurable function, we have

$$\alpha^{(n)}(\omega) \downarrow \alpha(\omega), \quad \text{as } n \uparrow \infty.$$

For any $A_1, \cdots, A_m \subseteq E$, set $B = \{x(t_1, \omega) \in A_1, \cdots, x(t_m, \omega) \in A_m, s \le \alpha(\omega)\}$, $K(n) = \max\{k : \frac{k}{2^n} < t\}$. Obviously, $K(n) \uparrow t$ as $n \uparrow \infty$. Then, for every $A \subseteq E$,

$$\int_{B\{\alpha<t\}} \mathcal{X}_{\{x(t,\omega)\in A\}} P_r(d\omega) = \lim_{n\to\infty} \sum_{k=0}^{K(n)} \int_{B\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}} \mathcal{X}_{\{x(t,\omega)\in A\}} P_r(d\omega).$$

Since $\alpha(\omega)$ is a stopping time,

$$\{\frac{k-1}{2^n} < \alpha \le \frac{k}{2^n}\} = \{\alpha \le \frac{k}{2^n}\} - \{\alpha \le \frac{k-1}{2^n}\} \in \mathcal{N}^+_{\frac{k}{2^n}}. \tag{2.9}$$

If $k \ge \min(k : 0 \le k \le K(n), \frac{k}{2^n} \ge s)$, by $t_1 \le \cdots \le t_m \le s \le \frac{k}{2^n}$, it follows:

$$B\{\frac{k-1}{2^n} < \alpha \le \frac{k}{2^n}\} \in \mathcal{N}^+_{\frac{k}{2^n}}.$$

If $k < \min(k : 0 \le k \le K(n), \frac{k}{2^n} \ge s)$, obviously,

$$B\{\frac{k-1}{2^n} < \alpha \le \frac{k}{2^n}\} = \emptyset \in \mathcal{N}^+_{\frac{k}{2^n}}.$$

Therefore, taking enough large n such that $\frac{1}{2^n} < t$, by Markov property, it follows:

$$\int_{B\{\alpha<t\}} \mathcal{X}_{\{x(t,\omega)\in A\}} P_r(d\omega)$$

$$\overset{1}{=} \lim_{n\to\infty} \sum_{k=0}^{K(n)} \int_{B\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}} P_r(x(t, \omega) \in A | \mathcal{N}^+_{\frac{k}{2^n}}) P_r(d\omega)$$

$$\overset{2}{=} \lim_{n\to\infty} \sum_{k=0}^{K(n)} \int_{B\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}} P_r(x(t,\,\omega)\in A|\mathcal{F}(x(\frac{k}{2^n})))P_r(d\omega)$$

$$\overset{3}{=} \lim_{n\to\infty} \sum_{k=0}^{K(n)} \int_{B\{\frac{k-1}{2^n}<\alpha\le\frac{k}{2^n}\}} \sum_{x\in E} P_r(x(t,\,\omega)\in A|x(\frac{k}{2^n},\,\omega)=x)\mathcal{X}_{\{x(\frac{k}{2^n},\omega)=x\}}P_r(d\omega)$$

$$\overset{4}{=} \lim_{n\to\infty} \sum_{k=0}^{K(n)} \int_{B\{\alpha<t\}} \sum_{x\in E} P_r(x(t,\,\omega)\in A|x(\frac{k}{2^n},\,\omega)=x)\mathcal{X}_{\{\alpha^{(n)}=\frac{k}{2^n},\,x(\frac{k}{2^n},\omega)=x\}}P_r(d\omega)$$

$$\overset{5}{=} \lim_{n\to\infty} \int_{B\{\alpha<t\}} \sum_{k=0}^{K(n)} \sum_{x\in E} p(\frac{k}{2^n},\,t;\,x,\,A)\mathcal{X}_{\{\alpha^{(n)}=\frac{k}{2^n},\,x(\frac{k}{2^n},\omega)=x\}}P_r(d\omega)$$

$$\overset{6}{=} \int_{B\{\alpha<t\}} \lim_{n\to\infty} \sum_{k=0}^{K(n)} p(\frac{k}{2^n},\,t;\,x(\frac{k}{2^n},\,\omega),\,A)\mathcal{X}_{\{\alpha^{(n)}=\frac{k}{2^n}\}}P_r(d\omega)$$

$$\overset{7}{=} \int_{B\{\alpha<t\}} \lim_{n\to\infty} p(\alpha^{(n)}(\omega),\,t;\,x(\alpha^{(n)}(\omega),\,\omega),\,A)P_r(d\omega)$$

$$\overset{8}{=} \int_{B\{\alpha<t\}} p(\alpha(\omega),\,t;\,x(\alpha(\omega),\,\omega),\,A)P_r(d\omega), \tag{2.10}$$

where $\mathcal{F}(x(\frac{k}{2^n}))$ is the σ-algebra generated by $x(\frac{k}{2^n},\,\omega)$; $P_r(x(t,\,\omega)\in A|\mathcal{F}(x(\frac{k}{2^n})))=\sum_{x\in E}P_r(x(t,\,\omega)\in A|x(\frac{k}{2^n},\,\omega)=x)\,\mathcal{X}_{\{x(\frac{k}{2^n},\omega)=x\}}$ follows from [2, Theorem 5.2.5]; the first equality follows from the definition of conditional expectation; the second equality follows from Markov property; the third equality follows from [2, Theorem 5.2.5]; the fourth and fifth equalities follow from the properties of integral; the sixth equality follows from $\sum_{x\in E} p(\frac{k}{2^n},\,t;\,x,\,A)\mathcal{X}_{\{x(\frac{k}{2^n},\omega)=x\}} = p(\frac{k}{2^n},\,t;\,x(\frac{k}{2^n},\,\omega),\,A)$ and dominated convergence theorem.

Again, by λ - π -system method, we obtain (2.10) holds for every $B \in \mathcal{N}_\alpha^+$. So by Lemma 2.7 and the definition of conditional expectation, we obtain

$$P_r(x(t) \in A|\mathcal{N}_\alpha^+) = p(\alpha(\omega), t; x(\alpha(\omega), \omega), A), \quad (P_r)_{\{\alpha<t\}} \text{ - a.e..} \qquad (2.11)$$

Since $\mathcal{F}(x(\alpha)) \subseteq \mathcal{N}_\alpha^+$ by [11, Theorem 3.6], and $p(\alpha(\omega), t; x(\alpha(\omega), \omega), A)$ is $\mathcal{F}(x(\alpha))$-measurable from Lemma 2.7, taking conditional expectation given $\mathcal{F}(x(\alpha))$ at two side of (2.11), by the property of conditional expectation, we have

$$P_r(x(t) \in A|\mathcal{F}(x(\alpha))) = p(\alpha(\omega), t; x(\alpha(\omega), \omega), A), \quad (P_r)_{\{\alpha<t\}} \text{ - a.e..} \qquad (2.12)$$

By (2.11) and (2.12), we obtain (2.8). □

Starting from Theorem 2.8, by the same proving method of [11, Theorem 4.14], we can obtain the following theorem:

Theorem 2.9. *Let* $x(t, \omega)$ *be the process constructed by Example* 1.2. *If* $x(t, \omega)$ *is a Markov process, then* $x(t, \omega)$ *must be strong Markov process, that is, for an arbitrary stopping time* $\alpha(\omega)$ *and* $A \subseteq E$, $t > 0$, *we have*

$$\begin{aligned} P_r(x(t + \alpha(\omega), \omega) \in A|\mathcal{N}_\alpha^+) &= p(\alpha(\omega), t + \alpha(\omega); x(\alpha(\omega), \omega), A) \\ &= P_r(x(t + \alpha(\omega), \omega) \in A|\mathcal{F}(x(\alpha))), \quad (P_r)_{\Omega_\alpha} \text{ - a.e.,} \end{aligned} \qquad (2.8)$$

where $(P_r)_{\Omega_\alpha}$ *means the equality holds for every* $\omega \in \{\alpha < \infty\}$ *except one null set of* $\{\alpha < \infty\}$.

3. Appendix: Theorems and Concepts Cited or Needed in this Paper

3.1. Theorems cited in this paper

For the convenience to reader, we list all theorems cited in this paper.

Theorem 3.1 ([2, Theorem 5.2.5]). *Let* ξ *be a random variable defined on the probability space* $(\Omega, \mathcal{F}, P)$, $\mathcal{C}$ *be a* σ*-subalgebra of* $\mathcal{F}$ *and* B *and be an arbitrary atom of* $\mathcal{C}$. *Then, for any* $\omega \in B$,

$$E(\xi|\mathcal{C})(\omega) \equiv constant.$$

Further, if $P(B) > 0$, *then*

$$E(\xi|\mathcal{C})(\omega) = \frac{1}{P(B)} \int_B \xi dP,$$

for every $\omega \in B$.

Theorem 3.2 ([Dominated convergence theorem; 2, Theorem 3.3.2]). *Let g be an integrable function and* μ *be a measure,* $|f_n| \leq g$, *a.e. for all n.*

If $f_n \xrightarrow{a.e.} f$ *or* $f_n \xrightarrow{\mu} f$, *then* $\int f_n d\mu \to \int f d\mu$.

Theorem 3.3 ([Radon-Nikodym's theorem; 2, Theorem 3.7.6]). *Let* μ *be a* σ*-finite measure on* σ*-algebra* $\mathcal{A}$ *of* Ω. *If the set function* φ *defined on* $\mathcal{A}$ *is* σ*-finite and* σ*-additive and* μ*-continuous, then there exists an* $\mathcal{A}$*-measurable finite function f defined on* $(\Omega, \mathcal{A})$ *such that* φ *is the indefinite integral of f on the measurable space* $(\Omega, \mathcal{A}, \mu)$, *and f is* $\mu_{\mathcal{A}}$*-almost surly uniquely determined by* φ.

3.2. The concepts of λ-system

The following we will introduce the concepts of λ-system, and the λ - π-system method mentioned in this paper, which are cited from [1, Appendix].

Definition 3.4. A system $\prod$ of subsets of a set Ω is called a π-*system*, if $A_1 \in \prod$, $A_2 \in \prod \Rightarrow A_1 A_2 \in \prod$.

Definition 3.5. A system Λ of subsets of a set Ω is called a λ-*system*, if it has the following properties:

(1) $\Omega \in \Lambda$;

(2) $A_1 \in \Lambda$, $A_2 \in \Lambda$, $A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \Lambda$;

(3) $A_1 \in \Lambda$, $A_2 \in \Lambda$, $A_1 \supset A_2 \Rightarrow A_1 - A_2 \in \Lambda$;

(4) $A_n \in \Lambda$, $A_n \uparrow A$, $n = 1, 2, \cdots \Rightarrow A \in \Lambda$.

Theorem 3.6. (1) *If the system $\mathcal{M}$ of subsets of a set Ω is a π-system, and is also a λ-system, then $\mathcal{M}$ is a σ-algebra.*

(2) *If λ-system Λ contains π-system $\prod$, then $\Lambda \supseteq \mathcal{F}(\prod)$.*

If we make use of this theorem, we call this method λ - π-system method.

3.3. The concepts of partial ordering

We will introduce the concepts of partial ordering and three important theorem, which are seen in bibliographies about real analysis (such as [10]).

Definition 3.7. Let $\mathcal{S}$ be an arbitrary set. $\mathcal{S}$ is said to be a *partially ordered set*, if there is a binary relation "$\preceq$" called a *partial ordering*, defined on $\mathcal{S}$ with the following properties:

(1) $x \preceq x$ for all $x \in \mathcal{S}$ (reflexive),

(2) $x \preceq y$, $y \preceq z \Rightarrow x \preceq z$ for all x, y, $z \in \mathcal{S}$ (transitive),

(3) $x \preceq y$, $y \preceq x \Rightarrow x = y$ for all x, $y \in \mathcal{S}$ (antisymmetric).

Definition 3.8. A partially ordered set $\mathcal{S}$ is called a *totally ordered set*, if it follows $x \preceq y$ or $x \preceq y$ for any x, $y \in \mathcal{S}$.

Definition 3.9. Let $\mathcal{S}$ be a partially ordered set, x_0 lies in $\mathcal{S}$. x_0 is said to be the *maximal element* of $\mathcal{S}$, if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x_0 \preceq x$; x_0 is said to be the *minimal element* of $\mathcal{S}$, if it follows $x = x_0$ for every $x \in \mathcal{S}$ with $x \preceq x_0$.

Definition 3.10. Let $\mathcal{S}$ be a partially ordered set, $\mathcal{M}$ be a subset of $\mathcal{S}$, α lies in $\mathcal{S}$. α is said to be an *upper bound* of $\mathcal{M}$ in $\mathcal{S}$, if it follows $x \preceq \alpha$ for all $x \in \mathcal{M}$; α is said to be a *lower bound* of $\mathcal{M}$ in $\mathcal{S}$, if it follows $\alpha \preceq x$ for all $x \in \mathcal{M}$.

Definition 3.11. Let $\mathcal{S}$ be a partially ordered set, $\mathcal{A}$ be a subset of $\mathcal{S}$. α is called a *minimum element* of $\mathcal{A}$, if α is a lower bound of $\mathcal{A}$ and α lies in $\mathcal{A}$; α is called a *maximum element* of $\mathcal{A}$, if α is an upper bound of $\mathcal{A}$ and α lies in $\mathcal{A}$.

Definition 3.12. A partial ordering "$\preceq$" on $\mathcal{S}$ is said to be a *well ordering,* if for every nonempty subset of $\mathcal{S}$ has the minimum element. $\mathcal{S}$ is called *well-ordered set*, if there is a well ordering defined on $\mathcal{S}$.

Theorem 3.13 ([Zorn's lemma])**.** *Let $\mathcal{S}$ be a partially ordered set. If every totally ordered subset $\mathcal{A}$ of $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal element.*

Theorem 3.14 ([Well order theorem])**.** *Every set can be well ordered.*

Theorem 3.15 ([Principle of transfinite induction])**.** *Let $(W, \preceq)$ be a well-ordered set. For any $a \in W$, let*

$$I(a) = \{x \in W : x \prec a\}.$$

If A is a subset of W such that $a \in A$ whenever $I(a) \subset A$, then $A = W$.

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