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HYERS-ULAM STABILITY OF WILSON'S FUNCTIONAL EQUATION

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Abstract

We study the Hyers-Ulam stability problem for the Wilson's equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G,$$

and the superstability of the d'Alembert's equation

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G,$$

where G is a group; $f, g : G \to \mathbb{C}$ are complex valued functions; and σ is an involution of G without imposing the Kannappan's condition to the function f.

1. Introduction

In 1940, Ulam proposed the following problem [18]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(.,.)$ such that

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$$d(f(xy),\ f(x)f(y)) \leq \epsilon.$$

Then does there exist a group homomorphism L and $\delta_\epsilon > 0$ such that

$$d(f(x),\ L(x)) \leq \delta_\epsilon,$$

for all $x \in G_1$?

This problem was solved affirmatively by Hyers [11] under the assumption that G_2 is a Banach space. In 1978, Rassias [14] firstly generalized the above result and since then, stability problems of many other functional equations have been investigated [1-4, 8, 9, 11, 12, 16-19]. The interested reader should refer to [12] for a thorough account on the subject of stability of functional equations.

There is a strong stability phenomenon, which is known as a superstability. An equation of homomorphism is called *superstable*, if each approximate homomorphism is actually a true homomorphism. This property was first observed by Baker et al. [3] in the following theorem:

Theorem. *Let V be a vector space. If a function $f : V \to \mathbb{R}$ satisfies the inequality*

$$|f(x + y) - f(x)f(y)| \leq \varepsilon,$$

for some $\varepsilon > 0$ and for all x; $y \in V$. Then either f is a bounded function or

$$f(x + y) = f(x)f(y), \quad x,\ y \in V.$$

Later this result was generalized by Baker [2] and Székelyhidi [16]. In the present paper, we shall extend the investigation given by Baker [2], Székelyhidi [16], Badora [1], and Elqorachi and Akkouchi [8, 9] to Equations (1.1) and (1.2) below.

We recall that a complex-valued function f on a group G is said to satisfy Kannappan's condition (K), if

$$f(xyz) = f(xzy), \text{ for all } x,\ y,\ z \in G. \tag{K}$$

In previous works treating the stability of d'Alembert and Wilson equations, the authors have imposed Kannappan's condition (K). The present paper differs from the previous ones by its discovery, without any condition on f, of the Hyers-Ulam stability problem of the Wilson's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \tag{1.1}$$

and the superstability of d'Alembert's functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x, y \in G, \tag{1.2}$$

where G is a group and σ is an involution of G, i.e., $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$.

2. Stability of the Equation (1.1)

Lemma 1. *Let $\delta > 0$ be given. Assume that the functions $f; g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad x, y \in G, \tag{2.1}$$

such that $f \neq 0$. If g is unbounded, then so is f.

Proof. Assume that g is unbounded function satisfying the inequality (2.1). If $f \neq 0$ is bounded, let $M = \sup|f|$ and choose $a \in G$ such that $f(a) \neq 0$, then we get from the inequality (2.1) that

$$|f(ay) + f(a\sigma(y)) - 2f(a)g(y)| \leq \delta, \quad y \in G,$$

from which we obtain that

$$|2f(a)g(y)| - |f(ay) + f(a\sigma(y))| \leq \delta, \quad y \in G,$$

so we conclude that

$$|g(y)| \leq \frac{1}{2f(a)}(\delta + 2M), \text{ for all } y \in G,$$

then g is bounded which contradicts our assumption. □

In Theorem 1 below, the Hyers-Ulam stability of Equation (1.1) will be investigated without any additional condition (i.e., without the fact that f satisfy Kannappan's condition (K)).

Theorem 1. *Let* $\delta > 0$ *be given. Assume that the functions* $f;\ g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad x,\ y \in G, \tag{2.2}$$

then

(i) $f;\ g$ *are bounded or*

(ii) f *is unbounded and* g *satisfies the d'Alembert's long functional equation*

$$g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) = 4g(x)g(y), \tag{2.3}$$

or

(iii) g *is unbounded and* f *satisfies the Equation* (1.1) (*if* $f \neq 0$, *then* g *satisfies the Equation* (2.3)).

Proof. (ii) Assume that $f;\ g$ satisfy inequality (2.2). First, we consider the case of f unbounded. For all $x,\ y,\ z \in G$, we have

$$\begin{aligned}
&2|f(z)||g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) - 4g(x)g(y)| \\
&= |2f(z)g(xy) + 2f(z)g(x\sigma(y)) + 2f(z)g(yx) + 2f(z)g(\sigma(y)x) - 8f(z)g(x)g(y)| \\
&\leq |f(zxy) + f(z\sigma(y)\sigma(x)) - 2f(z)g(xy)| \\
&\quad + |f(zx\sigma(y)) + f(zy\sigma(x)) - 2f(z)g(x\sigma(y))| \\
&\quad + |f(zyx) + f(z\sigma(x)\sigma(y)) - 2f(z)g(yx)| \\
&\quad + |f(z\sigma(y)x) + f(z\sigma(x)y) - 2f(z)g(\sigma(y)x)| \\
&\quad + |f(zxy) + f(zx\sigma(y)) - 2f(zx)g(y)|
\end{aligned}$$

$$+ |f(zyx) + f(zy\sigma(x)) - 2f(zy)g(x)|$$

$$+ |f(z\sigma(y)x) + f(z\sigma(y)\sigma(x)) - 2f(z\sigma(y))g(x)|$$

$$+ |f(z\sigma(x)y) + f(z\sigma(x)\sigma(y)) - 2f(z\sigma(x))g(y)|$$

$$+ 2|g(y)||f(zx) + f(z\sigma(x)) - 2f(z)g(x)| + 2|g(x)||f(zy) + f(z\sigma(y)) - 2f(z)g(y)|.$$

By virtue of inequality (2.2), we have

$$2|f(z)||g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) - 4g(x)g(y)|$$

$$\leq 8\delta + 2(|g(y)| + |g(x)|)\delta. \tag{2.4}$$

Since f is unbounded, from the preceding (2.4), we conclude that g is a solution of the long d'Alembert equation (2.3), which ends the proof in this case.

(iii) If g is unbounded, then for $f = 0$ the pair (f, g) is a trivial solution of the Equation (1.1).

Now assume that $f \neq 0$. For all $x, y, z \in G$, we have

$$2|g(z)||f(xy) + f(x\sigma(y)) - 2f(x)g(y)|$$

$$= |2f(xy)g(z) + 2f(x\sigma(y))g(z) - 4f(x)g(y)g(z)|$$

$$\leq |f(xyz) + f(xy\sigma(z)) - 2f(xy)g(z)|$$

$$+ |f(x\sigma(y)z) + f(x\sigma(y)\sigma(z)) - 2f(x\sigma(y))g(z)|$$

$$+ |f(xyz) + f(x\sigma(z)\sigma(y)) - 2f(x)g(yz)|$$

$$+ |f(xy\sigma(z)) + f(xz\sigma(y)) - 2f(x)g(y\sigma(z))|$$

$$+ |f(x\sigma(y)z) + f(x\sigma(z)y) - 2f(x)g(\sigma(z)y)|$$

$$+ |f(x\sigma(y)\sigma(z)) + f(xzy) - 2f(x)g(zy)|$$

$$+ |f(x\sigma(z)y) + f(x\sigma(z)\sigma(y)) - 2f(x\sigma(z))g(y)|$$

$$+ |f(xzy) + f(xz\sigma(y)) - 2f(xz)g(y)|$$

$$+ |2f(x)g(yz) + 2f(x)g(y\sigma(z)) + 2f(x)g(\sigma(z)y) + 2f(x)g(zy) - 8f(x)g(y)g(z)|$$

$$+ |2f(xz)g(y) + 2f(x\sigma(z))g(y) - 4f(x)g(y)g(z)|.$$

In virtue of inequality (2.2), we obtain

$$2|g(z)|\,|f(xy) + f(x\sigma(y)) - 2f(x)g(y)|$$

$$\leq 8\delta + 2\delta|g(y)| + |2f(x)|\,|g(yz) + g(y\sigma(z)) + g(zy) + g(\sigma(z)y) - 4g(y)g(z)|.$$

By using Lemma 1, we see that g is unbounded implies necessarily that f is unbounded hence according to Theorem 1 (ii), g is a solution of the Equation (2.3). So, we conclude that

$$|2g(z)|\,|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq 8\delta + 2\delta|g(y)|. \tag{2.5}$$

Since g is unbounded, from the preceding (2.5), we see that f satisfies the Equation (1.1) and it is easy to get that if $f \neq 0$, then g satisfies the Equation (2.3), which finished the proof of the Theorem 1. □

As a consequence of Theorem 1, we have the following result on the superstability of the d'Alembert equation (1.2) mentioned, according to our knowledge, for the first time without imposing the condition (K) to the function f.

Corollary 1. *Let* $\delta > 0$ *be given. Assume that the functions* $f : X \to \mathbb{C}$ *satisfies the inequality*

$$|f(xy) + f(x\sigma(y)) - 2f(x)f(y)| \leq \delta, \quad x,\ y \in G, \tag{2.6}$$

then either

$$|f(x)| \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad x \in G,$$

or

$$f(xy) + f(x\sigma(y)) = 2f(x)f(y), \quad x,\ y \in G.$$

Proof. Assume that f satisfies the inequality (2.6). If f is bounded, let $A = \sup|f|$, then we get for all $x \in G$ that

$$|2f(x)f(x)| \leq \delta + 2A,$$

from which we obtain that $2A^2 - 2A - \delta \leq 0$ such that

$$A \leq \frac{1 + \sqrt{1 + 2\delta}}{2}, \quad x \in G,$$

the rest of the proof is an immediate consequence of Theorem 1. □

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