Journal of Mathematical Sciences: Advances and Applications
Volume 18, Number 1-2, 2012, Pages 63-83

SOME TIME-OPTIMAL CONTROL PROBLEMS FOR $n \times n$ CO-OPERATIVE HYPERBOLIC SYSTEMS WITH DISTRIBUTED OR BOUNDARY CONTROLS

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Abstract

In this paper, we consider various time-optimal control problems for $n \times n$ co-operative hyperbolic linear system involving Laplace operator with distributed or boundary controls and with observations act in position or in velocity. For each problem, the optimal controls are characterized in terms of an adjoint system and shown to be unique and bang-bang.

1. Introduction

Time-optimal control of distributed parameter systems governed by a system of hyperbolic equations is of special importance for the active control of structural systems for which, the equations of motion are generally expressed by hyperbolic differential equations. A typical application of a hyperbolic equation is the vibrating system. Time-optimal

2010 Mathematics Subject Classification: 35K35, 49K20, 49J20, 93C20.

Keywords and phrases: time-optimal control problems, bang-bang controls, hyperbolic system, co-operative systems.

Received December 23, 2012

control of distributed parameter systems governed by a system of hyperbolic equations have been studied in many papers, we mention only [2], [3] in which time optimal distributed control problems of vibrating systems has been studied. In our paper [4], the results in [2] and [3] have been extended to the time optimal control problems for systems governed by $n \times n$ hyperbolic systems, involving Laplace operator with different cases of observations.

In this paper, we will consider various time-optimal control problems for the following $n \times n$ co-operative linear hyperbolic system involving Laplace operator (here and everywhere below the vectors are denoted by bold letters):

$$\left.\begin{aligned} \frac{\partial^2 y_i}{\partial t^2}(x,\,t) - (A(t)\mathbf{y})_i &= u_i(x,\,t) && \text{in } Q = \Omega \times]0,\,T[,\\ y_i(x,\,0) &= y_{i,0}(x) && \text{in } \Omega,\\ \frac{\partial y_i}{\partial t}(x,\,0) &= y_{i,1}(x) && \text{in } \Omega,\\ \frac{\partial y_i}{\partial \nu} &= v_i(x,\,t) && \text{on } \Sigma = \Gamma \times]0,\,T[, \end{aligned}\right\} \tag{1}$$

where $\Omega \subset R^N$ is a bounded open domain with smooth boundary Γ, $\frac{\partial}{\partial \nu}$ is the normal derivative at Γ, towards the exterior of Ω, $y_{i,0}$, $y_{i,1}$ are given functions, u_i represents either a distributed control or a given function defined in Q, v_i represents either a boundary control or a given function defined in Q, and $A(t)(t \in]0,\,T[)$ are a family of $n \times n$ continuous matrix operators

$$A(t)y = \begin{pmatrix} \Delta + a_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & \Delta + a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \Delta + a_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

with co-operative coefficient functions a_i, a_{ij} satisfying the following conditions:

$$\left.\begin{aligned} &a_i,\ a_{ij} \text{ are positive functions in } L^\infty(Q),\\ &a_{ij} = a_{ji} \text{ (symmetry conditions)},\\ &a_{ij}(x, t) \le \sqrt{a_i(x, t)a_j(x, t)}. \end{aligned}\right\} \tag{2}$$

This problems are, steering the initial vector state $(\mathbf{y}(0), \mathbf{y}'(0))$ for system (1), with a control $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ belonging to a given control set U_Q^n or with a vector control $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ belonging to a given control set U_Σ^n so that an observation $\mathbf{y}(t)$ or $\mathbf{y}'(t)$ hitting a given target set K_ε^n in minimum time

$$\begin{aligned} U_Q^n &= \{\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in (L^2(Q))^n : \|\phi_i\|_{L^2(Q)} \le \epsilon\},\\ U_\Sigma^n &= \{\phi = (\phi_1, \phi_2, \ldots, \phi_n) \in (L^2(\Sigma))^n : \|\phi_i\|_{L^2(\Sigma)} \le \epsilon\},\\ K_\varepsilon^n &= \{z = (z_1, z_2, \ldots, z_n) \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} \le \epsilon\}, \end{aligned} \tag{3}$$

and ϵ, $\varepsilon > 0$ and $z_{id} \in L^2(\Omega)$ are given.

First, we establish the well posedness of the system (1) under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem. Then, we formulate various time optimal control problems with distributed or boundary controls. In each problem, we derive the necessary and sufficient conditions, which the optimal controls must satisfy in terms of the adjoint.

2. $n \times n$ Co-operative Hyperbolic Systems

Let $H^1(\Omega)$ ([5]) be the usual Sobolev space of order one which consists of all $\phi \in L^2(\Omega)$, whose distributional derivatives $\frac{\partial \phi}{\partial x_i} \in L^2(\Omega)$ with the scalar product norm

$$\langle y, \phi \rangle_{H^1(\Omega)} = \langle y, \phi \rangle_{L^2(\Omega)} + \langle \nabla y, \nabla \phi \rangle_{L^2(\Omega)}, \quad \text{where } \nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}.$$

For $\mathbf{y} = (y_i)_{i=1}^n$, $\phi = (\phi_i)_{i=1}^n \in (H^1(\Omega))^n$, and $t \in]0, T[$, let us define a family of continues bilinear forms

$$\pi(t; ., .) : (H^1(\Omega))^n \times (H^1(\Omega))^n \to \Re \text{ by}$$

$$\pi(t; \mathbf{y}, \phi) = \sum_{i=1}^{n} \int_\Omega [(\nabla y_i)(\nabla \phi_i) - a_i(x, t) y_i \phi_i] dx - 2 \sum_{i>j}^{n} \int_\Omega a_{ij}(x, t) y_j \phi_i dx$$

$$= \sum_{i=1}^{n} \int_\Omega [(-\Delta y_i) - a_i(x, t) y_i] \phi_i dx - 2 \sum_{i>j}^{n} \int_\Omega a_{ij}(x, t) y_j \phi_i dx$$

$$= \sum_{i=1}^{n} \langle -(A(t)\mathbf{y})_i, \phi \rangle_{L^2(\Omega)}. \tag{4}$$

Lemma 1. *If Ω is a regular bounded domain in R^N, with boundary Γ, and if m is positive on Ω and smooth enough (in particular, $m \in L^\infty(\Omega)$), then the eigenvalue problem*

$$\left.\begin{aligned} -\Delta y = \lambda m(x) y \qquad & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = 0 \qquad & \text{on } \Gamma, \end{aligned}\right\},$$

possesses an infinite sequence of positive eigenvalues

$$0 < \lambda_1(m) < \lambda_2(m) \le \ldots \lambda_k(m) \ldots; \ \lambda_k(m) \to \infty, \quad \text{as } k \to \infty.$$

Moreover, $\lambda_1(m)$ *is simple, its associate eigenfunction* e_m *is positive, and* $\lambda_1(m)$ *is characterized by*

$$\lambda_1(m)\int_\Omega my^2 dx \le \int_\Omega |\nabla y|^2 dx. \tag{5}$$

Proof. See [6]. □

Now, let

$$\lambda_1(a_i) \ge n, \quad i = 1, 2, \ldots, n. \tag{6}$$

Lemma 2. *If* (2) *and* (6) *hold, then the bilinear form* (4) *satisfy the Gårding inequality*

$$\pi(t; \mathbf{y}, \mathbf{y}) + c_0\|y\|^2_{(L^2(\Omega))^n} \ge c_1\|y\|^2_{(H^1(\Omega))^n}, \quad c_0, c_1 > 0.$$

Proof. In fact,

$$\pi(t; \mathbf{y}, \mathbf{y}) = \sum_{i=1}^n \int_\Omega \left[|\nabla y_i|^2 - a_i(x, t)y_i^2\right]dx - \sum_{i,j=1}^n \int_\Omega a_{ij}(x, t)y_i y_j dx$$

$$\ge \sum_{i=1}^n \int_\Omega \left[|\nabla y_i|^2 - a_i(x, t)y_i^2\right]dx - 2\sum_{i>j}^n \int_\Omega \sqrt{a_i(x, t)a_j(x, t)}y_i y_j dx.$$

By Cauchy Schwartz inequality and (5), we obtain

$$\pi(t; \mathbf{y}, \mathbf{y}) \ge \sum_{i=1}^n \left(1 - \frac{1}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx$$

$$- 2\sum_{i>j}^n \int_\Omega \left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right)\left(\int_\Omega |\nabla y_i|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla y_j|^2 dx\right)^{\frac{1}{2}}$$

$$\ge \sum_{i=1}^n \left(\frac{\lambda_1(a_i) - n}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx.$$

From (6), we have

$$\pi(t;\ \mathbf{y},\ \mathbf{y}) \geq \alpha \left[\sum_{i=1}^{n} \int_{\Omega} |\nabla y_i|^2 dx \right], \quad \alpha > 0.$$

Add $\|\mathbf{y}\|_{(L^2(\Omega))^n}$ to two sides, then we have the result. □

For optimal control problems, it is of importance to consider the cases where the control u_i or v_i belongs to $L^2(Q)$ or $L^2(\Sigma)$. For these cases, we have the following results (by apply Theorem 1.1 and Remark 1.3, Chapter 4 in [1] with $V = (H^1(\Omega))^n$ and $H = (L^2(\Omega))^n$) :

Theorem 1. *Let* (2), (6) *be hold and let* $y_{i,0},\ y_{i,1},\ u_i,\ v_i$ *be given with*

$$y_{i,0} \in H^1(\Omega), \quad y_{i,1} \in L^2(\Omega), \quad u_i \in L^2(Q), \quad v_i \in H^{-1}(\Sigma).$$

Then, there exist a unique solution $\mathbf{y} \in \left\{ \mathbf{y} : \mathbf{y} \in L^2(0,\ T;\ (H^1(\Omega))^n),\ \frac{\partial \mathbf{y}}{\partial t} \in (L^2(Q))^n \right\}$ *satisfying the Neumann problem* (1). *Moreover,* $\mathbf{y}$ *is continuous from* $[0,\ T] \to (H^1(\Omega))^n$ *and* $\frac{\partial \mathbf{y}}{\partial t}$ *is continuous from* $[0,\ T] \to (L^2(\Omega))^n$.

By transposition (see [1], [7]), we deduce the following:

Theorem 2 (Transposition theorem)**.** *Let* (2), (6) *be hold and let* $y_{i,0},\ y_{i,1},\ u_i,\ v_i$ *be given with*

$$y_{i,0} \in L^2(\Omega), \quad y_{i,1} \in H^{-1}(\Omega), \quad u_i \in L^1(0,\ T;\ H^{-1}(\Omega)), \quad v_i \in L^2(\Sigma).$$

Then, there exist a unique solution $\mathbf{y} \in (L^2(Q))^n$ *satisfying the Neumann problem* (1) *such that*

$$\int_Q y_i \left[\frac{\partial^2 \phi_i}{\partial t^2} - (A(t)\phi)_i \right] dxdt = \int_\Omega y_{i,1}\phi_i(0)dx - \int_\Omega y_{i,0}\phi_i'(0)dx, \quad \forall \phi \in X, \tag{7}$$

$$X = \left\{ \begin{array}{l} \phi = (\phi)_{i=1}^{n} \in L^2(0, T; (H^1(\Omega))^n), \quad \dfrac{\partial \phi_i}{\partial t} \in L^2(Q), \\ \dfrac{\partial^2 \phi_i}{\partial t^2} + (A(t)\phi)_i \in L^2(Q), \\ \phi_i(x, T) = 0, \quad \dfrac{\partial \phi_i}{\partial t}(x, T) = 0, \quad \left.\dfrac{\partial \phi_i}{\partial \nu}\right|_{\Sigma} = 0. \end{array} \right\}.$$

In the next sections, we will denote by $\mathbf{y}(t; \mathbf{u})$ to the unique solution of (1), at time t corresponding to a given control $\mathbf{u} \in U_Q^n$ and a given functions $y_{i,0}, y_{i,1}, u_i, v_i$ satisfying the hypothesis of Theorem 1. Similarly, we will denote by $\mathbf{y}(t; \mathbf{v})$ to the unique solution of (1), at time t corresponding to a given control $\mathbf{v} \in U_\Sigma^n$ and a given functions $y_{i,0}$, $y_{i,1}, u_i, v_i$ satisfying the hypothesis of Theorem 2. Occasionally, we write $\mathbf{y}(x, t; \mathbf{u})$ or $\mathbf{y}(x, t; \mathbf{v})$ when the explicit dependence on x is required.

3. Distributed Control -Position Observation Problem

In this section, we consider the following first time-optimal control problem with distributed control $\mathbf{u}$ and position observation $\mathbf{y}(x, t; \mathbf{u})$:

$$\text{(TOP1):} \qquad \min\{t : \mathbf{y}(x, t; \mathbf{u}) \in K_\varepsilon^n, \mathbf{u} \in U_Q^n\}.$$

Theorem 3. *If* (2) *and* (6) *are hold, then the system whose state is given by* (1) *is controllable* ([8], [9]),

$$\textit{i.e., there exists } a\ \tau \in]0, T] \textit{ and } \mathbf{u} \in U_Q^n \textit{ with } \mathbf{y}(\tau; \mathbf{u}) \in K_\varepsilon^n. \qquad (8)$$

Proof. Let us first remark that by translation, we may always reduce the problem of controllability to the case where the system (1) with $y_{i,0} = y_{i,1} = v_i = 0$. We can show quite easily that (1) is approximately controllable in $(L^2(\Omega))^n$ in any finite time $\tau > 0$, if and only if $\{\mathbf{y}(\tau; \mathbf{u}) : \mathbf{u} \in (L^2(Q))^n\}$ is dense in $(L^2(\Omega))^n$. By the Hahn-Banach theorem, this will be the case if

$$\int_\Omega \bar{z}_i(x) y_i(x, \tau; \mathbf{u}) dx = 0, \quad \bar{z}_i \in L^2(\Omega), \tag{9}$$

for all $\mathbf{u} \in (L^2(Q))^n$, implies that $\bar{z}_i(x) = 0$, $i = 1, 2, \ldots, n$.

Let us introduce the adjoint state $p(t; \mathbf{u})$ by the solution of the following system:

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2}(t; \mathbf{u}) - (A(t)\mathbf{p}(t; \mathbf{u}))_i = 0 && \text{in } \Omega \times]0, T[, \\ &p_i(x, \tau) = 0 && \text{in } \Omega, \\ &\frac{\partial p_i}{\partial t}(x, \tau) = -\bar{z}_i(x) && \text{in } \Omega, \\ &\frac{\partial p_i}{\partial \nu}(x, t) = 0. && \text{on } \Gamma \times]0, \tau[. \end{aligned}\right\} \tag{10}$$

The existence of a unique solution for the problem (23) can be proved by using Theorem 1 with an obvious change of variables.

Multiply the first equation in (23) by $y_i(t; \mathbf{u})$ and using Green formula, we obtain the following identity:

$$\begin{aligned} 0 &= \int_0^\tau \int_\Omega \left[\frac{\partial^2 p_i}{\partial t^2} + (A(t)\mathbf{p}(t; \mathbf{u}))_i \right] y_i(t; \mathbf{u}) dx dt \\ &= \int_\Omega \frac{\partial p_i}{\partial t} y_i(t; \mathbf{u}) \Big|_0^\tau dx - \int_\Omega p_i \frac{\partial}{\partial t} y_i(t; \mathbf{u}) \Big|_0^\tau dx \\ &\quad + \int_0^\tau \int_\Omega p_i \left[\frac{\partial^2}{\partial t^2} y_i(t; \mathbf{u}) + (A(t) y(t; \mathbf{u}))_i \right] dx dt + \int_0^\tau \int_\Gamma \sum_{i=1}^n \frac{\partial p_i}{\partial \nu} y_i d\Gamma dt \\ &= \int_\Omega \bar{z}_i(x) y_i(x, \tau; \mathbf{u}) dx - \int_0^\tau \int_\Omega p_i(x, t; \mathbf{u}) u_i dx dt, \end{aligned}$$

and so, if (9) holds, then

$$\int_0^\tau \int_\Omega p_i(\tau; \mathbf{u}) u_i dx dt = 0, \quad \forall u_i \in L^2(Q),$$

hence $p_i(\tau; \mathbf{u}) = 0$. But from the continuity property, $p_i(\tau; \mathbf{u}) \equiv \frac{\partial p_i}{\partial t}(\tau; \mathbf{u}) \equiv 0$ and hence $\bar{z}_i(x) = 0$. □

Now, set

$$\tau_1^0 = \inf\{\tau : \mathbf{y}(\tau; \mathbf{u}) \in K_\varepsilon^n \text{ for some } \mathbf{u} \in U_Q^n\}. \tag{11}$$

The following result holds:

Theorem 4. *If* (2), (6) *are hold, then there exist an admissible control* $\mathbf{u}^0$ *to the problem* (*TOP*1), *which steering* $\mathbf{y}(t; \mathbf{u}^0)$ *to hitting a target set* K_ε^n *in minimum time* τ^0 (*defined by* (11)). *Moreover,*

$$\sum_{i=1}^{n} \int_\Omega (y_i(\tau^0; \mathbf{u}^0) - z_{id})(y_i(\tau^0; \mathbf{u}) - y_i(\tau^0; \mathbf{u}^0))dx \geq 0, \quad \forall \mathbf{u} \in U_Q^n. \tag{12}$$

Proof. Fixed x, we can choose $\tau^m \to \tau_1^0$ and admissible controls $\{\mathbf{u}^m\}$ such that

$$\mathbf{y}(\tau^m; \mathbf{u}^m) \in K_\varepsilon^n, \quad m = 1, 2, \ldots.$$

Set $\mathbf{y}^m = \mathbf{y}(\mathbf{u}^m)$. Since U_ϵ^n is bounded, we may verify that $\mathbf{y}^m$ (respectively, $\frac{d\mathbf{y}}{dt}$) ranges in a bounded set in $(L^2(0, T; (H^1(\Omega))^n))$ (respectively, $(L^2(0, T; (L^2(\Omega))^n) = (L^2(Q))^n)$.

We may then extract a subsequence, again denoted by $\{\mathbf{u}^m, \mathbf{y}^m\}$ such that

$$\left.\begin{aligned} &\mathbf{u}^m \to \mathbf{u}^0 \text{ weakly in } (L^2(Q))^n,\ u^0 \in U_\epsilon^n, \\ &\mathbf{y}^m \to \mathbf{y} \text{ weakly in } L^2\left(0, T; (H^1(\Omega))^n\right), \\ &\frac{d\mathbf{y}^m}{dt} \to \text{ weakly in } (L^2(Q))^n. \end{aligned}\right\} \tag{13}$$

We deduce from the equality

$$\frac{d^2\mathbf{y}^m}{dt^2} = \mathbf{u}^m - A(t)\mathbf{y}^m,$$

that

$$\frac{d^2\mathbf{y}^m}{dt^2} \to \frac{d^2\mathbf{y}}{dt^2} = \mathbf{u}^0 - A(t)\mathbf{y} \quad \text{in } L^2\big(0, T; (H^{-1}(\Omega))^n\big),$$

and

$$\mathbf{y}(0) = \mathbf{y}_0 \qquad \frac{d\mathbf{y}}{dt}(0) = \mathbf{y}_1.$$

But

$$\mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau_1^0; \mathbf{u}^0) = \mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau_1^0; \mathbf{u}^m) + \mathbf{y}(\tau_1^0; \mathbf{u}^m) - \mathbf{y}(\tau_1^0; \mathbf{u}^0),$$

then from (13), we have

$$\mathbf{y}(\tau_1^0; \mathbf{u}^m) \to \mathbf{y}(\tau_1^0; \mathbf{u}^0) \text{ weakly in } (H^1(\Omega))^n, \tag{14}$$

and

$$\|\mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau_1^0; \mathbf{u}^m)\|_{(L^2(\Omega))^n} = \left\| \int_{\tau^0}^{\tau^m} \frac{d}{dt}\mathbf{y}(t; \mathbf{u}^m)dt \right\|_{(L^2(\Omega))^n}$$

$$\leq \sqrt{\tau^m - \tau_1^0} \left(\int_{\tau_1^0}^{\tau^m} \left\| \frac{d}{dt}\mathbf{y}(t; \mathbf{u}^m)dt \right\|^2_{(L^2(\Omega))^n} dt \right)^{\frac{1}{2}}$$

$$\leq c\sqrt{\tau_n - \tau_1^0}. \tag{15}$$

Combine (14) and (15) show that

$$\mathbf{y}(\tau^m; \mathbf{u}^m) - \mathbf{y}(\tau_1^0; \mathbf{u}^0) \to 0 \quad \text{weakly in } (L^2(\Omega))^n. \tag{16}$$

Similarly, we can verify that

$$\mathbf{y}'(\tau^m; \mathbf{u}^m) - \mathbf{y}'(\tau_1^0; \mathbf{u}^0) \to 0 \quad \text{weakly in } (H^{-1}(\Omega))^n. \tag{17}$$

and so, $\mathbf{y}(\tau_1^0;\, \mathbf{u}^0) \in K_\varepsilon^n$ as K_ε^n is closed and convex, hence weakly closed. This shows that K_ε^n is reached in time τ_1^0 by admissible control $\mathbf{u}^0$.

For the second part of the theorem, really, from Theorem 1, the mapping $t \to \mathbf{y}(t;\, \mathbf{u})$ and $t \to \mathbf{y}'(t;\, \mathbf{u})$ from $[0,\, T] \to (H^1(\Omega))^n$ and $[0,\, T] \to (L^2(\Omega))^n$, respectively, are continuous for each fixed $\mathbf{u}$ and so $\mathbf{y}(\tau_1^0;\, \mathbf{u}) \notin \operatorname{int} K_\varepsilon^n$, for any $\mathbf{u} \in U_\epsilon^n$, by minimality of τ_1^0.

Using Theorem 1, it is easy to verify that the mapping $\mathbf{u} \to \mathbf{y}(\tau_1^0;\, \mathbf{u})$, from $(L^2(Q))^n \to (L^2(\Omega))^n$, is continuous and linear, then the set

$$\mathcal{A}(\tau_1^0) = \{\mathbf{y}(\tau_1^0;\, \mathbf{u}) : \mathbf{u} \in U_\epsilon^n\},$$

is the image under a linear mapping of a convex set hence $\mathcal{A}(\tau_1^0)$ is convex. Thus, we have $\mathcal{A}(\tau_1^0) \bigcap \operatorname{int} K_\varepsilon^n = \emptyset$ and $\mathbf{y}(\tau_1^0;\, \mathbf{u}^0) \in \partial K_\varepsilon^n$ (boundary of K_ε^n). Since $\operatorname{int} K_\varepsilon^n \neq \emptyset$ (from (8)), so there exists a closed hyperplane separating $A(\tau_1^0)$ and K_ε^n containing $\mathbf{y}(\tau_1^0;\, \mathbf{u}^0)$, i.e., there is a nonzero $\mathbf{g} \in (L^2(\Omega))^n$ such as

$$\sup_{\mathbf{y} \in A(\tau_1^0)} \left\langle \mathbf{g},\, \mathbf{y}(\tau_1^0;\, \mathbf{u}) \right\rangle_{(L^2(\Omega))^n} \leq \left\langle \mathbf{g},\, \mathbf{y}(\tau^0;\, \mathbf{u}^0) \right\rangle_{(L^2(\Omega))^n}$$

$$\leq \inf_{\mathbf{y} \in K_\varepsilon^n} \left\langle \mathbf{g},\, \mathbf{y}(\tau_1^0;\, \mathbf{u}) \right\rangle_{(L^2(\Omega))^n}. \tag{18}$$

From the second inequality in (22), $\mathbf{g}$ must support the set K_ε^n at $\mathbf{y}(\tau^0;\, \mathbf{u}^0)$, i.e.,

$$\langle \mathbf{g},\, (\mathbf{y}(\tau_1^0;\, \mathbf{u}) - \mathbf{y}(\tau_1^0;\, \mathbf{u}^0)) \rangle_{(L^2(\Omega))^n} \geq 0, \quad \forall \mathbf{u} \in U_\epsilon^n,$$

and since $(L^2(\Omega))^n$ is a Hilbert space, $\mathbf{g}$ must be of the form

$$\mathbf{g} = \lambda(\mathbf{y}(\tau_1^0;\, \mathbf{u}^0) - z_{id}) \text{ for some } \lambda > 0.$$

Dividing the inequality (22) by λ gives the desired result. □

The above condition (12) can be simplified by introducing the following adjoint equation. For each $\mathbf{u}^0 \in U_\epsilon^n$, we define $\boldsymbol{p}(x, t; \mathbf{u}^0)$ as the solution of the following system:

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2}(t; \mathbf{u}^0) - (A(t)\mathbf{p}(t; \mathbf{u}^0))_i = 0 && \text{in } \Omega \times]0, \tau_1^0[, \\ &p_i(x, \tau_1^0; \mathbf{u}^0) = 0 && \text{in } \Omega, \\ &p_i'(x, \tau_1^0; \mathbf{u}^0) = -(y_i(x, \tau_1^0; \mathbf{u}^0) - z_{id}) && \text{in } \Omega, \\ &\frac{\partial p_i}{\partial \nu}(x, t; \mathbf{u}^0) = 0 && \text{on } \Gamma \times]0, \tau_1^0[. \end{aligned}\right\} \tag{19}$$

As the proof of Theorem 3, we multiply the first equation in (23) by $y_i(t; \mathbf{u}) - y_i(t; \mathbf{u}^0)$ and integrate by parts from 0 to τ_1^0, we obtain the following identity:

$$\int_\Omega (y_i(x, \tau_1^0; \mathbf{u}^0) - z_{id})(y_i(x, \tau_1^0; \mathbf{u}) - y_i(x, \tau_1^0; \mathbf{u}^0))dx$$

$$= \int_0^{\tau_1^0} \int_\Omega p_i(u_i - u_i^0)dxdt.$$

Condition (12) then becomes

$$\sum_{i=1}^n \int_0^{\tau_1^0} \int_\Omega p_i(u_i - u_i^0)dxdt \geq 0, \quad \forall \mathbf{u} \in U_Q^n. \tag{20}$$

This result can be summarized as

Theorem 5. *We assume that* (2), (6) *hold. Then, there exist the adjoint state*

$$\mathbf{p} \in \left\{\mathbf{p} : \mathbf{p} \in L^2(0, \tau_1^0; (H^1(\Omega))^n), \frac{\partial \mathbf{p}}{\partial t} \in (L^2(\Omega))^n)\right\},$$

such that the optimal control $\mathbf{u}^0$ *of problem* (*TOP*1) *is characterized by* (19),(20) *together with* (1) (*with* $u_i = u_i^0$, $i = 1, 2, \ldots, n$).

The maximum conditions (20) of the optimal control leads to the following result:

Theorem 6 (Bang-bang theorem)**.** *We assume that* (2), (6) *hold. If the coefficients of the operator* $A(t)$ *are analytic in* $\overline{\Omega} \times [0, T]$ *and if* Ω *has analytic boundary, then the optimal control of* (*TOP*1) *is bang-bang* ($|u_i^0(x, t)| \equiv 1$, *a.e.*), *unique, and it is the unique solution of* (19), (20) *together with* (1) (*with* $u_i = u_i^0$, $i = 1, 2, \ldots, n$).

Proof. The theorem will follow from Theorem 12 if we can show that $p_i(x, t) \neq 0$ for almost all $(x, t) \in \Omega \times]0, \tau_1^0[$, $i = 1, 2, \ldots, n$. Accordingly, suppose that

$$p_i(x, t) = 0, \quad (x, t) \in \Omega \times]0, \tau_1^0[. \tag{21}$$

In $\Omega \times]0, \tau_1^0[$, $\mathbf{p}(x, t; \mathbf{u}^0)$, satisfies

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2}(t; \mathbf{u}^0) - (A(t)\mathbf{p}(t; \mathbf{u}^0))_i = 0 \qquad \text{in } \Omega \times]0, \tau_1^0[, \\ &\frac{\partial p_i}{\partial \nu}(x, t; \mathbf{u}^0) = 0 \qquad\qquad\qquad\quad \text{on } \Gamma \times]0, \tau_1^0[, \end{aligned}\right\},$$

and so by [12], $\mathbf{p}(x, t; \mathbf{u}^0)$ must be analytic in $\Omega \times]0, \tau_1^0[$. As $\mathbf{p}(x, t; \mathbf{u}^0)$ is zero in $\Omega \times]0, \tau_1^0[$, it must be identically zero in $\overline{\Omega} \times]0, \tau_1^0[$, From Theorem 1, the mapping $t \to p_i(t; \mathbf{u})$ is continuous from $[0, T] \to H^1(\Omega)$ and $t \to p_i'(t; \mathbf{u})$ is continuous from $[0, T] \to L^2(\Omega)$, and so

$$p_i'(x, \tau_1^0; \mathbf{u}^0) = -(y_i(x, \tau_1^0; \mathbf{u}^0) - z_{id}) = 0,$$

which contradicts the fact that $y_i(x, \tau_1^0; \mathbf{u}^0) \neq z_{id}$.

Since U_Q^n is strictly convex, then the optimal control is unique.

□

4. Distributed Control -Velocity Observation Problem

In this section, we consider the following second time-optimal control problem with distributed control $\mathbf{u}$ and velocity observation $\mathbf{y}'(x, t; \mathbf{u})$:

$$\text{(TOP2)}: \qquad \min\{t : \mathbf{y}'(x, t; \mathbf{u}) \in K_\varepsilon^n, \mathbf{u} \in U_Q^n\}.$$

As in the above section, we can prove that for large T, the following controllability condition is hold:

$$\text{There exists a } \tau \in]0, T] \text{ and } \mathbf{u} \in U_Q^n \text{ with } \mathbf{y}'(\tau; \mathbf{u}) \in K_\varepsilon^n, \tag{22}$$

and if we set

$$\tau_2^0 = \inf\{\tau : \mathbf{y}'(\tau; \mathbf{u}) \in K_\varepsilon^n \text{ for some } \mathbf{u} \in U_Q^n\}, \tag{23}$$

then similar to (TOP1), we can prove the following theorem:

Theorem 7. *If* (2) *and* (6) *are hold, then there exist an admissible control* $\mathbf{u}^0$ *to the problem* (*TOP*2), *which steering* $\mathbf{y}'(t; \mathbf{u}^0)$ *to hitting a target set* K_ε^n *in minimum time* τ_2^0 (*defined by* (23)). *Moreover*,

$$\sum_{i=1}^{n} \int_\Omega (y_i'(\tau^0; \mathbf{u}^0) - z_{id})(y_i'(\tau_2^0; \mathbf{u}) - y_i'(\tau_2^0; \mathbf{u}^0))dx \geq 0, \quad \forall \mathbf{u} \in U_Q^n. \tag{24}$$

Introduce the adjoint state $p(t; \mathbf{u}^0)$ by the solution of the following system:

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2}(t;\ \mathbf{u}^0) - (A(t)\mathbf{p}(t;\ \mathbf{u}^0))_i = 0 && \text{in } \Omega \times]0,\ \tau_2^0[,\\ &p_i(x,\ \tau_2^0;\ \mathbf{u}^0) = (y_i'(x,\ \tau_2^0;\ \mathbf{u}^0) - z_{id}) && \text{in } \Omega,\\ &p_i'(x,\ \tau_2^0;\ \mathbf{u}^0) = 0 && \text{in } \Omega,\\ &\frac{\partial p_i}{\partial \nu}(x,\ t;\ \mathbf{u}^0) = 0 && \text{on } \Gamma \times]0,\ \tau_2^0[. \end{aligned}\right\} \tag{25}$$

Since $(y_i'(x,\ \tau_2^0;\ \mathbf{u}^0) - z_{id}) \in L^2(\Omega)$, the existence of a unique solution for system (25) can be proved by using transposition theorem (Theorem 2 with an obvious change of variables); $\mathbf{p}(t,\ \mathbf{u})$ is the unique element of $(L^2(0,\ \tau_2^0;\ L^2(\Omega)))^n$ such that

$$\left.\begin{aligned} &\int_\Omega p_i'(\tau^0;\ \mathbf{u}^0)\phi_i(\tau_2^0)dx = \int_\Omega (y_i'(\tau_2^0;\ \mathbf{u}^0) - z_{id})\phi_i'(\tau_2^0)dx \quad \forall \phi,\\ &\left(\frac{\partial^2 \phi_i}{\partial t^2} + (A(t)\phi)_i\right) \in L^2(Q), \quad \phi_i(0) = 0, \quad \phi_i'(0) = 0. \end{aligned}\right\} \tag{26}$$

Therefore, in (26), we can take $\phi_i = y_i(t;\ \mathbf{u}) - y_i(t;\ \mathbf{u}^0)$ so that

$$\int_0^{\tau_2^0}\int_\Omega p_i(x,\ t;\ \mathbf{u}^0)(u_i - u_i^0)dx = \int_\Omega (y_i'(x,\ \tau_2^0;\ \mathbf{u}^0) - z_{id})(y_i'(x,\ \tau_2^0;\ \mathbf{u}) - y_i'(x,\ \tau_2^0;\ \mathbf{v}^0))dx.$$

Condition (24) then becomes

$$\sum_{i=1}^n \int_0^{\tau_2^0}\int_\Omega p_i(u_i - u_i^0)dxdt \geq 0, \quad \forall \mathbf{u} \in U_Q^n. \tag{27}$$

We have thus proved.

Theorem 8. *We assume that* (2), (6) *hold. Then, there exist the adjoint state*

$$\mathbf{p} = (p_i)_{i=1}^n \in L^2(0,\ \tau_2^0;\ (L^2(\Omega))^n),$$

such that the optimal control $\mathbf{u}^0$ *of problem* (*TOP*2) *is characterized by* (25),(27) *together with* (1) (*with* $u_i = u_i^0$, $i = 1, 2, \ldots, n$).

5. Boundary Control -Position Observation Problem

In this section, we consider the following third time-optimal control problem with boundary control $\mathbf{u}$, position observation $\mathbf{y}(x, t; \mathbf{v})$:

$$\text{(TOP3):} \qquad \min \{t : \mathbf{y}(x, t; \mathbf{v}) \in K_\varepsilon^n, \ \mathbf{v} \in U_\Sigma^n\}.$$

As in the above section, we can prove that for large T, the following controllability condition is hold:

$$\text{There exists a } \tau > 0 \text{ and } \mathbf{v} \in U_\Sigma^n \text{ with } \mathbf{y}(\tau; \mathbf{v}) \in K_\varepsilon^n, \tag{28}$$

and if we set

$$\tau_3^0 = \inf \{\tau : \mathbf{y}(\tau; \mathbf{v}) \in K_\varepsilon^n \text{ for some } \mathbf{v} \in U_\Sigma^n\}. \tag{29}$$

Then in this case, we can prove the following theorem:

Theorem 9. *If* (2) *and* (6) *are hold, then there exist an admissible control* $\mathbf{v}^0$ *to the problem* (*TOP*3), *which steering* $\mathbf{y}(t; \mathbf{v}^0)$ *to hitting a target set* K_ε^n *in minimum time* τ_3^0 (*defined by* (29)). *Moreover*,

$$\sum_{i=1}^{n} \int_\Omega (y_i(\tau_3^0; \mathbf{v}^0) - z_{id})(y_i(\tau_3^0; \mathbf{v}) - y_i(\tau_3^0; \mathbf{v}^0))dx \geq 0, \quad \forall \mathbf{v} \in U_\Sigma^n, \tag{30}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 10. *We assume that* (2), (6) *hold. The time-optimal control* $\mathbf{v}^0$ *of problem* (*TOP*3) *is characterized by the solution of the following systems of equations and inequalities*:

$$\left.\begin{aligned} &\frac{\partial^2 y_i}{\partial t^2}(t;\ \mathbf{v}^0) - (A(t)y(t;\ \mathbf{v}^0))_i = u_i,\ x \in \Omega,\ t \in\ ]0,\ \tau_3^0[,\\ &\qquad y_i(x,\ 0;\ \mathbf{v}^0) = y_{i,0}(x), \quad x \in \Omega,\\ &\qquad \frac{\partial y_i}{\partial t}(x,\ 0;\ \mathbf{v}^0) = y_{i,1}(x), \quad x \in \Omega,\\ &\qquad \frac{\partial y_i}{\partial \nu}(x,\ t;\ \mathbf{v}^0) = v_i^0, \qquad x \in \Gamma,\ t \in\ ]0,\ \tau_3^0[, \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} &\frac{\partial^2 p_i}{\partial t^2}(t;\ \mathbf{v}^0) - (A(t)p(t;\ \mathbf{v}^0))_i = 0,\ x \in \Omega,\ t \in\ ]0,\ \tau_3^0[,\\ &\qquad p_i(x,\ \tau_3^0;\ \mathbf{v}^0) = 0, \qquad x \in \Omega,\\ &\frac{\partial p_i}{\partial t}(x,\ \tau_3^0;\ \mathbf{v}^0) = (y_i(x,\ \tau_3^0;\ \mathbf{v}^0) - z_{id}),\ x \in \Omega,\\ &\qquad \frac{\partial p_i}{\partial \nu}(x,\ t;\ \mathbf{v}^0) = 0, \qquad x \in \Gamma,\ \ t \in\ ]0,\ \tau_3^0[, \end{aligned}\right\} \tag{32}$$

$$\sum_{i=1}^{n} \int_0^{\tau_3^0} \int_\Gamma p_i(v_i - v_i^0)\, d\Gamma dt \geq 0, \quad \forall \mathbf{v} = (v_1,\ v_2,\ \ldots,\ v_n) \in U_\Sigma^n,$$

together with

$$\left.\begin{aligned} &p_i(t;\ \mathbf{v}^0) \in L^2(0,\ \tau_3^0;\ H_0^1(\Omega)),\\ &y_i(t;\ \mathbf{v}^0),\ \frac{\partial p_i}{\partial t}(t;\ \mathbf{v}^0) \in L^2(0,\ \tau_3^0;\ L^2(\Omega)). \end{aligned}\right\} \tag{33}$$

6. Boundary Control -Velocity Observation Problem

In this section, we consider the following fourth time-optimal control problem with boundary control $\mathbf{v}$ and velocity observation $\mathbf{y}(x,\ t;\ \mathbf{v})$:

$$\text{(TOP4)}: \qquad \min\{t : \mathbf{y}'(x,\ t;\ \mathbf{v}) \in K_\varepsilon^n,\ \mathbf{v} \in U_\Sigma^n\}.$$

As in the above section, we can prove that for large T, the following controllability condition is hold:

$$\text{There exists a } \tau > 0 \text{ and } \mathbf{v} \in U_\Sigma^n \text{ with } \mathbf{y}'(\tau;\ \mathbf{v}) \in K_\varepsilon^n, \tag{34}$$

and if we set

$$\tau_4^0 = \inf\{\tau : \mathbf{y}'(\tau; \mathbf{v}) \in K_\varepsilon^n \text{ for some } \mathbf{v} \in U_\Sigma^n\}. \tag{35}$$

Then in this case, we can prove the following theorem:

Theorem 11. *If* (2), (6), *and* (34) *are hold, then there exist an admissible control* $\mathbf{v}^0$ *to the problem* (*TOP*4), *which steering* $\mathbf{y}'(t; \mathbf{v}^0)$ *to hitting a target set* K_ε^n *in minimum time* τ_4^0 (*defined by* (35)). *Moreover*,

$$\sum_{i=1}^{n} \int_\Omega (y_i'(\tau_4^0; \mathbf{v}^0) - z_{id})(y_i'(\tau_4^0; \mathbf{v}) - y_i'(\tau_4^0; \mathbf{v}^0))dx \geq 0, \quad \forall \mathbf{v} \in U_{\epsilon\Sigma}^n, \tag{36}$$

which can be interpreted as the above sections to obtaining the following theorem:

Theorem 12. *We assume that* (2) *and* (6) *hold. The time-optimal control* v^0 *of problem* (*TOP*4) *is characterized by the solution of the following systems of equations and inequalities*:

$$\left.\begin{aligned} \frac{\partial^2 y_i}{\partial t^2}(t; \mathbf{v}^0) - (A(t)y(t; \mathbf{v}^0))_i &= u_i,\ x \in \Omega,\ t \in]0, \tau_4^0[, \\ y_i(x, 0; \mathbf{v}^0) &= y_{i,0}(x), \quad x \in \Omega, \\ \frac{\partial y_i}{\partial t}(x, 0; \mathbf{v}^0) &= y_{i,1}(x), \quad x \in \Omega, \\ \frac{\partial y_i}{\partial \nu}(x, t; \mathbf{v}^0) &= v_i^0, \qquad x \in \Gamma,\ t \in]0, \tau_4^0[, \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} \frac{\partial^2 p_i}{\partial t^2}(t; \mathbf{v}^0) - (A(t)p(t; \mathbf{v}^0))_i &= 0,\ x \in \Omega,\ t \in]0, \tau_4^0[, \\ p_i(x, \tau_4^0; \mathbf{v}^0) &= (y_i'(x, \tau_4^0; \mathbf{v}^0) - z_{id}),\ x \in \Omega, \\ \frac{\partial p_i}{\partial t}(x, \tau_4^0; \mathbf{v}^0) &= 0, \qquad x \in \Omega, \\ \frac{\partial p_i}{\partial \nu}(x, t; \mathbf{v}^0) &= 0, \qquad x \in \Gamma, \quad t \in]0, \tau_4^0[, \end{aligned}\right\} \tag{38}$$

$$\sum_{i=1}^{n}\int_0^{\tau_3^0}\int_\Gamma p_i(v_i - v_i^0)d\Gamma dt \geq 0, \quad \forall \mathbf{v} = (v_1, v_2, \ldots, v_n) \in U_\Sigma^n,$$

together with

$$y_i(t; v^0),\ p_i(t; v^0) \in L^2(0, \tau_4^0; L^2(\Omega)). \tag{39}$$

7. Comments

• We note that, in this paper, we have chosen to treat a special systems involving Laplace operator, just for simplicity. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator:

$$L(x, .) = \sum_{i,j=1}^{n} b_{ij}(x, .)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x, .)\frac{\partial}{\partial x_j} + b_0(x, .),$$

with sufficiently smooth coefficients (in particular, $b_{ij}, b_j, b_0 \in L^\infty(Q)$, $b_j, b_0 > 0$) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i \eta_j \geq \sigma \sum_{i=1}^{n} \eta_i, \quad \forall (x, t) \in Q,$$

for all $\eta_i \in \Re$ and some constant $\sigma > 0$.

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator L (see [6]).

• In this paper, we have taken a simple target set K_ε^n. In (TOP1) (for example), if we take

$$K_\varepsilon^n = \left\{ z \in (L^2(\Omega))^n : \|z_i - z_{id}\|_{L^2(\Omega)} + \sum_{j=1}^{N} \| \frac{\partial z_i}{\partial x_j} - z_{id} \|_{L^2(\Omega)} \leq \epsilon \right\},$$

then the necessary optimality conditions coincide with (19), (20), (1) (with $v_i = v_i^0$, $i = 1, 2$) and $(y_i(x, \tau_1^0; \mathbf{v}^0) - z_{id})$ in (19) is replaced by $(-\Delta_x + I)(y_i(x, \tau_1^0; \mathbf{v}^0) - z_{id})$.

● The results in this paper, carry over to the optimal control problems with fixed-time ([1], Chapter 4), for example, the results of (TOP1) carry over to the fixed-time problem

$$\text{minimize} \sum_{i=1}^{n} \int_{\Omega} |y_i(x, T; u) - z_{id}(x)|^2 dx, \quad T \text{ fixed},$$

subject to (1) [except in the trivial case where $z_{id}(x) = y_i(x, T; \mathbf{v})$ for some admissible control $\mathbf{v} = (v_i)_{i=1}^n$]. This can proven in an analogous manner, as the necessary and sufficient conditions for optimality for this problem coincide with (19), (20), and (1) (with $v_i = v_i^0$, $i = 1, 2, \ldots, n$).

● As a final coment, we note that the control problem for the second order evolution system (1) can be reduced to a similar control problem first order system; in the usual way: set $\psi = \begin{pmatrix} \mathbf{y} \\ \dfrac{\partial \mathbf{y}}{\partial t} \end{pmatrix}$ and rewrite (1) in the first order form. However, the existing results on the time-optimal problem ([1], [10], [11]) pertain to the case, where the observation is only one case (position-velocity) but here we can take different cases.

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