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ON THE COEXISTENCE OF DISTRIBUTIONAL AND RATIONAL FUNCTION SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

G. I. MIRUMBE and V. A. SSEMBATYA

Department of Mathematics
Makerere University
P. O. Box 7062
Kampala
Uganda
e-mail: mirumbe@math.mak.ac.ug

Abstract

Given an ordinary differential equation with polynomial coefficients, Wiener and Cooke [5] gave a necessary and sufficient condition for the simultaneous existence of solutions to ordinary differential equations with polynomial coefficients in the form of finite order linear combination of the Dirac delta function and its derivatives and the rational function solutions by using the Laplace transform and functional differential equations techniques.

In this paper, we prove a similar result by using the theory of boundary values and the Cauchy transform. This method has an advantage over the method in Wiener and Cooke [5] as it gives a closed form expression for the resulting polynomial in case the finite order distributional solution and the rational function solution do not satisfy similar differential equations.

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1. Introduction

In Kanwal and Littlejohn [4], during the presentation of results on the distributional solutions of the hypergeometric differential equation that appear in the form of infinite series of the Dirac delta functions and their derivatives, noted some interesting observations about their coexistence, Wiener and Cooke [5].

Example 1.1. The ordinary differential equation

$$t(1-t)x''(t) + (1-3t)x'(t) - x(t) = 0,$$

has the distributional solution, $\delta(t-1)$ and the rational function solution, $\frac{1}{1-t}$. Both solutions exhibit intriguing similarities: $\frac{1}{1-t}$ has a pole of order 1 at $t = 1$ and similarly, the distributional solution has a simple pole at $t = 1$.

Example 1.2. The ordinary differential equation

$$t(1-t)x''(t) + (1-5t)x'(t) - 4x(t) = 0,$$

has the distributional solution, $\delta'(t-1) - \delta''(t-1)$ and the rational function $\frac{1+t}{(1-t)^3}$. Again each of the solutions have a pole of order 3 at $t = 1$.

Note 1.1. The variable t in the rational solutions is complex, whereas in the singular distribution it is real.

Wiener and Cooke [5] gave a necessary and sufficient condition for the simultaneous existence of solutions to ordinary differential equations with polynomial coefficients in the form of finite order linear combination of the Dirac delta function and its derivatives and the rational function solutions by using the Laplace transform and functional differential equations techniques.

In this paper, we explain the coexistence of rational function solutions and singular distributional solutions that are finite order linear combinations of the Dirac delta function and its derivatives by using the theory of boundary value distributions and the Cauchy transform of distributions with compact support on the closed subset of the real line.

2. Preliminaries

In this section, we state and prove some results that will be used to explain/prove the coexistence of rational and distributional solutions to ordinary differential equations with polynomial coefficients.

2.1. Boundary value distributions $(\mathfrak{Db})$ on the real line

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of rapidly decreasing and infinitely differentiable functions on the real line $(C^{\infty}(\mathbb{R}))$ and $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on the real line. If $f(z)$ is an analytic function defined in a domain near the real axis such as the open rectangle defined by

$$\{(x,\ y) : 0 < x < a : 0 < y < b\},$$

where a and b are small finite real numbers, then it is said to have moderate or temperate growth when we approach the real axis if each compact subinterval $\alpha \le x \le \beta$ of (a, b), there exists some integer $N \ge 0$ and a constant C such that

$$|f(x + iy)| \le C \cdot y^{-N}. \tag{2.1}$$

This moderate growth is a necessary and sufficient condition for the complex valued function $f(z)$ to have a boundary value $f(x \pm i0)$ given by a distribution $\mu(f) \in \mathcal{S}'(\mathbb{R})$ defined on the real axis by the limit

$$f(x \pm i0) = \lim_{\epsilon \to 0} \int_0^a g(x) f(x \pm i\epsilon) dx : g(x) \in C_0^{\infty}(0,\ a), \tag{2.2}$$

(see Hörmander [2], Theorem 3.1.13).

Definition 1. The limit integral in (2.2) yields a linear functional on test functions, which give a distribution denoted by $\mathfrak{b}(f)$ and called the boundary value distribution of the analytic function $f(z)$.

2.2. Schwartz reflection principle

Let $\Lambda_+ = \{x + iy, 0 < x < A : 0 < y < B\}$ and $\Lambda_- = \{x + iy, 0 < x < A : -B < y < 0\}$ be opposite rectangles in the upper and, respectively, lower half plane on the real line with $A, B > 0$. Let $g \in \mathcal{O}(\Lambda_+)$ and $h \in \mathcal{O}(\Lambda_-)$ satisfy the moderate growth condition. Then $g(z)$ and $h(z)$ will have boundary values $\mathfrak{b}g$ and $\mathfrak{b}h$, respectively (Hörmander [2], Theorem 3.1.11).

Theorem 1. *Let $g \in \mathcal{O}(\Lambda_+)$ and $h \in \mathcal{O}(\Lambda_-)$ be a pair such that $\mathfrak{b}g = \mathfrak{b}h$ holds as distributions. Then g and h are analytic continuations of each other, i.e., there exists an analytic function F defined in $\{-B < y < B : 0 < x < A\}$ such that $F = g$ in Λ_+ and $F = h$ in Λ_-.*

Proof. See Hörmander [2]. □

2.3. Distributions with compact support and their Cauchy transforms

Let $\mu \in \mathfrak{D}\mathfrak{b}$ be a boundary value distribution with compact support on the closed subset $[0, 1]$ of the real t-line (μ *could be a singular distribution*). If z is fixed, then the function

$$t \mapsto \frac{1}{z - t},$$

belongs to the class of $C^\infty(\mathbb{R})$ functions on the real t-line.

Definition 2. The Cauchy transform of μ denoted by $\mathcal{C}_\mu(z)$ is defined by the action of μ on a C^∞ function $\frac{1}{z - t}$. This function is holomorphic everywhere in the complex z-plane except on $[0, 1]$ and is defined by

$$\mathcal{C}_\mu(z) = \mu < \frac{1}{z-t} >, \quad z \in \mathbb{C}\setminus[0,1].$$

Example 2.1. If $\mu = \delta_a$ for $0 \le a \le 1$, then $\mathcal{C}_\mu(z) = \frac{1}{z-a}$ for $z \ne a$.

The Cauchy transform by definition is an analytic function in $\mathbb{C}\setminus[0,1]$. The Cauchy transform also satisfies the moderate growth condition (2.1) as we approach the real axis. If $f(x) \in C_0^\infty(\mathbb{R})$, then for every Cauchy transform of a boundary value distribution μ, $\mathcal{C}_\mu(z)$, there exists two boundary value distributions, $\mathcal{C}_\mu(x+i0)$ and $\mathcal{C}_\mu(x-i0)$ defined by

$$\mathcal{C}_\mu(x+i0) < f > = \lim_{\epsilon\to 0}\int_{x\in\mathbb{R}} \mu < \frac{1}{x-t+i\epsilon} > f(x)dx, \tag{2.3}$$

and

$$\mathcal{C}_\mu(x-i0) < f > = \lim_{\epsilon\to 0}\int_{x\in\mathbb{R}} \mu < \frac{1}{x-t-i\epsilon} > f(x)dx. \tag{2.4}$$

Lemma 1. *Given that* $\mu \in \mathfrak{Db}$ *with support on the interval* $[0,1]$ *including the case of singular support distributions, then the boundary value distributions of its Cauchy transform* $\mathcal{C}_\mu(z)$ *satisfies the equation*

$$\mu(f) = \frac{i}{2\pi}\{\mathcal{C}_\mu(x+i0) < f > - \mathcal{C}_\mu(x-i0) < f > \},$$

where $f(x) \in C_0^\infty(\mathbb{R})$.

Proof. Consider the difference $\mathcal{C}_\mu(x+i\epsilon) < f > - \mathcal{C}_\mu(x-i\epsilon) < f >$, which is equal to

$$= \int_{-\infty}^{\infty} \mu < \frac{1}{x+i\epsilon-t} > f(x)dx - \int_{-\infty}^{\infty} \mu < \frac{1}{x-i\epsilon-t} > f(x)dx$$

$$= \int_{-\infty}^{\infty} \mu < \frac{1}{x+i\epsilon-t} - \frac{1}{x-i\epsilon-t} > f(x)dx$$

$$= -2i\int_{-\infty}^{\infty} \mu < \frac{\epsilon}{(x-t)^2+\epsilon^2} > f(x)dx. \tag{2.5}$$

We consider μ in (2.5) as a distribution on the t-line for a fixed x and ϵ.

By linearity of the integral operator, the right hand side of (2.5) is equal to

$$-2i\mu\int_{-\infty}^{\infty} < \frac{\epsilon}{(x-t)^2+\epsilon^2} > f(x)dx. \tag{2.6}$$

We now use the fact that the function

$$f_\epsilon(t) = \int_{-\infty}^{\infty} \frac{\epsilon f(x)}{(x-t)^2+\epsilon^2}\, dx,$$

converges to $\pi f(t)$ in the space of C^∞ functions on the real t-line. Therefore, in the limit as $\epsilon \to 0$, $\lim_{\epsilon\to 0} \mu(f_\epsilon(t)) = \mu(f)$ and therefore,

$$\mathcal{C}_\mu(x+i0) < f > - \mathcal{C}_\mu(x-i0) < f >= -2\pi i f(t),$$

implying that

$$\frac{i}{2\pi}\{\mathcal{C}_\mu(x+i0) < f > - \mathcal{C}_\mu(x-i0) < f > \} = f(t).$$

□

Remark 1. Lemma 1 shows that we can recover a distribution with compact support (including the singular support distributions) via the boundary values of the Cauchy transform $\mathcal{C}_\mu(z)$.

2.4. The boundary value distributions of the analytic function $f(z) = z^\alpha$, where $\alpha = -k, \quad k \in \mathbb{N}$

Most of what is exposed in this subsection can be found, for example, in Hörmander [2]. By the analysis in the Subsection 2.1, the function $f(z) = z^\alpha$, defined in $\mathbb{C} \setminus \mathbb{R}_+$ as $e^{\alpha \log z}$, where $\log z$ is real for $z \in \mathbb{R}_+$

has boundary values $(x + i0)^{\alpha}$ as we approach the real axis from the upper half plane and, respectively, $(x - i0)^{\alpha}$ from the lower half plane. If $\Re\mathfrak{e}(\alpha) > 0$, then the functions $(x \pm i0)^{\alpha}$ are both entire in the complex z-plane. However, when α is a negative integer, $-k$ for $k \in \mathbb{N}$, then the boundary values are no longer entire functions. For its necessity in this chapter, we re-state and prove below a lemma, which could also be found in Hörmander [2].

Lemma 2. *Let k be a nonzero integer, i.e., $k \in \mathbb{N}$. The difference between the two boundary values $(x + i0)^{-k}$ and $(x - i0)^{-k}$ is given by*

$$(x + i0)^{-k} - (x - i0)^{-k} = -\frac{2\pi i(-1)^{k-1}}{(k-1)!}\delta_0^{(k-1)}, \tag{2.7}$$

where $\delta_0^{(k-1)}$ is the $(k-1)$-th derivative of the Dirac delta function at the origin.

Proof. We first prove that

$$(x + i0)^{-k} = x_+^{-k} + (-1)^k x_-^{-k} + \frac{i\pi(-1)^k \delta_0^{(k-1)}}{(k-1)!},$$

and

$$(x - i0)^{-k} = x_+^{-k} + (-1)^k x_-^{-k} - \frac{i\pi(-1)^k \delta_0^{(k-1)}}{(k-1)!},$$

and their difference will yield Equation (2.7).

Assume that $\phi(x) \in C_0^{\infty}(\mathbb{R})$. Let the distribution $< x_+^{\alpha}, \phi(x) >$ be denoted by $I_{\alpha}(\phi(x))$ and defined by

$$I_{\alpha}(\phi(x)) = \int_0^{\infty} x^{\alpha}\phi(x)dx. \tag{2.8}$$

The distribution in (2.8) is analytic for $\Re\mathfrak{e}(\alpha) > -1$. Applying the integration by parts formula once on the integral (2.8) gives that

$$I_\alpha(\phi(x)) = -\frac{1}{\alpha+1}\int_0^\infty x^{\alpha+1}\phi'(x)dx = -\frac{I_{\alpha+1}(\phi'(x))}{\alpha+1}.$$

Applying the integration by parts formula twice successively on the integral (2.8) yields

$$I_\alpha(\phi(x)) = \frac{1}{(\alpha+1)(\alpha+2)} I_{\alpha+2}(\phi''(x)).$$

Applying the same integration by parts formula k-times successively on the integral (2.8) gives

$$I_\alpha(\phi(x)) = \frac{(-1)^k I_{\alpha+k}\phi^{(k)}(x)}{(\alpha+1)(\alpha+2)\dots(\alpha+k)}. \tag{2.9}$$

Note that the right hand side of Equation (2.9) is analytic except for simple poles at $\alpha = -1, -2, \dots, -k$.

At $\alpha = -k$, $k \in \mathbb{N}$, the residue of the function $I_\alpha(\phi(x))$ is computed by the formula

$$\operatorname{res}(I_\alpha(\phi(x)), -k) = \lim_{\alpha\to -k}(\alpha+k)I_\alpha(\phi(x)).$$

Therefore,

$$\operatorname{res}(I_\alpha(\phi(x)), -k) = \lim_{\alpha\to -k}(\alpha+k)\cdot\frac{(-1)^k I_{\alpha+k}\phi^{(k)}(x)}{(\alpha+1)(\alpha+2)\dots(\alpha+k)},$$

$$\operatorname{res}(I_\alpha(\phi(x)), -k) = \frac{(-1)^k I_0(\phi^{(k)}(x))}{(-1)^k(k-1)(k-2)\dots 1} = \frac{(\phi^{(k-1)}(0))}{(k-1)!}.$$

Since $< \delta_0^{(k-1)}, \phi(x) >:= (-1)^{(k-1)}\cdot\frac{\phi^{(k-1)}(0)}{(k-1)!}$, then it means that

$$(\alpha+k)x_+^\alpha \to (-1)^{(k-1)}\cdot\frac{\delta_0^{(k-1)}}{(k-1)!} \text{ as } \alpha \to -k. \tag{2.10}$$

Note 2.1. The boundary values $(x + i0)^{\alpha}$ and $(x - i0)^{\alpha}$ are given, respectively, by $x_+^{\alpha} + e^{i\pi\alpha}x_-^{\alpha}$ and $x_+^{\alpha} + e^{-i\pi\alpha}x_-^{\alpha}$.

From Equation (2.10), it is true that

$$x_+^{\alpha} - (-1)^{(k-1)} \cdot \frac{\delta_0^{(k-1)}}{(\alpha + k)(k-1)!} \to x_+^{-k} \text{ as } \alpha \to -k. \tag{2.11}$$

The Taylor series expansion of $f(\alpha) = e^{\pm i\pi\alpha}$ about $\alpha = -k$ is

$$e^{\pm i\pi\alpha} = (-1)^k[1 \pm (\alpha + k)i\pi + O(\alpha + k)^2], \tag{2.12}$$

where $O(\alpha + k)^2$ are the terms with powers of $\alpha + k$ greater or equal to 2.

Note 2.2. $H_+(x) = -H_-(x)$. In case of the integral $\int_{-\infty}^{0} x_-^{\alpha} dx$, this means that

$$I_{\alpha}(\phi(x)) = -\int_0^{\infty} x^{\alpha}\phi(x)dx.$$

Using Note 2.2, Equations (2.11) and (2.12), then as $\alpha \to -k$, we have that

$$e^{i\pi\alpha}x_-^{\alpha} + \frac{(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!\,(\alpha + k)} + \frac{i\pi(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!} \to (-1)^k x_-^{-k}. \tag{2.13}$$

Similarly,

$$e^{-i\pi\alpha}x_-^{\alpha} + \frac{(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!\,(\alpha + k)} - \frac{i\pi(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!} \to (-1)^k x_-^{-k}. \tag{2.14}$$

Note 2.1 together with Equations (2.11), (2.13), and (2.14) gives

$$(x + i0)^{-k} = x_+^k + (-1)^k x_-^{-k} + \frac{i\pi(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!},$$

and

$$(x - i0)^{-k} = x_+^k + (-1)^k x_-^{-k} - \frac{i\pi(-1)^{(k-1)}\delta_0^{(k-1)}}{(k-1)!},$$

whose difference yields (2.7). □

3. Main Results

Given an ordinary differential equation with polynomial coefficients

$$\sum_{i=0}^{n} q_i(t) f^{n-i}(t) = 0. \tag{3.1}$$

Let z be a complex variable such that $z = x + iy$, where x and y are real variables. Let $f(z)$ be a rational function of the form

$$\sum_{k=0}^{m} a_k z^{-k-1}, \quad a_m \neq 0, \tag{3.2}$$

and ρ be a finite order (m) distributional function consisting of the Dirac delta function and its derivatives up to order m, i.e.,

$$\rho = \sum_{k=0}^{m} a_k \delta_0^{(k)}, \quad a_m \neq 0. \tag{3.3}$$

We state below our main theorem, whose proof heavily depends on the existence of boundary value distributions and the Cauchy transform of distributions with compact support.

Theorem 2. *If Equation* (3.1) *admits a rational function solution* $f(z)$ *of the form* (3.2), *then it also has a distributional solution of order m of the form* (3.3). *Conversely, if* (3.1) *admits a distributional solution* (3.3) *of order m, then there exists a polynomial* $q(t)$ *such that the equation*

$$\sum_{i=0}^{n} q_i(t) f^{n-i}(t) = q(t),$$

has a rational solution $f(z)$ *of the form* (3.2).

We give an illustration of Theorem 2.

Example 3.1. Consider the differential equation

$$t^2 f'' + tf' - f = 0. \tag{3.4}$$

In the complex domain, the differential equation will be

$$z^2 f'' + zf' - f = 0,$$

which has a rational solution $f(z) = \frac{1}{z}$. This rational solution satisfies the moderate growth condition (2.1) as we approach the real axis and therefore has boundary values $\frac{1}{x + i0}$ and $\frac{1}{x - i0}$ in the upper and, respectively, lower half plane. The linear combination of the two boundary values gives another solution, namely,

$$\frac{1}{x + i0} - \frac{1}{x - i0} = -2i\pi\delta_0,$$

which is a Dirac delta distribution of order 0 in agreement with order conditions in Theorem 2.

Example 3.2. Consider the singular distribution $f = \delta_0$ that satisfies the ordinary differential equation

$$P(f) = t^2 f'' + tf' + (t^2 - 1)f = 0.$$

The corresponding order zero rational function is $f = \frac{1}{z}$ for $z \neq 0$. Note that in the complex plane, the differential equation becomes

$$z^2 f'' + zf' + (z^2 - 1)f = 0,$$

and its action on the rational function gives

$$P(\frac{1}{z}) = z,$$

where $q(z) = z$. Therefore, the rational function $\frac{1}{z}$ is a solution to $P(f) = z$.

Proof of Theorem 2

Re-write the ordinary differential equation (3.1) with a complex variable $z = x + iy$, i.e.,

$$\sum_{i=0}^{n} p_i(z)\partial^{(n-i)}(f) = 0,$$

with complex polynomial coefficients. We prove that the existence of the rational solution

$$f(z) = \sum_{k=0}^{m} \frac{a_k}{z^{k+1}}, \quad a_m \neq 0,$$

implies the existence of m-th order distributional solution

$$\sum_{k=0}^{m} a_k \delta_0^{(k)}, \quad a_m \neq 0.$$

Note 3.1. The rational solution $f(z)$ satisfies the moderate growth condition (2.1) as we approach the real axis and therefore has boundary values $f(x + i0)$ and $f(x - i0)$ defined in the upper half plane (respectively, lower half plane).

Now consider the difference between the boundary values

$$f(x + i0) - f(x - i0) = \sum_{k=0}^{m} a_k \left\{ \frac{1}{(x + i0)^{k+1}} - \frac{1}{(x + i0)^{k+1}} \right\}, \quad a_m \neq 0.$$

By Lemma 2, we have

$$f(x + i0) - f(x - i0) = \sum_{k=0}^{m} a_k \delta_0^{(k)}, \quad a_m \neq 0.$$

From the fact that if two functions are solutions to a differential equation, then their linear combination is also a solution. Now, since $f(x + i0)$ and $f(x - i0)$ are both solutions to the differential equation (3.1), then

$\sum_{k=0}^{m} a_k \delta_0^{(k)}$, $a_m \neq 0$, is also a solution. This completes the proof of the first part of the theorem.

For the converse part, let the distributional solution be

$$\rho = a_0\delta_0 + a_1\delta_0^{(1)} + \ldots + a_m\delta_0^{(m)}. \tag{3.5}$$

Each component of ρ can be approximated by the residue formula (Lemma 2) thus,

$$(x + i0)^{-k} - (x - i0)^{-k} = -\frac{2\pi i(-1)^{k-1}}{(k-1)!}\delta_0^{(k-1)}.$$

Let $R(z)$ be the rational function obtained from the approximations by the residue formula, then

$$R(z) = \frac{d_0}{z} + \frac{d_1}{z^2} + \ldots + \frac{d_m}{z^m}.$$

By Lemma 2, we have that $\rho = R(x + i0) - R(x - i0)$. ρ is a solution to the differential equation implies that $Q(t, \partial_t)(\rho) = 0$, where $Q(t, \partial_t) = \sum_{i=0}^{n} q_i(t)\partial^{n-i}(t)$. Therefore, the boundary values of $R(z)$ satisfy

$$Q(t, \partial_t)(R(x + i0)) = Q(t, \partial_t)(R(x - i0)). \tag{3.6}$$

Since $R(z)$ is known to be rational, then (3.6) means that the two boundary values match or are equal as one moves from the upper half plane to the lower half plane and vice versa. By the Schwartz reflection principle and the fact that the rational function $R(z)$ has no poles on the real axis, therefore $Q(t, \partial_t)(R(z))$ is equal to a polynomial in z. □

3.1. An alternative proof of the converse part of Theorem 2

This alternative proof of the converse part of Theorem 2 is preferred for the reason that, it gives a formula for computing the polynomial $P(z)$ via the Cauchy transform of the distribution μ.

Theorem 3. *Let* μ *be a distribution with compact support in* $[0, 1]$ *and* $\mathcal{C}_\mu(z)$ *be its Cauchy transform defined by*

$$\mathcal{C}_\mu(z) = \int_0^1 \frac{d\mu(t)}{z - t},$$

where $z \in \mathbb{C} \setminus [0, 1]$ *and* t *is a real variable. If* $\gamma = Q(t, \partial_t)(\mu)$, *where* $Q(t, \partial_t)$ *is an m-th order ordinary differential equation with polynomial coefficients of the form*

$$Q(t, \partial_t) = \sum_{\nu \geq 0}^{m} q_\nu(t) \partial_t^\nu,$$

then

$$Q(z, \partial_z)(\mathcal{C}_\mu(z)) = \mathcal{C}_\gamma(z) + p(z),$$

where $p(z)$ *is a polynomial in* z. *If* $\gamma = 0$, *then* $Q(\mu) = 0$ *and therefore* $Q(z, \partial_z)(\mathcal{C}_\mu(z)) = p(z)$.

Some illustrations:

Example 3.3. If $\mu \in \mathfrak{Db}[0, 1]$ defined by

$$\mu(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & x \in \mathbb{R} \setminus [0, 1], \end{cases}$$

and $\phi(x) \in C^\infty(\mathbb{R})$, then the derivative of μ w.r.t. t in the distribution sense is given by

$$\partial_t(\mu) = -\int_0^1 \phi'(t) dt,$$

thus, $\partial_t(\mu) = \delta_0 - \delta_1$ and the differential equation $Q(t, \partial_t)$ that annihilates μ is

$$Q(t, \partial_t) = t(1 - t)\partial_t. \tag{3.7}$$

The Cauchy transform for μ is given by

$$\mathcal{C}_\mu(z) = \int_0^1 \frac{dt}{z-t}.$$

Its derivative w.r.t. z is given by

$$\partial_z \mathcal{C}_\mu(z) = -\int_0^1 \frac{dt}{(z-t)^2}. \tag{3.8}$$

Evaluating the definite integral on the right hand side of (3.8) and multiplying through by $z(z-1)$ gives the differential equation that annihilates the transform as

$$Q(z, \partial_z) = z(1-z)\partial_z(\mathcal{C}_\mu(z)) + 1,$$

where $\gamma = 0$ and $p(z) = -1$.

Proof of Theorem 3

The derivative of $\mathcal{C}_\mu(z)$ w.r.t. z is given by

$$\partial_z(\mathcal{C}_\mu(z)) = -\int_0^1 \frac{d\mu(t)}{(z-t)^2}. \tag{3.9}$$

Equation (3.9) is equivalent to $\mathcal{C}_{\partial_t(\mu)}(z)$. By induction, the k-th derivative of $\mathcal{C}_\mu(z)$ w.r.t. z is equal to $\mathcal{C}_{\partial_t^k(\mu)}(z)$, i.e.,

$$\partial_z^k(\mathcal{C}_\mu(z)) = \mathcal{C}_{\partial_t^k(\mu)}(z). \tag{3.10}$$

Multiplication by z and its higher powers on the Cauchy transform gives

$$z\mathcal{C}_\mu(z) = |\mu| + \int_0^1 \frac{t\,d\mu(t)}{z-t} = |\mu| + \mathcal{C}_{t\mu}(z),$$

where $|\mu| = \int_0^1 d\mu(t)$. Similarly, multiplication by z^2 gives

$$z^2\mathcal{C}_\mu(z) = z|\mu| + |t\mu| + \mathcal{C}_{t^2\mu}(z),$$

generally, the expression for the multiplication by the n-th power of z is

$$z^n\mathcal{C}_\mu(z) = \sum_{j=1}^{n}(z^{n-j}|t^{j-1}\mu|) + \mathcal{C}_{t^n\mu}(z). \tag{3.11}$$

If we consider a polynomial $q_\nu(z)$ of degree atmost k such that $q_\nu(0) = 0$, then from (3.11),

$$q_\nu(z)\mathcal{C}_\mu(z) = \mathcal{C}_{q_\nu(t)\mu}(z) + \sum_{k\geq 1}\sum_{j=1}^{k} a_k(z^{k-j}|t^{j-1}\mu|). \tag{3.12}$$

With $Q(z, \partial_z) = \sum_{\nu\geq 0}^{m} q_\nu(z)\partial_z^\nu$, then from (3.10) and (3.12), we have

$$Q(z, \partial_z)(\mathcal{C}_\mu(z)) = \mathcal{C}_{Q(t,\partial_t)(\mu)}(z) + \sum_{k\geq 1}\left\{\sum_{j=1}^{k} a_k(z^{k-j}|t^{j-1}\mu|)\right\},$$

where $p(z) = \sum_{k\geq 1}\left\{\sum_{j=1}^{k} a_k(z^{k-j}|t^{j-1}\mu|)\right\}$. □

Corollary 1. *In Theorem* 2, *if* $p(z) = 0$, *then* μ *and its Cauchy transform satisfy the same differential equations but in different variables. Further if* μ *is a singular distribution, whose support lies on the zeros of the polynomial* $p(z)$.

Proof. From Theorem 2, if $p(z) = 0$, then we have

$$Q(z, \partial_z)(\mathcal{C}_\mu(z)) = \mathcal{C}_\gamma(z). \tag{3.13}$$

If we assume that $\mathcal{C}_\mu(z)$ satisfy the differential equation $Q(z, \partial_z)$, then $\mathcal{C}_\gamma(z) = 0$ and this will immediately imply that μ satisfies $Q(t, \partial_t)$. □

4. Discussion and Conclusion

The coexistence of rational function solutions and the singular distribution solutions in form of the Dirac delta function and their derivatives to ordinary differential equations with polynomial coefficients is dependent on the degree of the leading polynomial coefficient. Special ordinary differential equations have been studied before, for example.

Kananthai [3] considered a third order Euler differential equations

$$t^3 f''' + t^2 f'' + tf' + mf, \tag{4.1}$$

where m is an integer and $t \in (-\infty, \infty)$. The condition $m = k^3 + 2k^2 + 2k$, $k = 1, 2, \ldots$ was derived for the Equation (4.1) to have a solution of the form of the Dirac delta function and their derivatives.

This condition could be derived immediately from Lemma 2 and Corollary 1 by representing the Dirac delta function and its derivatives by

$$\sum_{i=0}^{k} b_i \delta_0^{(i)}, \quad b_k \neq 0,$$

where $\delta_0^{(i)}$ is the i-th derivative of the Dirac delta distribution. The Cauchy transform of the distribution $\delta_0^{(k-1)}$ is $f(z) = \frac{1}{z^k}$. We find a condition on m for which $f(z) = \frac{1}{z^k}$ is a solution to Equation (4.1). Since

$$f'(z) = -kz^{-k-1}, \quad f''(z) = k(k+1)z^{-k-2},$$

and

$$f'''(z) = -k(k+1)z(k+2)^{-k-3},$$

substituting these expressions into Equation (4.1) and multiplying through by z^k, we get

$$-k(k+1)(k+2)+k(k+1)-k+m=0,$$

on simplification, we get the required condition $m=k^3+2k^2+2k$, $k=1, 2, 3, \ldots$.

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