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HYPERBOLIC PARABOLOID SPLINE FOR SPATIAL DATA FITTING AND APPROXIMATION

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Abstract

A method to construct the free spline curve and surface is presented in this paper, in which proposes four rational basis functions, each of them includes a common control handle and two variables. One of the variables is determined by a function in a set with a single parameter, and the other is defined by a multi-integral upper limit functions about the previous variable. For a spatial control polygon of four points, a spline curve can be produced by the basis functions, which can approximate the control edges at will via changing the control handle. Additionally, the authors consider the properties of shape-preserving and the spline curves at the endpoints. By constraining the control handle for the spline, the endpoint curvatures are less than some threshold at both endpoints. For a range of points, consequently, one approximately G^2-continuity curve can be connected by this spline curves without other conditions required.

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1. Introduction

There are mainly available to construct the spline curves and surfaces with parametric and algebraic forms [2, 10, 11, 16-18] in the area of computer aided geometric design (CAGD) and computation. The great advantages of the latter are its more freedom, intrinsically simpler representation, and better geometric operation etc, but its structure is difficult to understand and the computing is hard to apply widely. Therefore, at present, the former became an irreplaceable method in the geometric design due to the convenience of constructing spline curves/surfaces, specially, of which the spline function are a central issue. In the meantime, rational parametric spline is often used to construct the approximate spline for a control polygon, subdivision and the model reconstruction, which have excellent properties as nice effect of approximation, easy to implement, and with shape handles for human-machine interaction [5, 12] etc. The current research of rational spline becomes one of the hot spots in CAGD, and a number of authors have investigated the rational polynomial spline, such as the paper [1] introduces a rational spline that used for offset curve, different shapes spline curves are arisen in a family of a plane rational curves depending on a parameter. Besides, shape-preserving curves and surfaces design are also a key issue for the rational spline [8]. In addition, the complex rational splines are recalled in the literature [15], and the algebraic-trigonometric blended splines is studied in [19]. In order to produce a more fairing spline, in other words, to reach fairing as G^2-continuity, usually, the curvature of a spline curve (surface) is mainly considered at the connection points in each segment and even the whole splines [3]. A definition of the twisted cubic is developed was introduced in the literature [7].

The literature [4] points out that rational cubic allow modification of their fullness even when the end tangents are kept fixed, this is the reason why they are occasionally preferred to standard cubic in CAGD. This should serve as an easy and intuitive introduction and help the potential user to choose a suitable representation. This work starts from the papers [13, 14], where the authors present an approach to fit a range

of four spatial points. In those two papers, their main idea is to define a spline on a hyperbolic paraboloid in the tetrahedron via a factor and a function, and the method starts with algebraic form, but what to be eventually applied is its parametric form. The proceeding of implementation is transforming the unified spline curve to an arbitrarily quadrilateral by the barycentric coordinates [9]. Thus, this approach to construct a spline combines the advantage of algebraic and parametric form, nonetheless, the spline has its limitation that is not able to arbitrarily approximate the control polygon with four points. This deserves further study on the question about the rational form on the hyperbolic paraboloid. There is an intuitive argument for us: This kind of spline can be extended as a free spline to arbitrarily approximate a quadrilateral, and the method of implementation does as customary Bézier splines. However, it should be rational form with a shape handle and a similar function, this is what we are considering in this paper.

The rest of the paper is organized as follows. In Section 2, we analyze a ruled hyperbolic paraboloid with a pair of parameters, and define a unified spline on this ruled surfaced by a real number and a function in a set. In Section 3, the curvature's properties of this spline are studied, and these unit spline are employed to attach a roughly G^2-continuity curve to approximate a control polygon, meanwhile, there is a shape handle in each spline for curve control. In what is to follow, the surface employing this spline is considered to approximate a spatial subdivision in Section 4. Finally, the main conclusions will be drown in Section 5.

2. Basis Functions

First, the four bi-parameter functions are defined by follows:

$$\begin{cases} \alpha_1(u, v) = \dfrac{(1-u)(1-v)}{1+(\lambda-1)v}, \\ \alpha_2(u, v) = \dfrac{\lambda(1-u)v}{1+(\lambda-1)v}, \\ \alpha_3(u, v) = \dfrac{\lambda uv}{1+(\lambda-1)v}, \\ \alpha_4(u, v) = \dfrac{u(1-v)}{1+(\lambda-1)v}, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}^+$ is a nonnegative real number that works as a shape handle for a spline in the method hereinafter, and $u, v \in [0, 1]$ are two variables.

The four functions have the following characteristics:

(1) Non-negative: $\alpha_i(u, v) \geq 0$ $(i = 1, 2, 3, 4)$.

(2) Normalization: $\sum_{i=1}^{4} \alpha_i(u, v) = 1$.

(3) On a hyperbolic paraboloid: For arbitrarily $\lambda > 0$, the four functions without the use of α_4 (in fact, α_4 is determined by the others because of $\sum_{i=1}^{4} \alpha_i(u, v) = 1$) have relationship of

$$\alpha_2 = (\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3). \tag{2}$$

Notice that the surface determined by the Equation (2) is a sheet of hyperbolic paraboloid in the coordinate system of $(\alpha_1, \alpha_2, \alpha_3)$, which is in a tetrahedron with the four vertexes $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (see Figure 1), and those four vertexes all lie in the surface determined by the Equation (2). Therefore, we regard the formula (1) as a rational parametrization of the hyperbolic paraboloid.

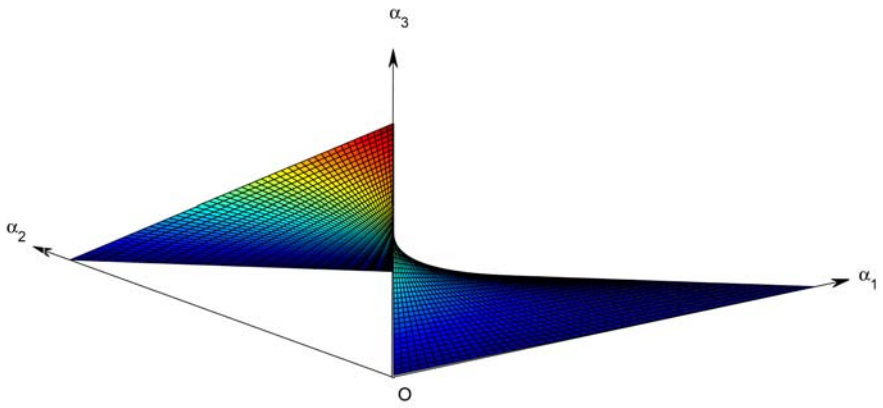


Figure 1. Hyperbolic paraboloid in tetrahedron.

If the two functions $v(t)$ and $u(t)$ are defined on the parameter t in $[a, b]$, in consequence, one curve is produced in the hyperbolic paraboloid. Next, we give the following parametric definition to $v(t)$, and determine the other parametric $u(t)$ with $v(t)$, to look for the splines on the ruled surface (1).

Definition 1. $\mathcal{V}[a, b]$ is a set of nonnegative real function, a function $v(t)$ with $t \in [a, b]$ in which satisfies for these conditions:

(1) $v(t) \in C^2[a, b]$;

(2) $v(a) = v(b) = 0$ and $v(t)$ is not constant zero for $t \in [a, b]$;

(3) $v(t) \in [0, 1]$ for any $t \in [a, b]$.

This is an integral series as following:

$$\begin{cases} u_0(t) = v(t), & t \in [a, b], \\ \delta_m = \dfrac{1}{\displaystyle\int_a^b (b - t)u_m(t)dt}, & m = 0, 1, \cdots, \\ u_m(t) = \delta_{m-1}\displaystyle\int_a^t (b - \tau)u_{m-1}(\tau)d\tau, & m = 1, 2, \cdots. \end{cases} \tag{3}$$

Given an integral number $m \geq 1$, let $u(t) = u_m(t)$, the function $u(t)$ has $u(a) = 0$, $u(b) = 1$, and $u'(t) = \delta_{m-1}(b - t)u_{m-1}(t)$, which is thus at least twice differentiable function, and $u'(a) = 0$, $u'(b) = 0$. At the same time, its second derivatives at the endpoints are

- case $m > 1$, $u''(a) = 0$, $u''(b) = -\delta_{m-1}\delta_{m-2}$,
- case $m = 1$, $u''(a) = \delta_0(b - a)v_a$, $u''(b) = 0$.

v_a is indicative of $v'(a)$, and so is v_b of $v'(b)$ below.

Furthermore, there are $\dfrac{dv}{du} = \dfrac{v'(t)}{\delta_{m-1}(b - t)u_{m-1}(t)}$ and $\lim\limits_{t \to a} \dfrac{v'(t)}{v(t)} = \infty$ if $v_a \neq 0$ in this scheme, which can refer to the literatures [13, 14].

Meanwhile, those spline curve in which was consider in a unified tetrahedron that could be transformed into different unit with four control points by one unified tetrahedron.

Chosen a $v(t)$ in $\mathcal{V}[a, b]$, function $u(t) = u_m(t)(m = 1, 2, \cdots)$ is determined by formula (3). We substitute $v(t), u(t)$ to the relative variables v, u in the formula (1), then the corresponding four basis functions are arisen, and they are $\alpha_1(t), \alpha_2(t), \alpha_3(t)$, and $\alpha_4(t)$, $t \in [a, b]$, respectively, each of which includes the common control handle λ. If $v(t)$ is taken as a polynomial, then $u(t)$ is also a polynomial and the four basis functions are all rational polynomials. For example, using

$$v(t) = (1 - t)t, \quad t \in [0, 1], \tag{4}$$

and working out $u(t) = u_1(t) = 3t^2 - 2t^3$ to the four basis functions, and their shape are illuminated in Figure 2. As a curve on the hyperbolic paraboloid, those four basis functions contain the properties as the four functions defined by formula (1).

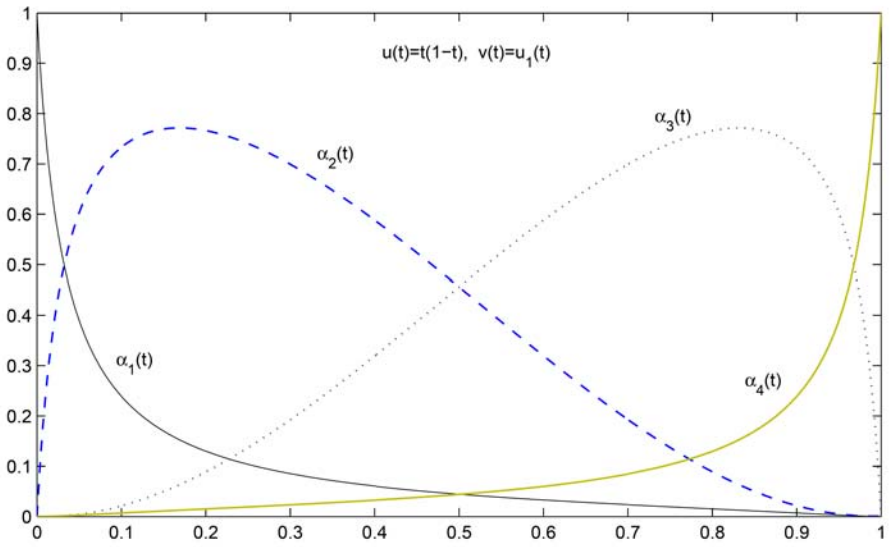


Figure 2. Four basis functions with $\lambda = 21$.

Furthermore, they have properties when t is a and b as following:

$$\alpha_1(a) = 1, \quad \alpha_1(b) = 0,$$

$$\alpha_2(a) = 0, \quad \alpha_2(b) = 0,$$

$$\alpha_3(a) = 0, \quad \alpha_3(b) = 0,$$

$$\alpha_4(a) = 0, \quad \alpha_4(b) = 1,$$

and

$$\alpha_1'(a) = -\lambda v_a, \quad \alpha_1'(b) = 0,$$

$$\alpha_2'(a) = \lambda v_a, \quad \alpha_2'(b) = 0,$$

$$\alpha_3'(a) = 0, \quad \alpha_3'(b) = \lambda v_b,$$

$$\alpha_4'(a) = 0, \quad \alpha_4'(b) = -\lambda v_b.$$

These results lead us to construct a spline curve or a surface if both v_a and v_b are not zeros.

3. Basis Functions for Spline Curve

Let $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4 \in \mathbb{R}^3$ be a range of points, which forms a quadrilateral with three control edges, and the authors also call the three control edges as a control quadrilateral. With the four basis functions above, we define the following spline for the quadrilateral:

$$\mathbf{C}(t, \lambda) = \sum_{i=1}^{4} \mathbf{P}_i \alpha_i(t), \quad t \in [a, b], \tag{5}$$

where t is a unique parametric variable, λ is a control handle for a spline in terms of formula (1), and $v(t)$ is a function in $\mathcal{V}[a, b]$ defined by Definition 1.

3.1. Properties of spline curve

With the definition of spline being of $\lambda > 0$ and function $v(t)$ in $\mathcal{V}[a, b]$, the following conclusions about spline curve $\mathbf{C}(t, \lambda)$ can be derived directionally:

(i) The spline is in the convex hull that forms by the four points.

(ii) The spline, arisen by expression (5) with arbitrarily $\lambda > 0$, interpolates the endpoints $\mathbf{P}_1$ and $\mathbf{P}_4$ (see Figure 3, which applies $v(t) = (1 - t)t$, $t \in [0, 1]$ and $u(t) = u_2(t) = 10t^3 - 17.5t^4 + 11t^5 - 2.5t^6$). At the same time, the splines with different m are of the same properties at the endpoints (see Figure 4).

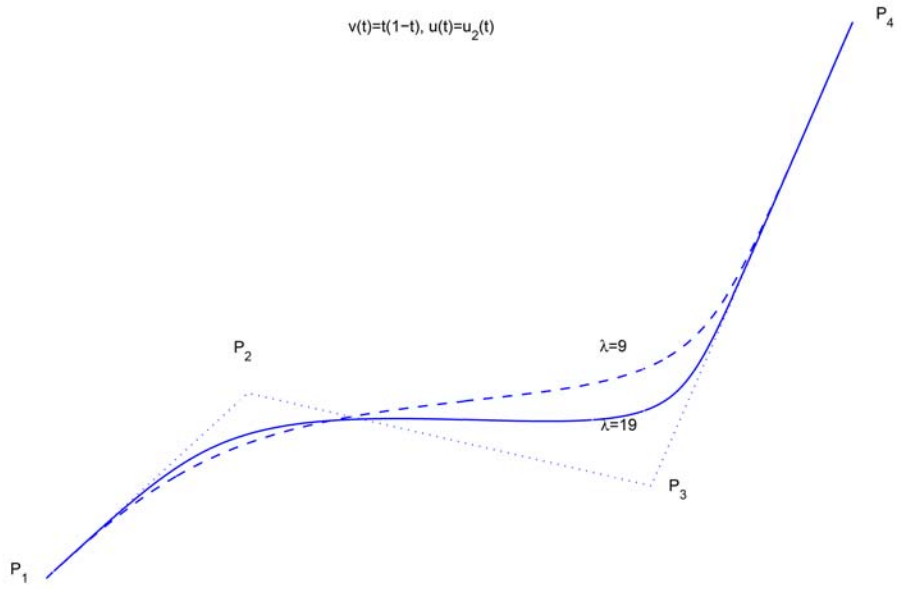


Figure 3. Spline curves with different λ for quadrilateral.

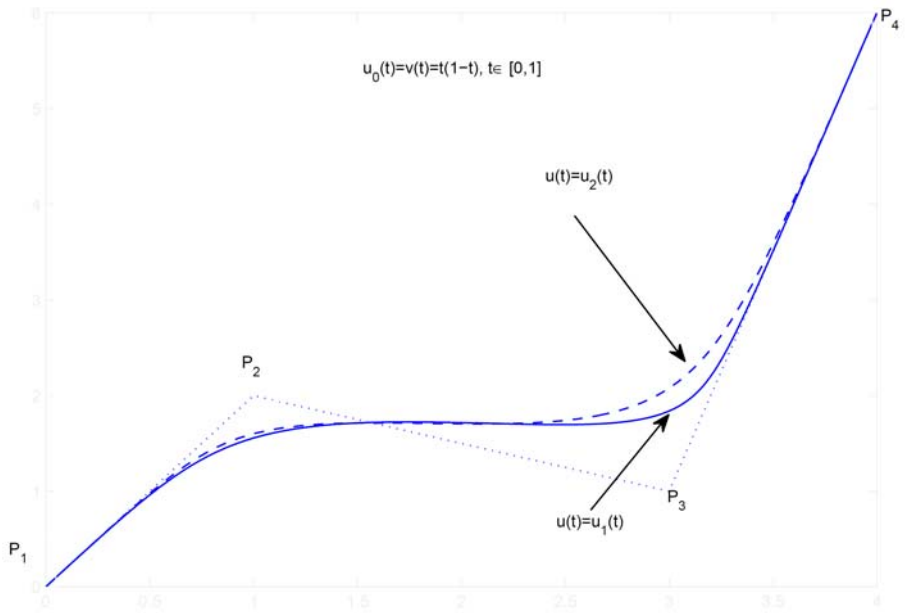


Figure 4. Spline curves with different times integral for quadrilateral and $\lambda = 19$.

(iii) It implies that the spline is tangent to $\overrightarrow{\mathbf{P}_1\mathbf{P}_2}$ at $\mathbf{P}_1$, and $\overrightarrow{\mathbf{P}_3\mathbf{P}_4}$ at $\mathbf{P}_4$. When λ is sufficiently small, the results of a experiment can be observed at the small neighbourhood of the two endpoints.

(iv) With $\lim_{\substack{\lambda\to+\infty \\ t\to a}} \alpha_2(t) = 1$ and $\lim_{\substack{\lambda\to+\infty \\ t\to b}} \alpha_3(t) = 1$, it implies that: the greater λ, the more approximate to the control polygon the spline becames, and that can approximate to the control polygon through infinite increasing the value of λ. Figure 3 shows the variation of the spline curves with two different λ and the same $v(t)$ with function (4).

3.2. Curvatures of splines

By the definitions of (1) and (3), we can obtain the second derivatives at the endpoints with $m = 1$,

$$\alpha_1''(a) = -(b-a)\delta_0 v_a - \lambda v''(a) + 2\lambda(\lambda-1)v_a^2,$$

$$\alpha_2''(a) = \lambda v''(a) - 2\lambda(\lambda-1)v_a^2,$$

$$\alpha_3''(a) = 0,$$

$$\alpha_4''(a) = (b-a)\delta_0 v_a,$$

and

$$\alpha_1''(b) = 0,$$

$$\alpha_2''(b) = 0,$$

$$\alpha_3''(b) = \lambda v''(b) - 2\lambda(\lambda-1)v_b^2,$$

$$\alpha_4''(b) = -\lambda v''(b) + 2\lambda(\lambda-1)v_b^2.$$

As a result, the absolute curvatures at the endpoints of the spline curve are

$$k(\mathbf{P}_1) = k(a) = \frac{(b-a)\delta_0}{\lambda^2|v_a|}\frac{|(\mathbf{P}_2-\mathbf{P}_1)\times(\mathbf{P}_4-\mathbf{P}_1)|}{|(\mathbf{P}_2-\mathbf{P}_1)|^3},$$

$$k(\mathbf{P}_4) = k(b) = 0.$$

The similar conclusion can be obtained with $m > 1$,

$$\alpha_1''(a) = -\lambda v''(a) + 2\lambda(\lambda-1)v_a^2,$$

$$\alpha_2''(a) = \lambda v''(a) - 2\lambda(\lambda-1)v_a^2,$$

$$\alpha_3''(a) = 0,$$

$$\alpha_4''(a) = 0,$$

and

$$\alpha_1''(b) = -\delta_{m-1}\delta_{m-2},$$

$$\alpha_2''(b) = 0,$$

$$\alpha_3''(b) = \lambda v''(b) - 2\lambda(\lambda - 1)v_b^2,$$

$$\alpha_4''(b) = \delta_{m-1}\delta_{m-2} - \lambda v''(b) + 2\lambda(\lambda - 1)v_b^2.$$

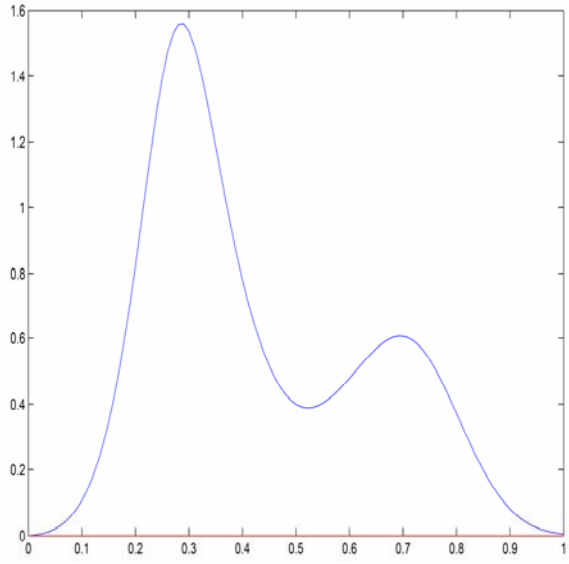
Then both curvatures at the endpoints of the spline curve are

$$k(\mathbf{P}_1) = k(a) = 0,$$

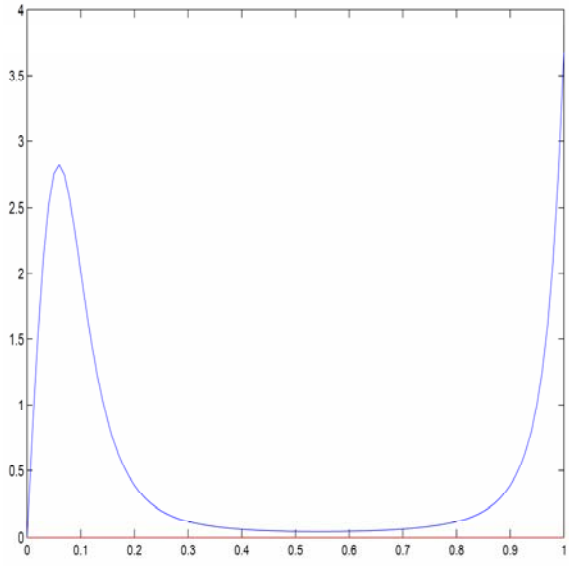
$$k(\mathbf{P}_4) = k(b) = \frac{\delta_{m-1}\delta_{m-2}}{\lambda^2 v_b^2} \frac{|(\mathbf{P}_3 - \mathbf{P}_4) \times (\mathbf{P}_4 - \mathbf{P}_1)|}{|(\mathbf{P}_3 - \mathbf{P}_4)|^3}.$$

In literatures [13, 14], there is $k(\mathbf{P}_1) = k(\mathbf{P}_4)$ if we consider in a unified form as $\mathbf{P}_1 = (0, 0, 0)^T$, $\mathbf{P}_2 = (1, 0, 0)^T$, $\mathbf{P}_3 = (0, 1, 0)^T$, $\mathbf{P}_4 = (0, 0, 1)^T$, and $v_a = -v_b$. For this reason, the curve connected by these splines with G^2-continuity is also approximately in transforming to different units with four control points.

Employing $v(t)$ as (4) and $u(t) = u_2(t)$, the plot of Figure 5(a) describes the curvature of a spline with the parameter t when $\lambda = 11$, and so is Figure 5(b) when $\lambda = 0.4$ (its data are $\mathbf{P}_1 = (0, 0, 0)^T$, $\mathbf{P}_2 = (1, 2, 5)^T$, $\mathbf{P}_3 = (3, 1, 4)^T$, and $\mathbf{P}_4 = (4, 6, 0)^T$).



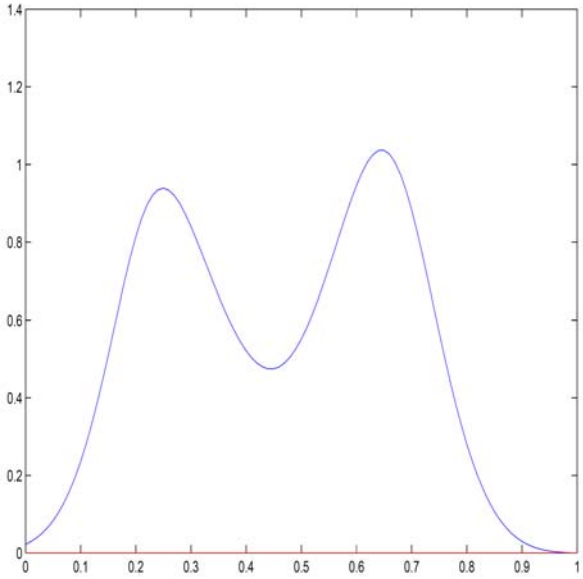
(a)



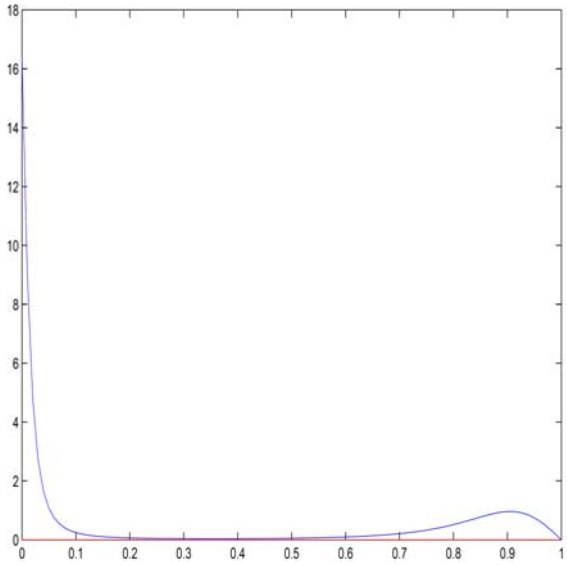
(b)

Figure 5. Curvatures on parameter t with different λ when $u(t) = u_2(t)$.

Figures 6(a) and 6(b) describe the curves being of $u(t) = u_1(t)$ with $\lambda = 0.4$ and $\lambda = 11$, respectively.



(a)



(b)

Figure 6. Curvatures on parameter t with different λ when $u(t) = u_1(t)$.

On the basis of the expression of curvature at the endpoints and what the figures describe above, with a comparison value 1, we can conclude that

(i) The curvature of one endpoint is zero that keeps fixed, and the other endpoint is approximate to zero with a great λ, which enables us to construct an approximately G^2-continuity curve via attaching some one as method of Bézier, namely, only consider that the point to be connected between two segments is three collinear with the two adjacent points.

(ii) With a small λ in the spline, the curvatures at endpoints are a lager alteration, and the curvature is almost equal at the others points. According to the simple energy integral formula $\int_a^b k^2(t)dt$, which only includes the curvature, the energy of this spline is thus relatively small, so, we can also use them to produce a fairly curve with many spline units.

3.3. Degree reduction and elevation

The method what we put up is not only suitable for four control points, which can also reduce (elevate) the degree to less (more) than four points in space.

If $\mathbf{P}_2$ and $\mathbf{P}_3$ are considered as one point, then

$$\mathbf{C}(t, \lambda) = \sum_{i=1}^{3} \mathbf{P}_i \alpha_i(t), \quad t \in [a, b],$$

where

$$\begin{cases} \alpha_1(u, v) = \dfrac{(1-u)(1-v)}{1+(\lambda-1)v}, \\ \alpha_2(u, v) = \dfrac{\lambda v}{1+(\lambda-1)v}, \\ \alpha_3(u, v) = \dfrac{u(1-v)}{1+(\lambda-1)v}. \end{cases}$$

After a short calculation, we obtain the tangent vector at the two endpoints

$$\mathbf{C}'(a, \lambda) = \lambda v_a(\mathbf{P}_2 - \mathbf{P}_1), \quad \mathbf{C}'(b, \lambda) = \lambda v_b(\mathbf{P}_2 - \mathbf{P}_3), \tag{6}$$

which possesses an analogy with endpoints properties of the four control points. Using the same $v(t)$ and $u(t) = u_2(t)$ by formula (4), Figure 7 describes the degree reduction for three points, which also can do with two control points only.

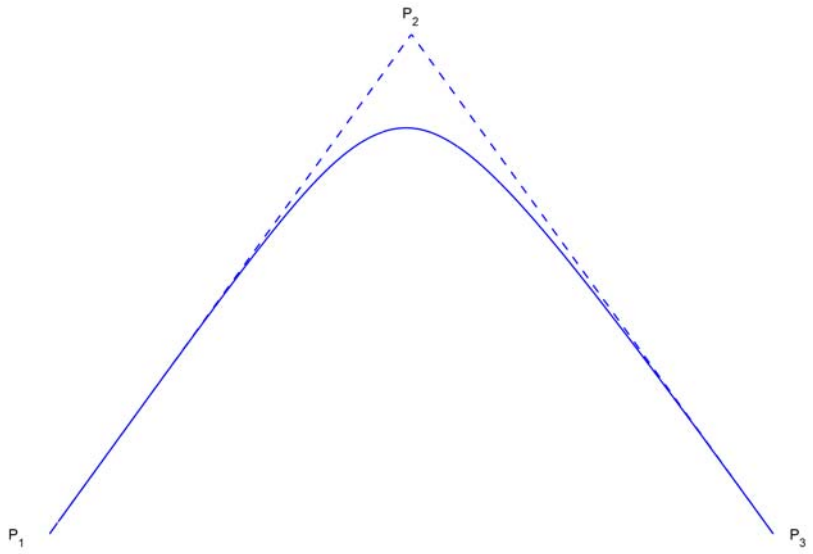


Figure 7. Degree reduction to control polygon with three points.

The scheme of degree elevation is analogous to the ordinary Bézier spline. Assume there are five points in space $\mathbf{P}_1, \cdots, \mathbf{P}_5$, one spline $\mathbf{C}_1^{(4)}(t, \lambda)$ can be produced by the previous four points, and another $\mathbf{C}_2^{(4)}(t, \lambda)$ is done by the back four points. One curve defined by

$$\mathbf{C}(t, \lambda) = (1 - u)\mathbf{C}_1^{(4)}(t, \lambda) + u\mathbf{C}_2^{(4)}(t, \lambda), \tag{7}$$

is the form of degree elevation for five points, and $u = u(t)$ has been defined above. The formula (7) possesses the endpoint properties as

$$\mathbf{C}'(a, \lambda) = \lambda v_a(\mathbf{P}_2 - \mathbf{P}_1), \quad \mathbf{C}'(b, \lambda) = \lambda v_b(\mathbf{P}_4 - \mathbf{P}_5),$$

which is analogous to the spline itself. The endpoint curvatures are

$$k(\mathbf{P}_1) = k(a) = \frac{(b-a)\delta_0}{\lambda^2|v_a|}\frac{|(\mathbf{P}_2 - \mathbf{P}_1)\times(\mathbf{P}_4 - \mathbf{P}_1)|}{|(\mathbf{P}_2 - \mathbf{P}_1)|^3},$$

$$k(\mathbf{P}_5) = k(b) = 0$$

while $m = 1$, and

$$k(\mathbf{P}_1) = k(a) = 0,$$

$$k(\mathbf{P}_5) = k(b) = \frac{\delta_{m-1}\delta_{m-2}}{\lambda^2 v_b^2}\frac{|(\mathbf{P}_4 - \mathbf{P}_5)\times(\mathbf{P}_5 - \mathbf{P}_2)|}{|(\mathbf{P}_4 - \mathbf{P}_5)|^3}$$

while $m > 1$. Figure 8 shows the degree elevation for five points with $v(t) = t(1-t)(t \in [0, 1])$ and $u(t) = u_2(t)$.

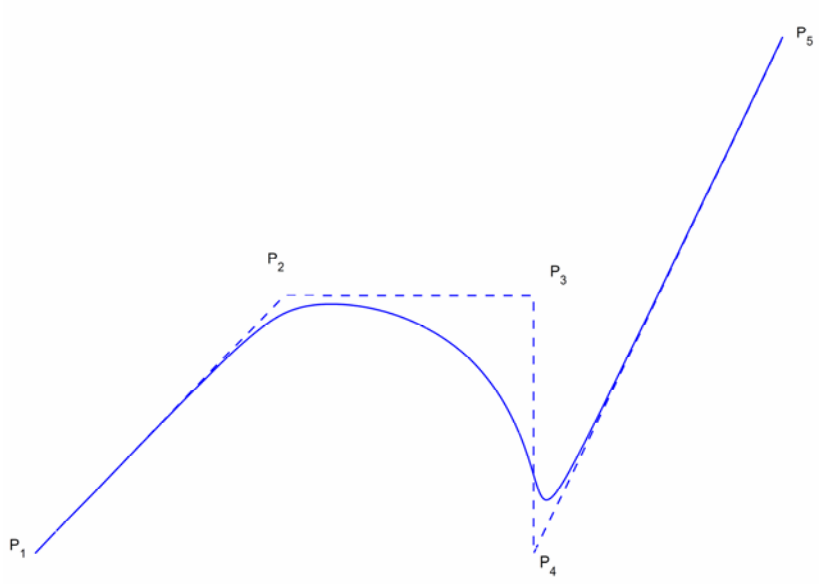


Figure 8. Degree elevation to control polygon with five points.

Utilizing the recursive method, we can generate a spline to a control polygon with more than five points. If the degree elevation formula is obtained for $n-1(n > 5)$ points, then we can also elevate degree to n

points as $\mathbf{P}_1, \cdots, \mathbf{P}_n$. A spline $\mathbf{C}_1^{(n-1)}(t, \lambda)$ can be produced by the previous $n-1$ points, and another $\mathbf{C}_2^{(n-1)}(t, \lambda)$ is obtained by the back $n-1$ points. Similarly, one degree elevation curve is defined by

$$\mathbf{C}(t, \lambda) = (1-u)\mathbf{C}_1^{(n-1)}(t, \lambda) + u\mathbf{C}_2^{(n-1)}(t, \lambda) = \sum_{i=1}^{4} \alpha_i(t)\mathbf{Q}_i, \tag{8}$$

where

$$\mathbf{Q}_i = \sum_{k=0}^{n-4} [C_{n-4}^k (1-u)^{n-4-k} u^k \mathbf{P}_{i+k}], \quad C_{n-4}^k = \frac{(n-4)!}{k!\,(n-4-k)!}.$$

The formula (8) possesses the endpoint properties of

$$\mathbf{C}'(a, \lambda) = \lambda v_a(\mathbf{P}_2 - \mathbf{P}_1), \quad C'(b, \lambda) = \lambda v_b(\mathbf{P}_{n-1} - \mathbf{P}_n),$$

and the curvatures are

$$k(\mathbf{P}_1) = k(a) = \frac{(b-a)\delta_0}{\lambda^2 |v_a|} \frac{|(\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_4 - \mathbf{P}_1)|}{|(\mathbf{P}_2 - \mathbf{P}_1)|^3},$$

$$k(\mathbf{P}_n) = k(b) = 0,$$

in the case of $m = 1$, while

$$k(\mathbf{P}_1) = k(a) = 0,$$

$$k(\mathbf{P}_n) = k(b) = \frac{\delta_{m-1}\delta_{m-2}}{\lambda^2 v_b^2} \frac{|(\mathbf{P}_{n-1} - \mathbf{P}_n) \times (\mathbf{P}_n - \mathbf{P}_{n-2})|}{|(\mathbf{P}_{n-1} - \mathbf{P}_n)|^3},$$

in the case of $m > 1$.

In expression (8), we rewrite the formula through changing the order of summation

$$\mathbf{C}(t, \lambda) = \sum_{i=1}^{4} \alpha_i(t)\mathbf{Q}_i = \sum_{i=1}^{n} \beta_i(t)\mathbf{P}_i,$$

where β represents $(\beta_1, \beta_2, \cdots, \beta_n)$, and there is a relationship between β and α by

$$\beta = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \alpha,$$

where $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_{n-4})^T$, and $\gamma_k = C_{n-4}^k (1-u)^{n-4-k} u^k (k = 0, 1, \cdots, n-4)$. In fact, $\gamma_k (k = 0, 1, \cdots, n-4)$ are the Bernstein basis functions.

3.4. Shape-preserving for a quadrilateral in a plane

In CAGD, what we employ to spline curve should be shape-preserving for a convex quadrilateral in a plane. Since a quadrilateral in space can be mapped into a normal square, for example, $\mathbf{Q}_1(0, 0, 0)$, $\mathbf{Q}_2(0, 1, 0)$, $\mathbf{Q}_3(1, 1, 0)$, and $\mathbf{Q}_4(1, 0, 0)$ in space, then the spline curve in terms of formula (5) is

$$\mathbf{C}(t, \lambda) = (\alpha_3(t) + \alpha_4(t), \alpha_2(t) + \alpha_3(t))^T, \quad t \in [a, b], \quad \mathbf{C}(t, \lambda) \subset \mathbb{R}^2, \tag{9}$$

here, $\alpha_3(t) + \alpha_4(t) = u(t)$, and we denote $\alpha_2(t) + \alpha_3(t) = \dfrac{\lambda v}{1 + (\lambda - 1)v}$ by $T(t)$, then the curve $\mathscr{L}$ in a plane determined by expression (9) can be considered as a locus with points $(u(t), T(t)), t \in [a, b]$, and the chain rule gives the second-order derivative expression of $\mathscr{L}$

$$\frac{d^2T}{du^2} = \frac{T''_{tt}u' - T'_t u''}{u'^3}. \tag{10}$$

The parametric curve

$$\begin{cases} u = u(t), \\ v = v(t), \quad t \in [a, b], \end{cases}$$

implies a function $v = v(u)$ between u and v.

Theorem 1. *If $v = v(u)$ is a convex function when $t \in [a, b]$ and $\lambda \geq 1$, then $\mathscr{L}$ is a convex curve in the normal square $\mathbf{Q}_1\mathbf{Q}_2\mathbf{Q}_3\mathbf{Q}_4$.*

Proof. Note that $v = v(u) \in C^2(0, 1)$ is a convex function, then $v''(u) \le 0$. By formula (10), we can derive that

$$\frac{d^2T}{du^2} = \frac{\lambda}{\lambda - 1}\frac{d}{du}\left[1 - \frac{1}{1 + (\lambda - 1)v}\right]$$

$$= \frac{\lambda v''_{uu}}{[1 + (\lambda - 1)v]^2} - \frac{2\lambda(\lambda - 1)v'^2_u}{[1 + (\lambda - 1)v]^3} \le 0.$$

Consequently, the conclusion holds. Similarly, so is the spline for a quadrilateral degenerated into a triangle. □

Thus, we should employ the convex functions $v = v(u)$ when $u \in [0, 1]$, consequently, the spline is produced by the basis functions with $\lambda \ge 1$, that is convex-preserving for a planar convex quadrilateral. In the meantime, the curve in $\mathbb{R}^2$ can bring about monotony if the given points are in monotone.

Theorem 2. *Let* $\mathbf{P}_i = \{x_i, y_i\}(i = 1, 2, 3, 4) \in \mathbb{R}^2$ *with* $x_{i+1} \ge x_i$, $y_{i+1} \ge y_i(i = 1, 2, 3)$, *then for any* $\lambda > 0$, *the curve* $\mathbf{C}(t, \lambda)$ *defined by formula* (5) *is monotony*.

Proof. Assume the curve $\mathbf{C}(t)$ be $y = y(x)$ in $\mathbb{R}^2$, and let $x = \mathbf{C}(t)[x] = \sum_{n=1}^{4} x_i\alpha_i(t)$ and $y = \mathbf{C}(t)[y] = \sum_{n=1}^{4} y_i\alpha_i(t)$, then $\frac{dy}{dx} = \frac{\mathbf{C}'(t)[y]}{\mathbf{C}'(t)[x]}$. The question is solved if we can prove that $\mathbf{C}'(t)[y] \cdot \mathbf{C}'(t)[x] \ge 0$. Denote $\Delta_i = x_{i+1} - x_i$, $\Lambda_i = y_{i+1} - y_i(i = 1, 2, 3)$, then

$$\mathbf{C}'(t)[x] = -(\Delta_1 + \Delta_2 + \Delta_3, \Delta_2 + \Delta_3, \Delta_3)\alpha'(t)$$

$$= -(\Delta_1, \Delta_2, \Delta_3)\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\alpha'(t)$$

$$= -\Delta^T\omega'(t),$$

$$\mathbf{C}'(t)[y] = -(\Lambda_1,\ \Lambda_2,\ \Lambda_3)\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}\alpha'(t)$$

$$= -\Lambda^T\omega'(t),$$

where $\alpha'(t)=(\alpha_1'(t),\alpha_2'(t),\alpha_3'(t))^T$, $\Delta=(\Delta_1,\Delta_2,\Delta_3)^T$, and $\Lambda=(\Lambda_1,\Lambda_2,\Lambda_3)^T$. Consequently,

$$\mathbf{C}'(t)[x]\cdot\mathbf{C}'(t)[y] = \frac{1}{2}\omega^T(t)\mathbf{A}\omega(t), \tag{11}$$

where $\mathbf{A} = (\Delta\Lambda^T + \Lambda\Delta^T)$ is a semi-definite matrix of rank less than 2. Here, we assume its nonzero eigenvalues be μ_1, μ_2, and at least one of them is nonzero. Hence, there exists an orthogonal matrix $\mathbf{B}$ of 3×3 to satisfy

$$\mathbf{A} = \mathbf{B}\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & 0\end{pmatrix}\mathbf{B}^T.$$

For the spline is a regular curve and $\omega'(t_0)\neq 0$ for arbitrarily $t_0\in[a, b]$. Thus, the value of (11) is non-negative for any non-negative $\Delta_i,\ \Lambda_i(i = 1, 2, 3)$, then the above conclusion holds. □

4. Basis Functions for Surface

There are points $\mathbf{P}_{ij}\in\mathbb{R}^3(i, j = 1, \cdots, 4)$, which form a subdivision in space. Using the four basis functions, we can employ following formula to construct a fitting spline surface:

$$\mathbf{S}[\mathbf{v}(t),\ \varsigma] = \alpha^T[v(t),\ \lambda]\mathbf{P}\alpha[\hat{v}(t),\ \hat{\lambda}],\quad t\in[a, b], \tag{12}$$

where $\mathbf{v}(t) = [u(t), \hat{u}(t)]^T$ is a vector with two parametric functions in $\mathcal{V}$, and $\varsigma = (\lambda, \hat{\lambda})^T$ is two shape handles for a spline surface, all of which are defined by Definition 1. In addition, $\mathbf{P}$ is a matrix with $\mathbf{P}_{ij}$, and $\alpha = (\alpha_1, \cdots, \alpha_4)^T$. As shown in Figure 9, the spline surfaces are produced with $v(t) = \hat{v}(t) = t(1-t)(t \in [0, 1])$, $u(t) = u_1(t)$ and $\hat{u}(t) = u_2(t)$ for the mesh $\mathbf{X} \times \mathbf{Y} = [0, 1, 2, 3] \times [0, 1, 2, 3]$ and $\mathbf{Z}. = \cos(\mathbf{X}.^5) + \cos(\mathbf{Y}.^3)$ (the right of the equation describe to use a point in the mesh, and the left is the relative 16 values of $\mathbf{Z}$), Figure 10 with $\lambda = 7$, $\hat{\lambda} = 11$ is produced to a triangle domain. If the norm of $\|\varsigma\|$ is great enough, the spline surface can approximate to the surface of meshes.

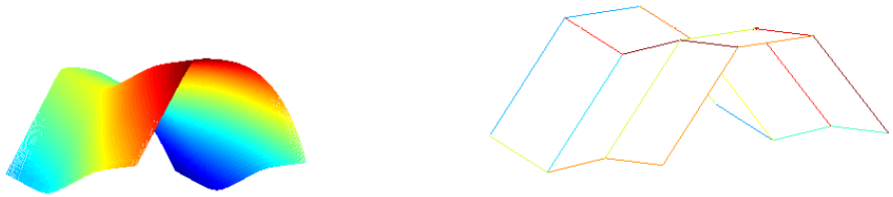


Figure 9. G^2-continuity surface for rectangle mesh.

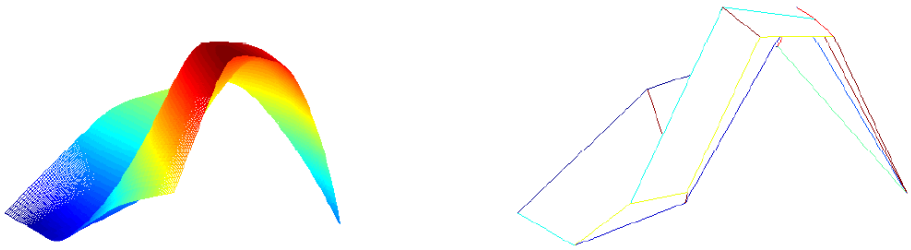


Figure 10. G^2-continuity surface for triangle mesh.

With customary Bézier surfaces method to stitch a surface as described in the literature [6], the osculating plane of this spline surface at boundary is the same plane for the two consecutive stitched surfaces,

and the Gaussian curvature of each angular point is roughly zero for every units. Furthermore, the Gaussian curvature of a point in the boundary is consistent for two consecutive surfaces. Therefore, the surface stitched by these spline surfaces is an approximately G^2-continuity surface. An approximately G^2-continuity surface, stitched by four unit spline surfaces with $\lambda = 7$, $\hat{\lambda} = 11$ and use the functions as Figure 9 and Figure 10, is illustrated in Figure 11.

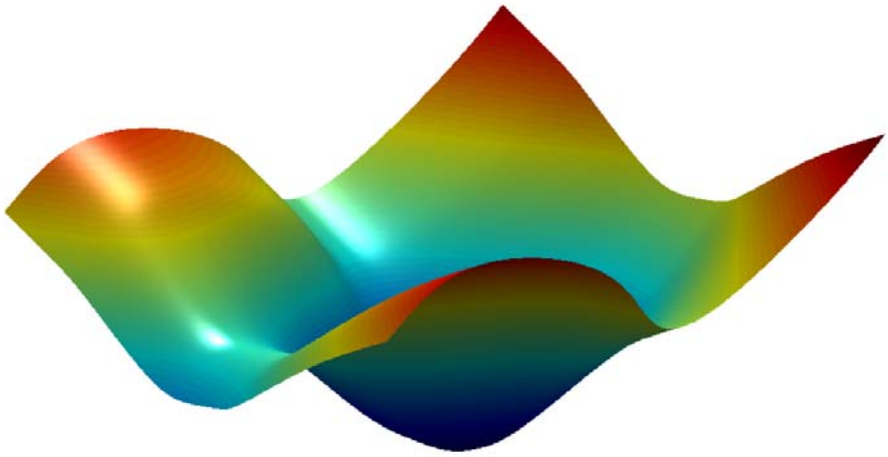
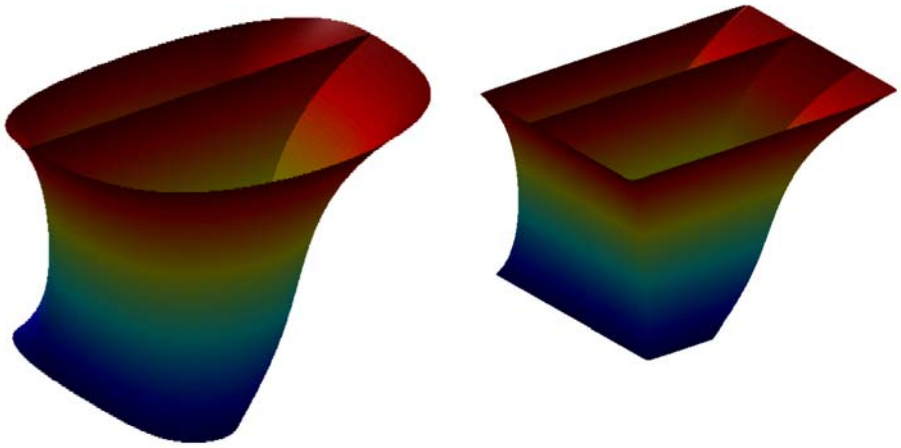


Figure 11. G^2-continuity surface is stitched by four spline surfaces.

Employing the scheme to construct surface, an equal-channel tube simulation with a small norm of $\|\varsigma\|$ is described in Figure 12(a), as well as Figure 12(b) with a greater λ and a smaller $\hat{\lambda}$, which are all stitched by three piece independent surface units.



(a) (b)

Figure 12. Two model surfaces are stitched with different λ.

5. Conclusion

In this paper, we have presented a series of parametric basis functions with a shape handle and a function in a family. Using the basis functions, an approximate curve for a quadrilateral can be produced, and so can do a surface for a subdivision of 4×4. We analyze the properties of its endpoints and their curvatures. By these unit splines applied to a range of points or a meshes in space, one approximately G^2-continuity curve or surface is arisen with some free control handles and functions, which do not need other constraits. At the same time, the curve and surface bear a fairing geometric meaning. Compared with the schemes in [13, 14], this method avoid to look for the function $v = v(u)$ with relatively complicated definition, and the approach of constructing curves and surfaces is identical as that of the customary parametric Bézier means.

The key idea to endow this representation with a shape handle is that the use of curve and surface arbitrarily approximate to a polygon or a subdivision, and the functions have the multiple of the handle and interior function $v(t)$. Additionally, notice that some of the four control points for a unit may collapse in a segment, in fact, the spline became a straight line when $\lambda = 0$, for this reason, we also can utilize it for a triangle or a straight line. On the other hand, if $v(t)$ is not a polynomial, the drawback of the provided spline is that it requires more calculation amount than rational polynomial spline. In view of this, if one employ a non-polynomial function $v(t)$ for a curve or a surface, then the function $\frac{1}{1+(\lambda-1)v}$ in basis functions (1) can be expanded to polynomial at $t = \frac{a+b}{2}$ with the Taylor formula, and the limited terms are chosen in terms of the required precision to take the place of $v(t)$.

This work puts its emphasis on introducing the spline for four points in space, however, it can be extended to arbitrary approximate a polygon with n points. We think that the quantification approaches is difficult to consider for a rational form. It is an interesting problem and currently under studying of the authors, and is left for the future study.

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