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ASYMPTOTIC BEHAVIOUR OF GENERALIZED APOSTOL-BERNOULLI AND APOSTOL-EULER POLYNOMIALS

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Abstract

In this paper, we obtain Fourier expansions of generalized Apostol-Bernoulli and Apostol-Euler polynomials. Using these expansions, we establish asymptotic behaviour of these polynomials for fixed positive integer μ and $z \in \mathbb{C}$, as $n \to \infty$. We also investigate the cases $\mu = 0, -1, -2, \ldots$, asymptotically.

1. Introduction

The generalized Apostol-Bernoulli and Apostol-Euler polynomials with complex variable z and complex order μ, are defined by means of the generating functions, respectively, ([8, 9])

$$\left(\frac{w}{\alpha e^w - 1}\right)^{\mu} e^{wz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{\mu}(z, \alpha) \frac{w^n}{n!}, \quad |w| < \mathbf{c}, \tag{1.1}$$

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$$\left(\frac{2}{\alpha e^{w}+1}\right)^{\mu} e^{zw} = \sum_{n=0}^{\infty} \mathcal{E}_n^{\mu}(z, \alpha)\frac{w^n}{n!}, \quad |w| < |\log(-\alpha)|, \qquad (1.2)$$

where $\log \alpha = \log|\alpha| + i \arg \alpha$ with $-\pi < \arg \alpha \leq \pi$ and $\mathbf{c} = |\log \alpha|$ or 2π depending on $\alpha \neq 1$ or $\alpha = 1$.

For $\mu = 1$, we have $\mathcal{B}_n^1(z, \alpha) = \mathcal{B}_n(z, \alpha)$ and $\mathcal{E}_n^1(z, \alpha) = \mathcal{E}_n(z, \alpha)$, which are called as *Apostol-Bernoulli* and *Apostol-Euler polynomials*. Using the Lipschitz summation formula, the Fourier expansion of Apostol-Bernoulli polynomials is first given in [11, Theorem 2.1] by

$$\mathcal{B}_n(z, \alpha) = -\delta_n(z, \alpha) - \frac{n!}{\alpha^z} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{2k\pi i z}}{(2k\pi i - \log \alpha)^n}, \qquad (1.3)$$

where $\delta_n(z, \alpha) = 0$ or $\dfrac{n!}{\alpha^z(-\log \alpha)^n}$, according as $\alpha = 1$ or $\alpha \neq 1$. This expansion is also established in [2] via residue theorem and in [14] depending on Riemann integral and algebraic properties of $\mathcal{B}_n(z, \alpha)$. Also, similar expression is proved for the Apostol-Euler polynomials in [2, 11, 14].

Navas et al. in [14, Theorem 3, p.7] derived asymptotic approximation of Apostol-Bernoulli polynomials with the help of (1.3). They also gave asymptotic behaviour for $\mathcal{B}_n(z, \alpha)$ by Corollary 6 and Proposition 7 in their paper. Similar expressions are obtained for Apostol-Euler polynomials by using relations between $\mathcal{B}_n(z, \alpha)$ and $\mathcal{E}_n(z, \alpha)$.

Explicit formulas and basic properties for the (generalized) Apostol-Bernoulli and Apostol-Euler polynomials can be found in [1, 3, 4, 7-12, 15, 17] and cited references therein.

These polynomials are extensions of generalized Bernoulli and Euler polynomials; $\mathcal{B}_n^{\mu}(z, 1) = B_n^{\mu}(z)$ and $\mathcal{E}_n^{\mu}(z, 1) = E_n^{\mu}(z)$ ([13]). Weinmann [16] defined $F_n^{\mu}(z) = \dfrac{1}{n!} B_n^{\mu}(z + \dfrac{1}{2}\mu)$ and derived asymptotic forms for

these polynomials for sufficiently large values of n when $z \in \mathbb{C}$ and $\mu \in \mathbb{Z}$ are fixed. Moreover, there were obtained asymptotic form for $F_{n+b}^{n+a}(z)$ as $n \to \infty$, where a, z being arbitrary complex numbers and $b \in \mathbb{Z}$.

López and Temme [5] obtained asymptotic forms of generalized Bernoulli and Euler polynomials for large values of μ, with n fixed. In [6], they gave asymptotic representations of the generalized Bernoulli and Euler polynomials for fixed z, $\mu \in \mathbb{C}$, as $n \to \infty$. Using two-point Taylor expansion, they also obtained new type expansions for $\mu \in \mathbb{C}$.

The aim of this paper is to obtain analogous asymptotic estimates for $\mathcal{B}_n^{\mu}(z, \alpha)$ and $\mathcal{E}_n^{\mu}(z, \alpha)$ valid for any $z \in \mathbb{C}$ and $\mu \in \mathbb{Z}$, as $n \to \infty$. In the following section, utilizing the Cauchy's integral formula and the generating function, we obtain the Fourier expansion of generalized Apostol-Bernoulli polynomials. Using this expansion, we derive the asymptotic behaviour of $\mathcal{B}_n^{\mu}(z, \alpha)$ for positive integer μ. We also establish estimates of $\mathcal{B}_n^{\mu}(z, \alpha)$ for $\mu = 0, -1, -2, \ldots$, asymptotically. In the last section, similar results are obtained for the generalized Apostol-Euler polynomials by the same way, as well.

From now on, we assume $\mu \in \mathbb{Z}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Also, we recall that the falling factorial denoted by $(x)_m$ and defined by $(x)_m = \prod_{a=0}^{m-1} (x - a)$ for positive integer m, with the convention $(x)_0 = 1$.

2. The Polynomials $\mathcal{B}_n^{\mu}(z, \alpha)$

By Cauchy's integral formula, for $n > \mu$, $\mathcal{B}_n^{\mu}(z, \alpha)$ can be written as follows:

$$\mathcal{B}_n^{\mu}(z, \alpha) = \frac{n!}{2\pi i} \int_C \frac{w^{\mu} e^{wz}}{(\alpha e^w - 1)^{\mu}} \frac{dw}{w^{n+1}}, \tag{2.1}$$

where C is a circle around the origin, with radius less than $\mathbf{c}$ given in (1.1). Note that $\mathcal{B}_n^{\mu}(z, \alpha) = 0$ for $\alpha \neq 1$ and $n < \mu$, $\mu = 1, 2, 3, \ldots$.

We first state the asymptotic behaviour of $\mathcal{B}_n^m(z, \alpha)$ when m is positive integer.

Theorem 2.1. *Let $z \in \mathbb{C}$ and m be positive integer. For $n \to \infty$, when $\mu = m$ and z are fixed, we have*

$$\mathcal{B}_n^m(z, \alpha) = n! \frac{(-1)^{n+m}}{\alpha^z (\log \alpha)^n} \binom{n-1}{m-1} \times \left[\sum_{v=0}^{m-1} \frac{(\log \alpha)^v}{(n-1)_v} \binom{m-1}{v} B_v^m(z) + \mathcal{O}\left(\frac{1}{\left(\min \left| 1 \pm \frac{2\pi i}{\log \alpha} \right| \right)^n} \right) \right], \quad (2.2)$$

for $\alpha \in \mathbb{C} \setminus (-\infty, 0) \cup \{1\}$, and

$$\mathcal{B}_n^m(z, \alpha) = n! \frac{(-1)^{n+m}}{\alpha^z (\log \alpha)^n} \binom{n-1}{m-1} \left[\sum_{v=0}^{m-1} \frac{(\log \alpha)^v}{(n-1)_v} \binom{m-1}{v} B_v^m(z) \times \left\{ 1 + \left(\frac{\log|\alpha| + \pi i}{\log|\alpha| - \pi i} \right)^{n-v} e^{2\pi i z} \right\} + \mathcal{O}\left(\frac{1}{\left| 1 + \frac{2\pi i}{\log \alpha} \right|^n} \right) \right], \quad (2.3)$$

for $\alpha \in (-\infty, 0)$.

Proof. Let $m \geq 1$ and $n \geq m$. We start with the integral

$$\frac{n!}{2\pi i} \int_{\gamma_K} \frac{w^m e^{wz}}{(\alpha e^w - 1)^m} \frac{dw}{w^{n+1}},$$

where γ_K is a circle around the origin, with radius $|(2K+1)\pi i - \log \alpha|$ ($0 < K$ an integer). It can be shown that the integral tends to 0 over γ_K,

as $K \to \infty$, provided $0 < z < 1$ if $n = m$ and $0 \le z \le 1$ if $n > m$. Let the integrand be denoted by $f_m(w) := f_m(w, z, \alpha)$. By residue theorem and formula (2.1), it follows that $\mathcal{B}_n^m(z, \alpha) = n!\, Res(f_m(w), 0)$. So, we can write

$$\mathcal{B}_n^m(z, \alpha) = -n! \sum_{k \in \mathbb{Z}^*} Res(f_m(w), w_k) = -n! \sum_{k \in \mathbb{Z}^*} \xi_k^m(z, n, \alpha) \frac{e^{w_k z}}{w_k^n}, \quad (2.4)$$

where

$$w_k = (2k\pi i - \log \alpha), \quad \mathbb{Z}^* = \begin{cases} \mathbb{Z} \setminus \{0\}, & \text{if } \alpha = 1, \\ \mathbb{Z}, & \text{if } \alpha \neq 1, \end{cases}$$

and $\xi_k^1(z, n, \alpha) = 1$ for all k (see (1.3)). To obtain $\xi_k^m(z, n, \alpha)$ for $m > 1$, we need the residues of $f_m(w)$ at $w_k = (2k\pi i - \log \alpha)$. To do this, it is enough to compute the coefficients b_r given by

$$\frac{(w - w_k)^m w^m e^{wz}}{(\alpha e^w - 1)^m w^{n+1}} = \sum_{r=0}^{\infty} b_r (w - w_k)^r. \quad (2.5)$$

Differentiating both sides of (2.5) with respect to the variable w by $(m-1)$ times, we get

$$b_{m-1} = \xi_k^m(z, n, \alpha) \frac{e^{w_k z}}{w_k^n}. \quad (2.6)$$

Also, by writing $w = s + w_k$ in (2.5), we conclude that

$$\sum_{r=0}^{\infty} b_r s^r = \frac{e^{w_k z}}{w_k^{n+1-m}} \frac{s^m e^{sz}}{(e^s - 1)^m} \left(\frac{s}{w_k} + 1 \right)^{m-n-1}$$

$$= \frac{e^{w_k z}}{w_k^{n+1-m}} \sum_{r=0}^{\infty} \left(\sum_{v=0}^{r} \frac{B_v^m(z)}{v!} \binom{m-n-1}{r-v} w_k^{v-r} \right) s^r.$$

From this, it follows that

$$b_{m-1} = \frac{e^{w_k z}}{w_k^n} \sum_{v=0}^{m-1} \frac{B_v^m(z)}{v!} \binom{m-n-1}{m-1-v} w_k^v. \tag{2.7}$$

By (2.6) and (2.7), we obtain

$$\xi_k^m(z, n, \alpha) = \sum_{v=0}^{m-1} \frac{B_v^m(z)}{v!} \binom{m-n-1}{m-1-v} w_k^v. \tag{2.8}$$

Furthermore, since

$$\begin{aligned}\binom{m-n-1}{m-1-v} \frac{1}{v!} &= \frac{(m-n-1)\cdots(m-n-1-(m-2-v))}{(m-1-v)!\,v!} \frac{(m-1)!}{(m-1)!} \\ &= \binom{m-1}{v} (-1)^{m-1-v} \frac{(n-1-v)!}{(m-1)!\,(n-m)!} \\ &= (-1)^{m-1-v} \binom{n-1}{m-1} \binom{m-1}{v} \frac{(n-1-v)!}{(n-1)!},\end{aligned}$$

and by (2.8), we find

$$\xi_k^m(z, n, \alpha) = (-1)^{m-1} \binom{n-1}{m-1} \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{(-w_k)^v}{(n-1)_v} B_v^m(z). \tag{2.9}$$

From (2.4) and (2.9), we get

$$\mathcal{B}_n^m(z, \alpha) = (-1)^m n! \binom{n-1}{m-1} \sum_{k \in \mathbb{Z}^*} \sum_{v=0}^{m-1} \binom{m-1}{v} \frac{(-1)^v B_v^m(z)}{(n-1)_v} \frac{e^{w_k z}}{w_k^{n-v}}, \tag{2.10}$$

which is a Fourier expansion for $0 < z < 1$ when $n > m$. For other values of z, we can use this for asymptotic expansion as $n \to \infty$. It is enough to examine the values of α in two cases, $\alpha \in \mathbb{C} \setminus (-\infty, 0) \cup \{1\}$ and $\alpha \in (-\infty, 0)$. In both cases, it is obvious that

$$0 < |w_{\pm k}| \leq |w_{\pm(k+1)}|, \tag{2.11}$$

for all $k \geq 0$.

For $\alpha \in \mathbb{C} \setminus (-\infty, 0) \cup \{1\}$, we have $|w_0| < |w_{\mp 1}|$ and $\left|\frac{w_{\mp 1}}{\log \alpha}\right| = \left|1 \pm \frac{2\pi i}{\log \alpha}\right| > 1$. So, in (2.10), only the term with $k = 0$ is in accordance for the asymptotic behaviour of $\mathcal{B}_n^m(z, \alpha)$ when m and z are fixed as $n \to \infty$. This and formula (2.10) yield (2.2).

For $\alpha \in (-\infty, 0)$, we have $|w_0| = |-\log \alpha| = |2\pi i - \log \alpha| = |w_1| < |w_{-1}|$ and $\left|\frac{w_{-1}}{\log \alpha}\right| = \left|1 + \frac{2\pi i}{\log \alpha}\right| > 1$. Then, in (2.10), only the terms with $k = 0, 1$ are in accordance for the asymptotic behaviour. Thus, the formula (2.3) follows from (2.10). □

Note that formula (2.9) may be used for the asymptotic behaviour of coefficients $\xi_k^m(z, n, \alpha)$. For sufficiently large values of n, in (2.9), the main term occurs at $v = 0$, this gives

$$\xi_k^m(z, n, \alpha) = (-1)^{m-1} \frac{n^{m-1}}{(m-1)!}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]. \tag{2.12}$$

Moreover, asymptotic forms given by (2.2) and (2.3) in Theorem 2.1 can be simplified as follows:

$$\mathcal{B}_n^m(z, \alpha) = (-1)^{n+m} \frac{n!}{\alpha^z (\log \alpha)^n} \frac{n^{m-1}}{(m-1)!}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], \tag{2.13}$$

for $\alpha \in \mathbb{C} \setminus (-\infty, 0) \cup \{1\}$, and

$$\mathcal{B}_n^m(z, \alpha) = n! \frac{(-1)^{n+m}}{\alpha^z (\log \alpha)^n} \frac{n^{m-1}}{(m-1)!}\left[1 + \left(\frac{\log|\alpha| + \pi i}{\log|\alpha| - \pi i}\right)^n e^{2\pi i z} + \mathcal{O}\left(\frac{1}{n}\right)\right], \tag{2.14}$$

for $\alpha \in (-\infty, 0)$, respectively.

Remark 2.2. As a result of (2.10), we obtain the well-known Fourier expansion of the generalized Bernoulli polynomials. In addition, for $m = 1$ in Theorem 2.1, we get Corollary 6 and Proposition 7 derived in [14] except for order constant.

With the help of (2.1) and the binomial theorem, we have the following:

Theorem 2.3. *Let $z = x + iy \in \mathbb{C}$ and m be nonnegative integer. For $n \to \infty$, when $\mu = -m$ and z are fixed, we have the following. If $x > -\frac{m}{2}$, then*

$$\mathcal{B}_n^{-m}(z, \alpha) = \alpha^m \frac{n!}{(n+m)!}(z+m)^{n+m}\left[1 + \mathcal{O}\left(\left|\frac{z+m-1}{z+m}\right|^{n+m}\right)\right], \tag{2.15}$$

if $x < -\frac{m}{2}$, then

$$\mathcal{B}_n^{-m}(z, \alpha) = (-1)^m \frac{n!}{(n+m)!} z^{n+m}\left[1 + \mathcal{O}\left(\left|\frac{z+1}{z}\right|^{n+m}\right)\right], \tag{2.16}$$

and if $x = -\frac{m}{2}$, then

$$\mathcal{B}_n^{-m}(z, \alpha) \sim \frac{(-1)^m n!}{(n+m)!}\left[\left(-\frac{m}{2} + iy\right)^{n+m} + (-\alpha)^m\left(\frac{m}{2} + iy\right)^{n+m}\right]. \tag{2.17}$$

Proof. For $\mu = -m$, the formula (2.1) gives

$$\begin{aligned}\mathcal{B}_n^{-m}(z, \alpha) &= n! \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \alpha^r \frac{1}{2\pi i} \int_C e^{w(z+r)} \frac{dw}{w^{n+1+m}} \\ &= \frac{n!}{(n+m)!} \sum_{r=0}^{m} \binom{m}{r} (-1)^{m-r} \alpha^r (z+r)^{n+m}. \end{aligned} \tag{2.18}$$

When $n \to \infty$, the maximal values of $|z + r|$ give a large contribution to sum in (2.18). To get the maximal value(s) of $|z + r|$, it is enough to investigate the maximal values of the function $h(r) = (x + r)^2$ on interval

$[0, m]$. When $x > -\frac{m}{2}$, we have $\max_{0\le r\le m} h(r) = h(m)$ and then the proof of (2.15) follows. In the case $x < -\frac{m}{2}$, $\max_{0\le r\le m} h(r) = h(0)$, which gives (2.16). Finally, if $x = -\frac{m}{2}$, then $\max_{0\le r\le m} h(r) = h(0) = h(m)$, so we get (2.17). □

Note that the formulas (2.4), (2.5), and (2.6) given in [6, p. 198] are special cases of Theorem 2.3.

3. The Polynomials $\mathcal{E}_n^{\mu}(z, \alpha)$

The proofs of next theorems follow precisely along the same arguments as in Section 2. So, we omit details.

By Cauchy integral formula and (1.2), $\mathcal{E}_n^{\mu}(z, \alpha)$ can be written as

$$\mathcal{E}_n^{\mu}(z, \alpha) = \frac{n!}{2\pi i}\int_C \frac{2^{\mu}e^{wz}}{(\alpha e^{w}+1)^{\mu}}\frac{dw}{w^{n+1}}, \quad \alpha \neq -1, \tag{3.1}$$

where C is a circle around the origin, with radius less than $|\log(-\alpha)|$.

When μ is positive integer, we obtain the following:

Theorem 3.1. *Let $z \in \mathbb{C}$ and m be positive integer. For $n \to \infty$, when $\mu = m$ and z are fixed, we have*

$$\mathcal{E}_n^{m}(z, \alpha) = \frac{2^m}{\alpha^z}\frac{n!}{(\pi i - \log\alpha)^{n+m}}\binom{n+m-1}{m-1} \times\left[\sum_{v=0}^{m-1}\frac{(\log\alpha - \pi i)^v}{(n+m-1)_v}\binom{m-1}{v}B_v^m(z)e^{\pi i z} + \mathcal{O}\left(\left|\frac{\pi i - \log\alpha}{\pi i + \log\alpha}\right|^n\right)\right], \tag{3.2}$$

for $0 < \arg\alpha \le \pi$ and $\alpha \neq -1$,

$$\mathcal{E}_n^m(z,\,\alpha)=\frac{2^m}{\alpha^z}\frac{(-1)^{n+m}n!}{(\pi i+\log\alpha)^{n+m}}\binom{n+m-1}{m-1}\Bigg[\sum_{v=0}^{m-1}\frac{(\log\alpha+\pi i)^v}{(n+m-1)_v}\binom{m-1}{v}$$

$$\times\,B_v^m(z)\left\{e^{-\pi iz}+\left(\frac{\log\alpha+\pi i}{\log\alpha-\pi i}\right)^{n+m-v}e^{\pi iz}\right\}+\mathcal{O}\left(\left|\frac{\pi i+\log\alpha}{3\pi i-\log\alpha}\right|^n\right)\Bigg],\tag{3.3}$$

for $\alpha\in(0,\,\infty)$, *and*

$$\mathcal{E}_n^m(z,\,\alpha)=\frac{2^m}{\alpha^z}\frac{(-1)^{n+m}n!}{(\pi i+\log\alpha)^{n+m}}\binom{n+m-1}{m-1}$$

$$\times\left[\sum_{v=0}^{m-1}\frac{(\log\alpha+\pi i)^v}{(n+m-1)_v}\binom{m-1}{v}B_v^m(z)e^{-\pi iz}+\mathcal{O}\left(\left|\frac{\pi i+\log\alpha}{\pi i-\log\alpha}\right|^n\right)\right],\tag{3.4}$$

for $-\pi<\arg\alpha<0$.

Proof. We write $\mu=m$ in (3.1). For $m\geq 1$, we get

$$\mathcal{E}_n^m(z,\,\alpha)=-n!\sum_{k\in\mathbb{Z}}Res(h_m(w),\,w_k)=-n!\sum_{k\in\mathbb{Z}}\mathcal{F}_k^m(z,\,n,\,\alpha)\frac{e^{w_kz}}{w_k^{n+1}},\tag{3.5}$$

where $h_m(w):=h_m(w,\,z,\,\alpha)$ denotes the integrand in (3.1) and $w_k=(2k+1)\pi i-\log\alpha$, $k\in\mathbb{Z}$. Note that $\mathcal{F}_k^1(z,\,n,\,\alpha)=-2$, $\forall k$ (see [7, Theorem 2.2], [2, Theorem 1.2] or [14, formula (31)]). This follows from the fact that the integral in (3.1) tends to 0 over C_K, as $K\to\infty$, where C_K is a circle around the origin, with radius $|2K\pi i-\log\alpha|$ ($0<K$ an integer). To obtain $\mathcal{F}_k^m(z,\,n,\,\alpha)$ for $m>1$, we ought to compute the coefficients a_r given by

$$\frac{(w-w_k)^m2^me^{wz}}{(\alpha e^w+1)^mw^{n+1}}=\sum_{r=0}^{\infty}a_r(w-w_k)^r.\tag{3.6}$$

After elementary calculations, we get

$$a_{m-1} = \mathcal{F}_k^m(z, n, \alpha)\frac{e^{w_k z}}{w_k^{n+1}}. \tag{3.7}$$

Also, write $w = s + w_k$ in (3.6) to obtain

$$a_{m-1} = \frac{(-2)^m e^{w_k z}}{w_k^{n+1}} \sum_{v=0}^{m-1} \frac{B_v^m(z)}{v!}\binom{-n-1}{m-1-v} w_k^{v-m+1}. \tag{3.8}$$

Formulas (3.7), (3.8), and the following equation:

$$\binom{-n-1}{m-1-v}\frac{1}{v!} = (-1)^{m-1-v}\binom{m-1}{v}\binom{n+m-1}{m-1}\frac{(n+m-v-1)!}{(n+m-1)!},$$

give

$$\mathcal{F}_k^m(z, n, \alpha) = \frac{-2^m}{(w_k)^{m-1}}\binom{n+m-1}{m-1}\sum_{v=0}^{m-1}\binom{m-1}{v}\frac{(-w_k)^v}{(n+m-1)_v} B_v^m(z),$$

so that

$$\mathcal{E}_n^m(z, \alpha) = 2^m n!\binom{n+m-1}{m-1}\sum_{k\in\mathbb{Z}}\sum_{v=0}^{m-1}\binom{m-1}{v}\frac{(-1)^v B_v^m(z)}{(n+m-1)_v}\frac{e^{w_k z}}{w_k^{n+m-v}}. \tag{3.9}$$

It can be verified that if $0 \le \arg\alpha \le \pi$ and $\alpha \ne -1$, then

$$0 < |w_{-k}| \le |w_k| \le |w_{-(k+1)}| \le |w_{k+1}|, \tag{3.10}$$

for all $k \ge 0$, and if $-\pi < \arg\alpha < 0$, then

$$0 < |w_{-k}| \le |w_{-(k+1)}| \le |w_k| \le |w_{k+1}| \text{ with } |w_{-1}| < |w_0| < |w_{-2}|, \tag{3.11}$$

for all $k \ge 1$.

For $0 < \arg\alpha \le \pi$ and $\alpha \ne -1$, we have $|w_0| = |\pi i - \log\alpha| < |w_{-1}| \le \cdots$. So, the main term occurs at $k = 0$ in (3.9) for the asymptotic behaviour of $\mathcal{E}_n^m(z, \alpha)$ when m and z are fixed, as $n \to \infty$. Then the formula (3.2) follows from (3.9).

For $\alpha \in (0, \infty)$, we have $|w_0| = |w_{-1}| < |w_1| \leq \cdots$ and in (3.9), the terms with $k = 0, -1$ give the main terms. Thus, we obtain (3.3).

For $-\pi < \arg \alpha < 0$, we have $|w_{-1}| < |w_0| \leq \cdots$. In this case, the main term occurs at $k = -1$ in (3.9), which gives (3.4). □

Remark 3.2. The expressions (3.2), (3.3), and (3.4) given in Theorem 3.1 can be also simplified similar to (2.13) and (2.14). Also, (3.9) gives the Fourier expansion of generalized Euler polynomials for $\alpha = 1$, and Theorem 3.1 reduces to formulas (34) and (35) in [14, p.16] for $m = 1$.

By the same manner in Theorem 2.3, we can prove the following, which generalize the formulas (3.3), (3.5), and (3.4) given in [6, p. 205], respectively.

Theorem 3.3. *Let* $z = x + iy \in \mathbb{C}$, $\alpha \neq -1$, *and* m *be nonnegative integer. For* $n \to \infty$, *when* $\mu = -m$ *and* z *are fixed, we have the following.*

If $x > -\frac{m}{2}$, *then*

$$\mathcal{E}_n^{-m}(z, \alpha) = 2^{-m}\alpha^m(z+m)^n\left[1 + \mathcal{O}\left(\left|\frac{z+m-1}{z+m}\right|^n\right)\right], \tag{3.12}$$

if $x < -\frac{m}{2}$, *then*

$$\mathcal{E}_n^{-m}(z, \alpha) = 2^{-m}z^n\left[1 + \mathcal{O}\left(\left|\frac{z+1}{z}\right|^n\right)\right], \tag{3.13}$$

and if $x = -\frac{m}{2}$, *then*

$$\mathcal{E}_n^{-m}(z, \alpha) \sim 2^{-m}\left[\left(-\frac{m}{2} + iy\right)^n + \alpha^m\left(\frac{m}{2} + iy\right)^n\right]. \tag{3.14}$$

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