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L^p SPACES FOR RELATIVELY WEAKLY COMPACT VECTOR MEASURES AND S-COMPACT SET-VALUED MEASURES

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Abstract

We introduce a new concept of L^p spaces, for an s-compact set-valued measure or set-valued Radon measure, a relatively weakly compact vector measure, or a weakly compact vector Radon measure. We establish that the quotient spaces are reflexive Banach spaces, since there norms are uniformly convex. We prove that these spaces are also uniformly smooths, therefore there norms are uniformly strongly differentiable.

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1. Introduction

The set-valued measures have been the subject of many thesis. In the school of Pallu De La Barrière, we have the one's of: Thiam, Costé and Siggini; in the school of C. Castaing, the one of Godet-Thobie; and in the school of D. S. Thiam, the one's of Dia [4], Thiam [27], and Ndiaye. In the set-valued integration field, the works are mainly done, for set-valued measures with weakly compact values. We had got the project to look at integration with respect to set-valued measures with closed convex values. The work in [11], is done for a set-valued measure with closed convex values, or a set-valued Radon measure with closed convex bounded values. For that, we had introduced in [11], respectively, the concept of sequentially compact (s-compact, for short), set-valued measures and set-valued Radon measures, to go beyond the weakly compact case of the previous works [3], [18], [25]. The concept of s-compact is equivalent to σ-finite of the scalar case. The results we have obtained from [9] to [16], are applied here. We consider a set-valued Daniell integral I, associated to: A monotone s-compact set-valued measure, or set-valued Radon measure, or a relatively weakly compact vector measure, or a weakly compact vector Radon measure. In previous papers, we had defined and studied the spaces $\mathrm{L}^1(I)$ and $L^\infty(I)$: see [9] to [16]. Here, we define the spaces $\mathrm{L}^p(I)$, with $p \in]1, +\infty[$. This notion do not appear in [11], [3], [5], [18], [25]. Let us denote by $\mathrm{L}^{(p)}(I)$, the quotient space for the I-almost everywhere (I-ae) equivalence relation. We show first that $\mathrm{L}^{(p)}(I)$ is a Banach space. Secondly, we show that $\mathrm{L}^{(p)}(I)$ is uniformly convex, therefore by the Milman's criterion, it is reflexive. We establish also that the dual of $\mathrm{L}^{(p)}(I)$ is $\mathrm{L}^{(q)}(I)$, where p and q are conjugate. From that, we deduce that $\mathrm{L}^{(p)}(I)$ is uniformly smooth, therefore, it's norm is uniformly strongly differentiable. Something relevant is: most of the proofs, are direct adaptations, of the methods and

techniques, used in the classical $L^{(p)}$ spaces. We wonder if these adaptations, could be possible, from the previous existing works in this field, before the results we presented in [11]. Working with a set valued integral, like in the positive scalar case, is quite surprising and unusual.

2. Notations and Preliminaries

Let E be a normed space, and the associated topological dual is denoted by E' and the usual duality between E and E' by $\langle \cdot, \cdot \rangle$. Respectively by $(cc(E), +)$, $(cfb(E), \dot{+})$, $(cf(E), \dot{+})$, we shall indicate the collection of all the nonempty weak compact convex subsets, the nonempty closed convex bounded subsets, and the nonempty closed convex subsets of E. For any A and B in $cf(E)$, we set: $\delta^*(y / A) = \sup\{\langle x, y\rangle, x \in A\}$, and $A \dot{+} B = \overline{(A + B)}$, to define, respectively, for the set A the *support function* $\delta^*(. / A)$, and for the sets A, B the *sum*. The topology in $cf(E)$ is the Hausdorff one cf. [1]. The order relation is the inclusion. We consider a nonempty set T, $\mathbb{R}^T$ the set of the collection of all the numerical functions defined on T, and H a Riesz subspace of $\mathbb{R}^T$. If T is a topological space, we shall denote, for a numerical function f, defined on T, i.e., $f \in \overline{\mathbb{R}}^T$, it's carrier as $\operatorname{supp}(f) = \overline{\{t \in T, f(t) \neq 0\}}$. We set $f^+ = \sup(f, 0)$, and $f^- = \sup(-f, 0)$. We denote by $\overline{C}_O(A)$, the closed convex hull of a subset A of E. If O is a family of closed convex subsets of E, we have $\inf\{A : A \in O\} = \bigcap\{A : A \in O\}$, and $\sup\{A : A \in O\} = \overline{C}_O \bigcup\{A : A \in O\}$. We set $H_+ = \{h \in H : h \geq 0\}$. Let $I : H_+ \to cc(E)$ be a set-valued function. We say that I is a secondary Daniell integral if I is additive, $0 \in I(h)$ for each $h \in H_+$, i.e., I is monotone, and if (h_n) is a decreasing sequence of H, which converges pointwise to 0 (we denote $h_n \downarrow 0$), then $\inf_n I(h_n) = 0$.

We go now throughout some situations we could have to do with. We denote by Ω a clan of subsets of T.

Case 1. Let $M : \Omega \longrightarrow cf(E)$ be a set-valued mapping, M is a weak set-valued measure if $\delta^*(y / M(.))$ is a scalar measure for each $y \in E'$, with $M(\emptyset) = \{0\}$. M is said to be sequentially compact (s-compact), if there exists $T_n \in \Omega$ an increasing sequence with $\bigcup_n T_n = T$ (we denote $T_n \uparrow T$) such that $M(T_n) \in cc(E)$, for each $n \in \mathbb{N}^*$.

We consider now a weak set valued measure M, which is monotone and s-compact. We set $\Lambda = \{A \in \Omega : M(A) \in cc(E)\}$. If we take $I(h) = \sum a_i M(A_i)$, where $h = \sum a_i 1_{A_i}$ is a nonnegative step function based on the clan Λ, and if H is the Riesz subspace of $\mathbb{R}^T$, of the collection of all the steps functions based on the clan Λ, then $I : H_+ \to cc(E)$, is a secondary Daniell integral.

Case 2. We suppose that E is a Banach space. We consider $m : \Omega \to E$ a vector measure, this mean's $M(\emptyset) = 0$, and if $\{A_n, n = 1, 2, \dots\}$ is an Ω-partition of $A \in \Omega$, then $m(A) = \lim_{n \to +\infty} \sum_{i=1}^{i=n} m(A_i)$, for the topology of E. We say that m is relatively weakly compact, if for each $A \in \Omega$, $\{m(B), B \in \Omega, B \subset A\}$ is relatively weakly compact. By the Theorem 3 ([10], page 53), a vector measure is relatively weakly compact, if and only if it can be extended to a δ-clan; but this is equivalent to say that m is with finite variation. Like in [25], page 140, see also [10] or [11]: if $A \in \Omega$, let $M_m(A)$ be the closure in E of all the sum $\sum_{i \in I} \lambda_i m(A_i)$, where $\{A_i, i \in I\}$ is an Ω-partition of A and where $\lambda_i \in R$ with $|\lambda_i| \leq 1$.

We have $\delta^*(y : M_m(A)) = |m_y|(A)$, where $|m_y|$ is the total variation of the scalar measure $m_y =< m(.), y >$, $y \in E'$ (cf. [25], Lemma 5.1, page 141). We have therefore that $M_m : \Omega \to cc(E)$, is a weak set-valued measure, which is s-compact and monotone. We have the situation of Case 1.

In the two following cases, T will be a locally compact space, which is metrizable and separable. We denote by $K(T)$, the collection of all the numerical continuous functions, defined on T, with compact carrier.

Case 3. The set-valued function, $I : K_+(T) \to cf(E)$, is said to be s-compact, if there exists an increasing sequence (h_n) of $K_+(T)$, which is convergent pointwise to 1_T (we denote $h_n \uparrow 1_T$), and such that for each $n \in N^*$, $I(h_n) \in cc(E)$. The set-valued function I is said to be monotone, if $0 \in I(f)$, for each $f \in K_+(T)$. If we take on $K_+(T)$ the inductive limit topology, defined by the spaces $K(K)$, where K is a compact subset of T, and on $cfb(E)$ the Hausdorff one, a set-valued Radon measure I, is a mapping $I : K_+(T) \to cfb(E)$, which is additive, homogeneous, and continuous. For a set-valued Radon measure I, let us set $H = \{f \in K(T), I(|f|) \in cc(E)\}$, and $I_1 : H_+ \to cc(E)$, $I_1(f) = I(f)$. The set H is a Riesz subspace of $K(T)$. By the Proposition III-1-1 [11], see also Proposition 3.2 [16], I_1 is a secondary set-valued Daniell integral.

Case 4. We suppose that E is a Banach space, and $C_0(T)$ is the collection of all the continuous functions null at infinity. Let $u : K(T) \to E$, be a weakly compact Radon measure, i.e., which transform the unit ball of $C_0(T)$, to a relatively weak compact subset of E. Like in [25], page 155, or [16], if we set $M_u(f) = \overline{\{u(g) : g \in K(T), |g| \leq f\}}$, with $f \in K_+(T)$, then $M_u : K_+(T) \to cc(E)$, is a secondary Daniell integral, since $\delta^*(y, M_u(f)) = |u_y|(f)$, where $|u_y|$ is the absolute value of the scalar Radon measure u_y defined by $u_y(g) =< u(g), y >, g \in K(T)$. We have given in [16] the link with [28] and [18].

In all these four cases, we have the following situation:

We consider H a Riesz subspace of $\mathbb{R}^T$, and I a secondary Daniell integral, defined on H_+ with values in $cc(E)$, which is s-compact; I is

said to be s-compact, if there exists an increasing sequence (h_n) of H_+, which is convergent pointwise to 1_T (we denote $h_n \uparrow 1_T$), and such that for each $n \in \mathbb{N}^*$, $I(h_n) \in cc(E)$.

We denote by $H^\vee$ (resp., $H^\wedge$), the collection of all the functions, which are increasing (resp., decreasing) limits of functions of $H_+ = \{h \in H, h \geq 0\}$. If $h_n \uparrow f$ and $k_n \downarrow g$, with $h_n \in H_+$ and $k_n \in H_+$, we set $I^\vee(f) = \sup_n I(h_n)$, and $I^\wedge(g) = \inf_n I(k_n)$. We denote by I^*, the upper integral of I; see [9], [11] or [15]: $I^*(f) = \inf\{I^\vee(\psi), \psi \in H^\vee, \psi \geq f\}$, if this set is nonempty, otherwise, we take $I^*(f) = E$, and by $I_*(f)$: $I_*(f) = \sup\{I^\wedge(\phi), \phi \in H^\wedge, \phi \leq f\}$, and we set, $L^1_+(I) = \{f \in \overline{\mathbb{R}}^T_+, I^*(f) = I_*(f) \in cfb(E)\}$, for the space of integrable functions, which is a reticulated convex cone, see [9] or [11]. If $f \in L^1_+(I)$, $\int fI = I^*(f)$ is additive and positively homogeneous. We will denote this by $\int(.)I$ or I. We set if $h \in H_+$, $I_y(h) = \delta^*(y, I(h))$, and if $h \in H$, $I_y(h) = I_y(h^+) - I_y(h^-)$. If we set $H^* = \{f \in \mathbb{R}^T, I^*(|f|) \in cfb(E)\}$, then $(H^*, +, \leq)$ is a Riesz subspace of $\mathbb{R}^T$. We denote by $V(0)$ the collection of all the neighbourhoods of 0 in E, and if $V \in V(0)$, we set $V^* = \{f \in \mathbb{R}^T, I^*(|f|) \subset V\}$. We denote by $V^*(0)$, the collection of all the V^*, where $V \in V(0)$. The set $V^*(0)$, is a filter basis of 0 of H^*. We have defined in H^* a topology for which H^* is a locally convex space.

Definition 2.1. (1) A function f of $\overline{\mathbb{R}}^T$ is I-negligible, if $I^*(|f|) = \{0\}$.

(2) A subset A of T is I-negligible, if 1_A is I-negligible.

(3) A propriety $p(t)$ is true almost everywhere (I-a.e or a.e), with respect to I, if the set of t where [non $p(t)$] is true is negligible.

(4) A function defined I-a.e on T, with values in $\overline{\mathbb{R}}$, is I-measurable, if it is limit I-a.e of a sequence of H.

(5) A subset A of T is I-measurable, if 1_A I-measurable.

(6) We set $\pounds^1_+(I) = \{f$ defined I-a.e on T, $\exists g \in L^1_+(I)$, $f = g$ I-a.e$\}$.

(7) We set also $F(T, \overline{\mathbb{R}}) = \{f$ defined I-a.e on T, $I^*(|f|) \in cfb(E)\}$.

(8) Wet denote by $\ell^1(I)$, the closure of H in H^*.

For $f \in F(T, \overline{\mathbb{R}})$, we denote $N_1(f) = \sup\{\delta^*(y / I^*(|f|)), y \in E', \|y\| \leq 1\}$. A sequence $(f_n)_{n\in\mathbb{N}^*}$ of $F(T, \overline{\mathbb{R}})$ converges in mean to f, if $\lim_n N_1(f_n - f) = 0$. The topology of $F(T, \overline{\mathbb{R}})$, defined by the semi-norm N_1, is the topology of the convergence in mean. The topology of H^* is the one induced by N_1. Now, we set $L^1(I) = [\overline{H}]_{N_1}$ = the closure of H in $F(T, \overline{\mathbb{R}})$. The following Proposition 2.2, gives successively, the Propositions 7 and 13, the Theorems 6 and 1, we had obtained in [9].

Proposition 2.2. *We consider* $f \in \overline{\mathbb{R}}^T$, *and* $y \in E'$;

(1) *We have* $(I_y)^*(|f|) = \delta^*(y / I^*(|f|))$.

(2) *If* $I_*(|f|) \in cfb(E)$, *then we get* $(I_y)_*(|f|) = \delta^*(y / I_*(|f|))$.

(3) *We have also* $L^1_+(I) = \cap_{y\in E'} L^1_+(I_y)$, *and if* $f \in L^1_+(I)$, *then*

$$\delta^*(y / \int f I) = \int f \delta^*(y / I(.)) = \int f I_y.$$

(4) *If* (f_n) *is an increasing sequence of* $\overline{\mathbb{R}}^T_+$, *we have* $\sup_n I^*(f_n) = I^*(\sup_n (f_n))$.

Remark 2.3. We note that $N_1(f) = \sup\{\delta^*(y / I^*(|f|)),\ y \in E',\ \|y\| \le 1\}$ $= \sup\{\|x\|,\ x \in I^*(|f|)\} = DH(I^*(|f|), 0)$, where DH is the Hausdorff distance. The reason why we use the first expression is to do not loose the statements 1, 2, and 3 of Proposition 2.2, i.e., the relation between the L^1 over set-valued measures and scalar measures. Indeed, this will be very important in this paper, specially in the proof of Theorem 3.13.

The Theorem 2.3 is Theorem 3.3 of [11]. The Theorem 2.4 is Theorem 4.17 of [12], the Corollary 2.5 is Corollary 4.19 of [12]. The Theorem 2.6 is Theorem 2 of [10], the Proposition 2.7 is Proposition 3.7 of [16], and Proposition 2.8 is Proposition 3.5 of [16].

Theorem 2.4. (1) *If f is I-measurable and nonnegative, and if $I^*(f) \in cfb(E)$, then $f \in \mathfrak{L}^1_+(I)$.*

(2) *If f is I-measurable and if $I^*(|f|) \in cc(E)$, then $f \in \mathfrak{L}^1(I)$.*

Theorem 2.5. (1) *Let (f_n) be an increasing sequence I-a.e of $\mathfrak{L}^1_+(I)$, such that there exists $g \in F(T, \overline{\mathbb{R}})$, with $f_n \le g$ I-a.e for each n, and $I^*(g) \in cc(E)$, then $f = \sup_n f_n \in \mathfrak{L}^1_+(I)$, f_n converges to f in mean, $\int f I \in cc(E)$.*

(2) *If f_n is a decreasing I-a.e sequence of $\mathfrak{L}^1_+(I)$, with $\int f_n I \in cc(E)$, then $f = \inf_n f_n \in \mathfrak{L}^1_+(I)$, $\int f I \in cc(E)$, and f_n converges in mean to f.*

(3) *Let (f_n) be a sequence of $\mathfrak{L}^1_+(I)$, such that there exists $g \in F(T, \overline{\mathbb{R}})$, with $|f_n| \le g$ I-a.e for each $n \in \mathbb{N}^*$, and $I^*(g) \in cc(E)$. If (f_n) converges pointwise I-a.e to f, we have $f \in \mathfrak{L}^1_+(I)$ and (f_n) converges in mean to f. In all these three cases, we have $\lim_{n\to+\infty} \int f_n I = \int f I$, for topology of $cc(E)$.*

Corollary 2.6. *Let* (f_n) *be a sequence of* $L^1(I)$, *which converges I-a.e to f, if there exists* $g \in L^1_+(I)$, *with* $\int gI \in cc(E)$, *and* $|f_n| \leq g$ *I-a.e for each* $n \in \mathbb{N}^*$, *then* $f \in L^1(I)$ *and* $\int fI = \lim_{n\to+\infty} \int f_n I$ *for the Hausdorff topology of* $cc(E)$ *and* (f_n) *converges in mean to f.*

Theorem 2.7. *We denote by* $S(M)$, *the collection of all the additive sections of M.*

(1) *We assume that M is monotone for each y of E', there exists* $\mu \in S(M)$, *such that* $\mu_y = \delta^*(y / M(.))$, *where* $\mu_y = \langle \mu(.), y \rangle$.

(2) *If E is complete, then M is with values in* $cc(E)$, *if and only if* $S(M) \neq \emptyset$.

Proposition 2.8. (1) *If we take the closure in* H^*, *we will have*

$$K_+(T) \subset \ell^1_+(I_1) = \overline{K_+(T)} \subset L^1_+(I_1) \subset \mathfrak{L}^1_+(I_1), \text{ and } K(T) \subset \ell^1(I_1) = \overline{K(T)}.$$

(2) *In the same way, if we take the closure in* $F(T, \overline{\mathbb{R}})$, *we will have*

$$K_+(T) \subset L^1_+(I_1) = \overline{K_+(T)}, \text{ and } K(T) \subset L^1(I_1) = \overline{K_+(T)}.$$

Proposition 2.9. (1) *If E is complete, then I is with weakly compact values.*

(2) *If E is a normed space, we have*

(i) *The inductive limit topology on* $K(T)$ *is finer than the topology induced on* $K(T)$, *by* H^* *or* $F(T, \overline{\mathbb{R}})$, *i.e., the topology of the convergence in mean*;

(ii) $K_+(T) \subset \ell^1(I)$, $H_+ \subset K_+(T) \subset H^\vee$ *and therefore* $H^\vee = [K_+(T)]^\vee$. $\int(.)I_1$ *extend I*;

(iii) *I is uniformly continuous, since so is* $\int(.)I_1$.

We give now examples of s-compact set-valued measures:

Example 2.10. Let $\beta(\mathbb{R}^n)$ be the Borelian tribe of $\mathbb{R}^n$ and λ a σ-finite measure on $\beta(\mathbb{R}^n)$. We set

$$M : \beta(\mathbb{R}^n) \longrightarrow cf(\mathbb{R}), \text{ defined by}$$

$$M(A) = [0, \lambda(A)], \text{ if } \lambda(A) < +\infty,$$

$$\text{and } M(A) = [0, +\infty[, \text{ otherwise,}$$

M is a monotone set-valued measure, which is s-compact. The additivity comes from the Riesz decomposition property, see Perissini [19], page 8, or Thiam [25], page 173.

Example 2.11. Let Ω be the clan generated by the compact subset of a locally compact metrizable and separable topological space. If $M : \Omega \longrightarrow cc(E)$ is a monotone set-valued measure, therefore M is s-compact.

Example 2.12. Let $T = \mathbb{R}^n$, $E = L^1(\mathbb{R}^n, dx)$ and Ω be the clan generated by the compact subset of $\mathbb{R}^n$. For $A \in \Omega$, let us set

$$N(A) = \{f \in E, 0 \leq f \leq 1_A\},$$

N is a set-valued measure because

$$\delta^*(y / N(A)) = \int 1_A f^+ dx, \quad \forall f \in L^\infty(dx).$$

Moreover, N is monotone and s-compact: From Valadier [29], we have $N(A) \in cc(E)$.

3. The Spaces $L^{(p)}(I),\ p \in]1, +\infty[$

Definition 3.1. We set

(1) $F_p = \{f$ defined I-a.e on $T \to \overline{\mathbb{R}},\ I^*(|f|^p) \in cfb(E)\}$, with $p \geq 1$.

(2) $N_p : F_p \to \mathbb{R}_+$, if $f \in F_p$, $N_p(f) = [\sup_{y \in E', \|y\| \leq 1} \delta^*(y / I^*(|f|^p)]^{\frac{1}{p}} = [N_1(|f|^p)]^{\frac{1}{p}}$.

Proposition 3.2. *If $f, g \in F_p$ and $a \in \mathbb{R}$, we have $f + g \in F_p$ and $af \in F_p$.*

Proof. If $a, b \in \mathbb{R}$, we have $|a + b| \leq |a| + |b| \leq 2 \max [|a|, |b|]$. We deduce that $|a + b|^p \leq [2 \max [|a|, |b|]]^p \leq 2^p [|a|^p + |b|^p]$. We obtain for $f, g \in F_p$: $|f + g|^p \leq 2^p [|f|^p + |g|^p]$, I-a.e. Since I^* is increasing, sub-additive, and positively homogeneous, we get $I^*[|f + g|^p] \subset 2^p [I^*(|f|^p) + I^*(|g|^p)]$. □

Theorem 3.3. *Hölder's inequality*:

Let $f \in F_p$, $g \in F_q$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in]1, +\infty[$, we have $fg \in F_1$, and $N_1(fg) \leq N_p(f) N_q(g)$.

Proof. Let $a, b \in \mathbb{R}_+$, we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. We get therefore $|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$ I-a.e, and $I^*(|fg|) \subset \frac{1}{p} I^*(|f|^p) \dot{+} \frac{1}{q} I^*(|g|^q) \in cfb(E)$, and $fg \in F_1$. If we set $a = \frac{|f|}{N_p(f)}$ and $b = \frac{|g|}{N_q(g)}$, we obtain $\frac{|fg|}{N_p(f) N_q(g)} \leq \frac{|f|^p}{p[N_p(f)]^p} + \frac{|g|^q}{q[N_q(g)]^q}$, I-a.e, thus $\frac{I^*(|fg|)}{N_p(f) N_q(g)} \subset \frac{I^*(|f|^p)}{p[N_p(f)]^p}$

$+\dfrac{I^*(|g|^q)}{q[N_q(g)]^q}$. This means $\dfrac{N_1(fg)}{N_p(f)N_q(g)} \le \dfrac{N_1(|f|^p)}{p[N_p(f)]^p} + \dfrac{N_1(|g|^q)}{q[N_q(g)]^q}$

$= \dfrac{1}{p} + \dfrac{1}{q} = 1.$ □

Theorem 3.4. *Minkowski's inequality*:

Let $f, g \in F_p$, *we have* $N_p(f+g) \le N_p(f) + N_p(g)$.

Proof. Let $p \in]1, +\infty[$ and q so that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, since

$|f+g|^p \le |f|\,|f+g|^{p-1} + |g|\,|f+g|^{p-1}$ I-a.e, we get

$$N_1[|f+g|^p] \le N_1[|f|(|f+g|)^{p-1}] + N_1[|g|(|f+g|)^{p-1}].$$

We apply now the Hölder's inequality Theorem 3.3 to get

$$N_1[|f+g|^p] \le N_p(f)N_q[(|f+g|)^{p-1}] + N_p(g)N_q[(|f+g|)^{p-1}],$$

and this end the proof, since $N_q[(|f+g|)^{p-1}] = N_1[(|f+g|)^p]^{1-\frac{1}{p}}$. □

Definition 3.5. We set $\mathfrak{L}_+^p(I) = \{f$ defined I-a.e on T, $|f|^p \in \mathfrak{L}_+^1(I)\}$,

$\mathrm{L}^p(I) = [\overline{H}]_{Np}$ = the closure of H in F_p, $p \ge 1$, for the topology of the convergence in mean of order p, defined by the semi-norm N_p.

The reason of the difference in the notation of the two spaces in the Definition 3.5, can be found in [12] or [13].

Theorem 3.6. (1) $\mathfrak{L}_+^p(I)$, $\mathrm{L}^p(I)$ *are complete for the topology of the convergence in mean of order* p, *defined by the semi-norm* N_p.

(2) *Let* (f_n) *be a sequence of* $\mathfrak{L}_+^p(I)$, *or* $\mathrm{L}^p(I)$, *which converges to* f *in mean of order* p, *therefore there exists a subsequence* (f_{n_k}) *of* (f_n), *which converges* I-*a.e to* f.

Proof. For $p = 1$, this is the Theorem 4.6 of [12] or [13]. Let (f_n) be a Cauchy sequence of $\mathfrak{L}^p_+(I)$, there exists a subsequence (f_{n_k}) of (f_n) such that $N_1[|f_{n_{k+1}} - f_{n_k}|^p] < 3^{-k}$, $\forall k \geq 1$. We deduce that $\delta^*(y/I^*(|f_{n_{k+1}} - f_{n_k}|^p)) < 3^{-k}\|y\|$, $\forall y \in E'$, $\forall k \geq 1$. Since $[|f-g| + |h-k|]^p \leq 2^p[|f-g|^p + |h-k|^p]$, we obtain $\int \sum_{k=1}^{n} |f_{n_{k+1}} - f_{n_k}|^p I_y \leq 2^p \sum_{k=1}^{n} \int |f_{n_{k+1}} - f_{n_k}|^p I_y$, $\forall y \in E'$, and $\sum_{k=1}^{+\infty} \int |f_{n_{k+1}} - f_{n_k}|^p I_y \leq 2^p \sum_{k=1}^{+\infty} 3^{-k}\|y\| < +\infty$. By the theorem of integration term by term of a series, and by the Theorem I-1-6 of [11], or Proposition 2.2 assertion (3), we get $\sum_{k=1}^{+\infty} |f_{n_{k+1}} - f_{n_k}|^p \in \mathfrak{L}^1_+(I)$, thus $\sum_{k=1}^{+\infty} |f_{n_{k+1}} - f_{n_k}|^p$ is finite I-a.e, by Proposition 3.4 statement (iii) of [12] or [13]. Let us set $M_k = \{t \in T / |f_{n_{k+1}} - f_{n_k}| \geq 2^{-\frac{k}{p}}\}$. We have $2^{-k} 1_{M_k} \leq |f_{n_{k+1}} - f_{n_k}|^p$ I-a.e, and $2^{-k} \int 1_{M_k} I \subset \int |f_{n_{k+1}} - f_{n_k}|^p I$, i.e., $2^{-k} N_1(1_{M_k}) \leq N_1(|f_{n_{k+1}} - f_{n_k}|^p) < 3^{-k}$. Let N be a nonnegative integer. If $t \notin \bigcup_{k=N+1}^{+\infty} M_k$, we have $|f_{n_{k+1}}(t) - f_{n_k}(t)| < 2^{-\frac{k}{p}}$, i.e., $\sum_{k=1}^{+\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < +\infty$, $\forall t \notin \bigcup_{k=N+1}^{+\infty} M_k$. Let us set $M = \bigcap_N \bigcup_{k=N+1}^{+\infty} M_k$, we get $\sum_{k=1}^{+\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < +\infty$, $\forall t \notin M$. Since $M \subset M_k$, and $0 \leq N_1(1_{M_k}) \leq (\frac{2}{3})^k$, we obtain $0 \leq N_1(1_M) \leq (\frac{2}{3})^k$, $\forall k$. This mean's that M is I-negligible, thus $\sum_{k=1}^{+\infty} |f_{n_{k+1}} - f_{n_k}| < +\infty$, I-a.e. From the relation $f_{n_k} = f_{n_1} + (f_{n_2} - f_{n_1}) + \ldots + (f_{n_k} - f_{n_{k-1}})$, we deduce $f_{n_k} \to f_{n_1} + g$, I-a.e, where

$g(t) = \sum_{k=1}^{+\infty} (f_{n_{k+1}}(t) - f_{n_k}(t))$, if $\sum_{k=1}^{+\infty} |f_{n_{k+1}}(t) - f_{n_k}(t)| < +\infty$, otherwise, we take $g(t) = 0$. Now, if we set $h = f_{n_1} + g$, we obtain $|f_{n_k} - h|^p \to 0$ I-a.e. Since $|f_{n_k} - h|^p$ is a Cauchy sequence of $\mathfrak{L}^1_+(I)$, and since, by Theorem 4.6 of [12] or [13], $\mathfrak{L}^1_+(I)$ is complete, we obtain $\lim_{n_k \to +\infty} N_1[(|f_{n_k} - h|)^p] = 0$, i.e., $\lim_{n_k \to +\infty} N_p[(|f_{n_k} - h|)] = 0$. From the inequality $|f_n - h|^p \leq 2^p[|f_n - f_{n_k}|^p + |f_{n_k} - h|^p]$ I-a.e, we deduce that $N_1(|f_n - h|^p) \leq 2^p[N_1(|f_n - f_{n_k}|^p) + N_1(|f_{n_k} - h|^p)]$. If we take $n \leq n_k$, and if $n \to +\infty$, then $\lim_{n \to +\infty} N_1(|f_n - h|^p) = 0$, i.e., $\lim_{n \to +\infty} N_p(|f_n - h|) = 0$. We have also $f_{n_k} \to h$, I-a.e, and for the statement 2, $f = h$ I-a.e. □

Proposition 3.7. (1) $\mathsf{L}^p(I) \subset \{f$ *I-measurable*, $|f|^p \in \mathfrak{L}^1_+(I)\}$.

(2) $\mathsf{L}^p(I)$ *is reticulated and* $\mathsf{L}^p(I) = \mathsf{L}^p_+(I) - \mathsf{L}^p_+(I)$.

(3) $\mathsf{L}^p_+(I) = \overline{[H_+]}$ *in* F_p.

(4) *The mapping* $I : \mathfrak{L}^p_+(I) \to cfb(E)$, *is continuous.*

(5) *and* $\mathsf{L}^p_+(I) \subset \mathfrak{L}^p_+(I)$.

Proof. If $f \in \mathsf{L}^p(I)$, then f is I-measurable by Theorem 3.6, and using Theorem II-3-1 of [11], or 3.3 of [15], see also Theorem 2.4, we obtain the statement 1. If $a, b, c, d \in \mathbb{R}^T$, we have $|\sup(a, b) - \sup(c, d)| \leq |a - c| + |b - d|$, and $|\inf(a, b) - \inf(c, d)| \leq |a - c| + |b - d|$. Therefore, we get $|\sup(a, b) - \sup(c, d)|^p \leq 2^p[|a - c|^p + |b - d|^p]$, and $|\inf(a, b) - \inf(c, d)|^p \leq 2^p[|a - c|^p + |b - d|^p]$. If a, b, c, d, are taken in F_p, these inequalities old I-a.e. For $\varepsilon > 0$, and $f_1, f_2 \in \mathsf{L}^p(I)$, there

exists $h_1, h_2 \in H$, such that $N_p(f_i - h_i) < \dfrac{\varepsilon^p}{2^{p+1}}$, for $i = 1, 2$. We obtain $N_1[(\mathrm{Sup}(f, g) - \mathrm{Sup}(h_1, h_2))^p] \leq 2^p(N_1(f - h_1)^p + N_1(f - h_2)^p] \leq \varepsilon^p$, we have therefore $N_p(\mathrm{Sup}(f, g) - \mathrm{Sup}(h_1, h_2)) \leq \varepsilon$, thus $\mathsf{L}^p(I)$ is reticulated and we get $\mathsf{L}^p(I) = \mathsf{L}^p_+(I) - \mathsf{L}^p_+(I)$. If $f \in \mathsf{L}^p_+(I)$, $\forall \varepsilon > 0$, $\exists h \in H$, such that $N_P(f - h) < \varepsilon$, since $|f - h^+| \leq |f - h|$ I-a.e, we obtain $N_p(f - h^+) < \varepsilon$. We have therefore $\mathsf{L}^p_+(I) \subset [\overline{(H_+)}$ in F_p, thus $\mathsf{L}^p_+(I) = [\overline{(H_+)}]$ in F_p. Since $\mathfrak{L}^p_+(I)$ is complete, therefore closed in F_p, and since $H_+ \subset \mathfrak{L}^p_+(I)$, we get $\mathsf{L}^p_+(I) \subset \mathfrak{L}^p_+(I)$. For the statement 4, if (f_n) converges to f in $\mathfrak{L}^p_+(I)$, we have $|f_n - f|^p$ converges to 0 in $\mathfrak{L}^1_+(I)$ thus $I^*(|f_n - f|^p)$ converges to 0 in $cfb(E)$. □

Theorem 3.8. (1) *Let* (f_n) *be an increasing sequence of* $\mathfrak{L}^p_+(I)$ *and* $g \in F_p$ *such that* $I^*(|g|^p) \in cc(E)$, *with* $f_n \leq g$ *I-a.e for all* $n \in \mathbb{N}^*$, *we have* $f = \sup_n f_n \in \mathfrak{L}^p_+(I)$, $\int |f|^p I \in cc(E)$, *and* (f_n) *converges to f in mean of order p.*

(2) *Let* (f_n) *be a decreasing sequence of* $\mathfrak{L}^p_+(I)$, *with* $\int f_n I \in cc(E)$, *then we have* $f = \inf_n f_n \in \mathfrak{L}^p_+(I)$, *and* (f_n) *converges to f in mean of order p.*

(3) *Let* (f_n) *be a sequence of* $\mathfrak{L}^p_+(I)$, *which converges I-a.e to f, and* $g \in F_p$ *such that* $I^*(|g|^p) \in cc(E)$, *with* $f_n \leq g$ *I-a.e for each n, then* $\int f^p I \in cc(E)$, *and* (f_n) *converges to f in mean of order p.*

(4) *In all these cases, we have* $\lim_{n \to +\infty} \int |f_n|^p I = \int |f|^p I$, *in* $cc(E)$.

Proof. We apply the Theorem II-1-3 of [11], see Theorem 2.5, to the sequence $(|f_n|)^p$, to obtain in the three cases $\lim_{n\to+\infty} N_1(|f_n|^p - |f|^p) = 0$, $|f|^p \in \mathfrak{L}^p_+(I)$, and $\lim_{n\to+\infty} \int |f_n|^p I = \int |f|^p I$, for the Hausdorff topology of $cc(E)$. Applying again the Theorem II-1-3 to the sequence $|f_n - f|^p$, we get $\lim_{n\to+\infty} N_1[|f_n - f|^p] = 0$, i.e., $\lim_{n\to+\infty} N_p(f_n - f) = 0$. □

Corollary 3.9. *Let (f_n) be a sequence of $\mathsf{L}^p(I)$ such that there exists $g \in \mathsf{L}^p_+(I)$, with $\int |g|^p I \in cc(E)$, and $|f_n| \le g$ I-a.e $\forall n$. If (f_n) converges to f I-a.e, then $f \in \mathsf{L}^p(I)$, and $\int |f|^p I = \lim_{n\to+\infty} \int |f_n|^p I$, for the Hausdorff topology of $cc(E)$. Moreover, we have $\lim_{n\to+\infty} N_p(f_n - f) = 0$.*

Proof. We apply the Corollary II-1-1 of [11], see also 2.6 to the sequence $(|f_n|^p)$, and to the sequence $(|f_n - f|^p)$, to obtain the result. □

Theorem 3.10. *We assume that E is a Banach space.*

(1) *Let f be an I-measurable function such that $I^*(|f|^p) \in cc(E)$, therefore $f \in \mathsf{L}^p(I)$.*

(2) *We have $\mathsf{L}^p(I) = \{f$ I-measurable, $|f| \in \mathfrak{L}^p_+(I)$, $I^*(|f|^p) \in cc(E)\}$.*

Proof. If $f \ge 0$, $\exists h_n \in H_+$, such that (h_n) converges I-a.e to f. If we set $\Phi_{n_k} = \inf\{h_n, \ldots, h_{n+k}\}$, we have $\Phi_{n_k} \downarrow \Phi_n = \inf_{k\ge n} h_k$. From the Corollary 3.9, we get $\Phi_n \in \mathsf{L}^p(I)$. Since $\Phi_n \uparrow f$, and $I^*(|f|^p) \in cc(E)$, using again the Corollary 3.9, we get $f \in \mathsf{L}^p(I)$. If f is not positive, since we have $f = f^+ - f^-$, thus $f \in \mathsf{L}^p_+(I) - \mathsf{L}^p_+(I) = \mathsf{L}^p(I)$, we obtain $\{f$ I-measurable, $|f| \in \mathfrak{L}^p_+(I)$, $I^*(|f|^p) \in cc(E)\} \subset \mathsf{L}^p(I)$.

The inclusion $L^p(I) \subset \{f\ I\text{-measurable},\ |f| \in \mathfrak{L}^p_+(I)\}$ comes from Proposition 3.7. Since E is complete, by Theorem 2.2 of [10], or Theorem I-2-2 of [11], see also 2.7, if $f \in \mathfrak{L}^p_+(I)$, then $\int (|f|^p)I \in cc(E)$. The Theorem I-2-2 of [11] comes from Theorem 1, page I-5 of [3], and the theorem of James Pryce [20]. □

Notation. $L^p(I) =$ the quotient space of $L^p(I)$, for the equivalence relation I-a.e.

Theorem 3.11. *Setting* $\|\bar{f}\|_p = N_p(f)$, *for* $f \in \bar{f}$, *we have* $[L^{(p)}(I), \|.\|_p]$ *is a Banach space.*

Proof. This comes from the Theorem 3.6. □

Theorem 3.12. *The space* $L^{(p)}(I)$ *is uniformly convex, therefore strictly convex and reflexive.*

Proof. The proof is the same as the one in page 358 of [6], he uses a method of McShane [7]. Let $\varepsilon > 0$, and $f, g \in L^{(p)}(I)$, such that $\|f\|_p \leq 1$, $\|g\|_p \leq 1$, with $\|f - g\|_p \geq \varepsilon$. Let us denote by M the collection of all $t \in T$, such that

$$|f(t) - g(t)|^p \geq \frac{\varepsilon^p}{4}[|f(t)|^p + |g(t)|^p] \geq \frac{\varepsilon^p}{4} \sup [|f(t)|^p, |g(t)|^p]\ I\text{-a.e.} \quad (3.1)$$

If we take $E = \mathbb{R}$ in (9'), page 357 of [6], we obtain the following inequality on the I-measurable set M:

$$|\frac{1}{2}[f(t) + g(t)]^p \leq [1 - \delta_p(\frac{\varepsilon}{\frac{1}{4^p}})][\frac{1}{2}[|f(t)|^p + |g(t)|^p]]. \quad (3.2)$$

If we set $N = T - M$, we obtain from 3.1, $\int |f - g|^p 1_N I \subset \frac{\varepsilon^p}{4}\int [|f|^p + |g|^p]I$, thus

$$N_1(|f-g|^p 1_N) \le \frac{\varepsilon^p}{4}[N_1(|f|^p) + N_1(|g|^p)] \le \frac{\varepsilon^p}{2}.$$

We obtain $\varepsilon^p \le N_1[|f-g|^p] \le N_1[|f-g|^p 1_M] + N_1[|f-g|^p 1_N] \le N_1[|f-g|^p 1_M] + \frac{\varepsilon^p}{2}$, thus $\frac{\varepsilon^p}{2} \le N_1[|f-g|^p 1_M] = N_p[|f-g|1_M]^p$, we get

$$[N_p[|f-g|1_M]] \ge \frac{\varepsilon}{2^{\frac{1}{p}}}. \tag{3.3}$$

Since $N_1[(|f-g|)^p 1_M] \le 2^p(\sup[N_p(|f|1_M), N_p(|g|1_M)])^p$, we get from 3.3,

$$\frac{\varepsilon}{2^{\frac{1}{p}}} \le [N_p[|f-g|1_M]] \le 2\sup[N_p(|f|1_M), N_p(|g|1_M)], \tag{3.4}$$

and

$$\sup[N_p(|f|1_M), N_p(|g|1_M)] \ge \frac{\varepsilon}{2^{1+\frac{1}{p}}}. \tag{3.5}$$

Using the statement 3.2, we obtain

$$\frac{1}{2}[(N_p(f))^p + (N_p(g))^p] - [\frac{1}{2}N_p(f+g)]^p \ge \delta_p \frac{\varepsilon}{4\frac{1}{p}} a, \tag{3.6}$$

where $a = \frac{1}{2}[(N_p(f))^p + (N_p(g))^p] \ge \frac{1}{2}\sup[(N_p(f))^p, (N_p(g))^p]$.

From 3.4 or 3.5, we deduce $\sup[(N_p(f1_M))^p, (N_p(g1_M))^p] \ge \frac{\varepsilon^p}{2^{p+1}}$, thus $\sup[(N_p(f))^p, (N_p(g))^p] \ge \frac{\varepsilon^p}{2^{p+1}}$. With the statement 3.6, we obtain

$$\frac{1}{2}[(N_p(f))^p + (N_p(g))^p] - [\frac{1}{2}N_p(f+g)]^p \ge \delta_p \frac{\varepsilon}{4^{\frac{1}{p}}} \frac{\varepsilon^p}{2^{p+2}}.$$

Since $\frac{1}{2}[(N_p(f))^p + (N_p(g))^p] \leq 1$, we get $[\frac{1}{2}\|f+g\|_p]^p \leq [1 - \delta_p \frac{\varepsilon}{4^{\frac{1}{p}}} \frac{\varepsilon^p}{2^{p+2}}]$, and $[\frac{1}{2}\|f+g\|_p] \leq [1 - \delta_p \frac{\varepsilon}{4^{\frac{1}{p}}} \frac{\varepsilon^p}{2^{p+2}}]^{\frac{1}{p}}$, for all $f, g \in L^{(p)}(I)$. Such that, $\|f\|_p \leq 1, \|g\|_p \leq 1$, with $\|f-g\|_p \geq \varepsilon$. Therefore, by the Clarkson criterion [2], $L^{(p)}(I)$ is uniformly convex. □

Theorem 3.13. *We assume that E is a Banach space. Let* $p, q \in]1, +\infty[$ *be conjugated, i.e.,* $\frac{1}{p} + \frac{1}{q} = 1$. *Therefore, the topological dual of* $L^{(p)}(I)$ *is* $L^{(q)}(I) : [L^{(p)}(I)]' = L^{(q)}(I)$.

Proof. By the Theorem 3.10, we have

$$L^p(I) = \{f \text{ } I\text{-measurable}, |f| \in \mathfrak{L}^p_+(I), \text{ with } I^*(|f|^p) \in cc(E)\}$$

$$= \{f \text{ } I\text{-measurable}, |f| \in L^p_+(I), \text{ with } I^*(|f|^p) \in cc(E)\}$$

$$= \{f \text{ } I\text{-measurable}, |f| \in L^p_+(I)\}.$$

Moreover, by the Remark II-1-8, page 36 of [11], see also Remark 4.16 [12] or [13], we have $\mathfrak{L}^1_+(I) = \bigcap_{y \in E'} \mathfrak{L}^1_+(I_y) = L^1_+(I)$. We get $L^p_+(I) = \bigcap_{y \in E'} L^p_+(I_y)$, and $L^p(I) = L^{(p)}_+(I) - L^{(p)}_+(I) = \bigcap_{y \in E'} L^{(p)}(I_y)$, since $f = f^+ - f^-$. We obtain $[L^{(p)}(I_y)]' \subset [\bigcap_{y \in E'} L^{(p)}(I_y)]' = [L^{(p)}(I)]'$, we deduce that $\bigcap_{y \in E'} [L^{(p)}(I_y)]' \subset [L^{(p)}(I)]'$, i.e., $L^{(q)}(I) = \bigcap_{y \in E'} L^{(q)}(I_y) \subset [L^{(p)}(I)]'$. Now, we set $G = L^{(q)}(I)$, which is closed in $[L^{(p)}(I)]' : [L^{(p)}(I_y)]' = L^{(q)}(I_y)$ is closed in $[L^{(p)}(I)]'$.

Let $\phi : [L^{(p)}(I)]'' \to [L^{(q)}(I)]' = G'$, i.e., $\phi : L^{(p)}(I) \to [L^{(q)}(I)]'$, be the canonical injection. The orthogonal of G is the kernel of ϕ, i.e., $G^{\perp} = \ker(\phi)$.

We get, since $\ker(\phi) = \{0\}$, that the quotient space $[\mathrm{L}^{(p)}(I)]'' / G^{\perp}$, is by (3), page 93 of [6], topologically isomorphic to G'. On the other hand, $[\mathrm{L}^{(p)}(I)]'' = \mathrm{L}^{(p)}(I)$, we get $\mathrm{L}^{(p)}(I) = [\mathrm{L}^{(q)}(I)]'$. □

Corollary 3.14. *Let E be a Banach space. If $p \in]1, +\infty[$, we have $\mathrm{L}^{(p)}(I)$ is uniformly smooth, this is equivalent to say that it's norm is uniformly strongly differentiable.*

Proof. By the Theorem 3.13 $[\mathrm{L}^{(q)}(I)]' = \mathrm{L}^{(p)}(I)$, since $\mathrm{L}^{(q)}(I)$ is uniformly convex, by (9), page 365 of [6], $\mathrm{L}^{(p)}(I)$ is uniformly smooth. This is equivalent, by (6), page 364 of [6], that it's norm is uniformly strongly differentiable.

Theorem 3.15. *Let T be a locally compact space, which is metrizable and separable, and I be an s-compact set-valued Radon measure with values in $cfb(E)$. We have $\mathrm{L}^{(p)}(I) = [\overline{K(T)}]_{N_p}$, i.e., $K(T)$ is dense in $\mathrm{L}^{(p)}(I)$, for $p \geq 1$.*

Proof. Since we are in the quotient space, by N_p, we denote the norm. By Proposition III-1-4 of [11], see also Proposition 2.9, we have the result for $p = 1$.

$$\text{If } f \in K(T), \text{ then } |f|^p \in K(T) \subset \mathrm{L}^{(1)}(I),$$

by Proposition III-1-3 of [11]. We deduce that $K(T) \subset \mathrm{L}^{(p)}(I)$. Since $\mathrm{L}^{(p)}(I)$ is complete, thus closed in the quotient space of F_p, we obtain $[\overline{K(T)}]_{N_p} \subset \mathrm{L}^{(p)}(I)$. In [15], we had set $H = \{f \in K(T), I(|f|) \in cc(E)\}$, and had obtain in Proposition 3.7 $\mathrm{L}^1(I) = \overline{K(T)}$, in the same way, we have $\mathrm{L}^p(I) = [\overline{K(T)}]_{N_p}$. Since $H \subset K(T)$, we get the result. We note that if E is complete, then $H = K(T)$, since by Proposition 2, page I-15 of [17], $cc(E)$ is closed in $cfb(E)$. □

Proposition 3.16. *The inductive limit topology on* $K(T)$, *is finer than the one induced on* $K(T)$ *by* N_p.

Proof. Let K be a compact subset of T, and $f \in K(T)$ such that $\text{Supp}(f) \subset K$. We have $N_1(|f|^p) \leq \|f^p\|_u N_1(\varphi_K)$, where

$$\varphi_K \in K(T),\ \varphi_K(t) \in [0, 1] \forall t \in T, \text{ with } \varphi_K(t) = 1 \text{ if } t \in K.$$

We get $[N_p(f)]^p \leq \|f^p\|_u [N_p(\varphi_K)]^p$. Since there exists $t_0 \in K$ such that $[\|f^p\|_u]^{\frac{1}{p}} = |f(t_0)| = \|f\|_u$, we obtain $N_p(f) \leq \|f\|_u N_p(\varphi_K)$. □

Remark 3.17. We have here, new kind of $L^{(p)}$ spaces, whose properties need to be investigated more deeply. For example, a problem is : how to define the duality $\langle L^{(p)}(I), L^{(q)}(I) \rangle$.

Remark 3.18. For other references about the professor Thiam, one may consult [21], [22], [23], [24], and [26].

Remark 3.19. Something relevant is: most of the proofs, are direct adaptations, of the methods and techniques, used in the classical $L^{(p)}$ spaces. We wonder if these adaptations, could be possible, from the previous existing works in this field, before the results we presented in [11]. Working with a set valued integral, like in the positive scalar case, is quite surprising and unusual.

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