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MESHLESS SOLUTIONS OF 2D UNILATERAL CONTACT PROBLEMS BY PENALITY AND MOVING LEAST SQUARE APPROACH

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Abstract

In this paper, the applicability of a meshless approximation method, namely, the MLS method, to two-dimensional unilateral contact problems is investigated. For the analysis, the element free Galerkin method and Galerkin formulation for two dimensional elasticity problems are considered. Then, the penalty method for imposition of contact constraint is proposed and the Lagrange multiplier method is used to impose essential boundary conditions. The implementation of the penalty/Lagrange based contact algorithm is described and the efficiency for our algorithm is illustrated with various examples.

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1. Introduction

Most of engineering and physical problems are solved by using finite differences methods, finite elements methods, finite volume methods, or boundary element methods. These numerical methods are all based on a mesh-grid discretization that has to be generated in advance or dynamically modified as the solution progresses (adaptive meshing). The difficulties encountered in conventional computational mesh-based methods are, e.g., meshing complex geometries, mesh distortion due to large deformation, and re-meshing in moving boundary problems. Also, the construction of a mesh in three or more dimensions is a non-trivial problem.

During recent years, there has been a rapid development of so-called mesh-free methods, which are also referred to in the literature as meshless, element-free, gridless or cloud methods (see [3, 9] for an overview). In meshless methods, the process of mesh generation has been replaced by the process of node generation. The process of node generation takes less time and memory in mathematical programming compared to the mesh generation process. Recently, some attentions have been paid to the meshless methods based on moving least squares (MLS) method, for numerical solution of partial differential equations and remarkable progress has been achieved. Using this approach, some new meshless methods, such as element-free Galerkin (EFG) method [1], boundary node method (BNM) [20], hp-cloud method [6], meshless local boundary integral equation (LBIE) method [19, 26, 30], meshless local Petrov-Galerkin (MLPG) method [2, 5], and other relative methods are constructed. The present study is concerned with the development of variational inequality/meshless method to solve frictionless contact problems.

The contact problems appear within a large number of engineering problems, such as extrusion, metal forming, machining and crash and is one of the mostly used models in the theory of variational inequality (see [7, 14]). Kikuchi and Oden [14], made a detailed analysis of the contact problem in elasticity with the mathematical model and numerical implementation of the models. Wang [28], improved the duality methods in the mixed nite element approximation. The phenomenon of contact in mechanical systems introduces dynamical boundary conditions that are normally inactive, but are enforced when parts of the moving mechanical structure come in contact. This introduces a greater level of complexity because it is not always known where or how contact occurs. Thus, unknown contact boundary and its change during the loading, and also the sliding of two bodies in frictional contact make the contact problem one of the most difficult nonlinear problems in the solid mechanics. Many studies have tried to bypass the difficulties of contact simulation. Most of the methods have used the finite element method to solve the problem (see [10, 21, 29]). Typically, two main constraint methods that have been employed in the finite element solution of contact problems are the method of Lagrange multipliers and the penalty approach. In Lagrange multipliers approach, the contact forces are taken as primary unknowns and the non-penetration condition is enforced [13, 25]. These extra variables add to the computational effort of the solution process. In the penalty method, the penetration between two contacting boundaries is introduced and the normal contact force is related to the penetration by a penalty parameter [21]. To best of our knowledges, few number of the contact analysis' use meshless methods [11, 15, 17]. Li et al. [17] have used a meshless method to study the two dimensional frictional contact problems.

The aim of this paper is to utilize the strength of both finite element method and meshless methods to overcome the difficulties of the contact problem. Thus, a combination of the finite element method and a meshless approximation method, namely, the MLS method is proposed to simulate the two dimensional unilateral contact problems. For the analysis, the element free Galerkin method and Galerkin formulation for two dimensional elasticity problems are considered. Then, the penalty method for imposition of contact constraint is proposed and the Lagrange multiplier method is used to impose essential boundary conditions. The implementation of the penalty/Lagrange based contact algorithm is described and the efficiency for our algorithm is illustrated with various examples.

The paper is organized as follows: In the next section, a review of the MLS approximation is presented and the applicability of the element free Galerkin method and Galerkin formulation for two dimensional elasticity problems is investigated. The model problem, its weak formulation, the penalty method for implementation of contact constraint and the Lagrange multiplier method for imposition of essential boundary conditions are described and analyse in Section 3. In Section 4, we present some numerical examples to conform the validity and efficiency of our approach. Finally, in Section 5, some conclusions are drawn about the described method and some possible future developments.

2. MLS Approximation

2.1. Review of the MLS approximation

The MLS as an approximation method has been introduced by Shepard [24] and Lancaster and Salkauskas [16]. It consists of three components: A basis function, a weight function associated with each node, and a set of coefficients that depends on node position.

Using MLS approximation, a data values $\mathbf{u} = \{u_i\}_{i=1}^{N}$ at nodes $\mathbf{x}_i$ is approximated by a function $u^h \in C^s(R^d)$ in a weighted square sense, namely,

$$u^h(\mathbf{x}) = \sum_{i=1}^{m} p_i(\mathbf{x}) a_i(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}), \tag{1}$$

where $p_i(\mathbf{x}),\ i = 1, 2, \ldots, m$ are monomial basis functions, $\mathbf{p}^T(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})]$, m is the number of terms in the basis, and $a_i(\mathbf{x})$ are the coefficients of the basis functions. In general, the basis functions are as follows: For example, for a 1-D problem, the linear basis is $\{1, x\}$, and the quadratic basis is $\{1, x, x^2\}$. For a 2-D problem, the linear basis is $\{1, x, y\}$, and the quadratic basis is $\{1, x, y, x^2, xy, y^2\}$, where $\mathbf{x} = (\mathrm{x}, y)$.

The coefficient vector $\mathbf{a}(\mathbf{x})$ is determined by minimizing a weight discrete square norm, which is defined as

$$J(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})(\mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) - u_i)^2, \tag{2}$$

where $w_i(\mathbf{x})$ is the weight function associated with the node i, n is the number of nodes in Ω for which the weight function $w_i(\mathbf{x}) > 0$, and u_i are the fictitious nodal values, but not the nodal values of the unknown trial function $u^h(\mathbf{x})$, i.e., $u^h(x_i) \neq u_i$.

Equation (2) can be written as

$$J = [P \cdot \mathbf{a}(\mathbf{x}) - \mathbf{u}]^T \cdot W \cdot [P \cdot \mathbf{a}(\mathbf{x}) - \mathbf{u}],$$

where

$$P = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix},$$

and

$$W = \begin{bmatrix} w_1(\mathbf{x}) & 0 & \cdots & 0 \\ 0 & w_2(\mathbf{x}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_n(\mathbf{x}) \end{bmatrix}.$$

The stationary point of J, in Equation (2), with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and $\mathbf{u}$:

$$A(\mathbf{x})\mathbf{a}(\mathbf{x}) = B(\mathbf{x})\mathbf{u}, \tag{3}$$

where the matrices $A(\mathbf{x})$ and $B(\mathbf{x})$ are defined by

$$A(\mathbf{x}) = P^T WP = B(\mathbf{x})P = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i), \tag{4}$$

$$B(\mathbf{x}) = P^T W = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \ldots, w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)]. \tag{5}$$

The matrix A is often called the *moment matrix*, it is of size $(m+1)\times(m+1)$. Computing $\mathbf{a}(\mathbf{x})$ by using Equation (3) and substituting it into Equation (1), give

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}) \cdot \mathbf{u} = \sum_{i=1}^{n} \Phi_i(\mathbf{x}) \cdot u_i, \quad \mathbf{x} \in \overline{\mathbf{\Omega}}, \tag{6}$$

where

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})A^{-1}(\mathbf{x})B(\mathbf{x}), \tag{7}$$

or

$$\phi_i(\mathbf{x}) = \sum_{k=0}^{m} p_k(\mathbf{x})[A^{-1}(\mathbf{x})B(\mathbf{x})]_{ki}, \tag{8}$$

where $\phi_i(\mathbf{x})$ are called the shape functions of the MLS approximation, corresponding to nodal point $\mathbf{x}_i$. If $w_i(\mathbf{x}) \in C^r(\Omega)$ and $p_k(\mathbf{x}) \in C^s(\Omega)$, $i = 1, 2, \ldots, n$, $k = 0, 1, \ldots, m$, then $\phi_i(\mathbf{x}) \in C^{\min(r,s)}(\Omega)$.

The cubic splin weight function is applied in the present work as

$$w_i(\mathbf{x}) = \begin{cases} \frac{2}{3} - 4r^2 + 4r^3, & r \leq \frac{1}{2}, \\ \frac{4}{3} - 4r + 4r^2 - \frac{4}{3}r^3, & \frac{1}{2} < r \leq 1, \\ 0, & r > 1, \end{cases}$$

with $r = \dfrac{\|\mathbf{x} - \mathbf{x}_i\|}{d_i}$, where d_i is the support size of node $\mathbf{x}_i$.

2.2. MLS method for two-dimensional elasticity problems

2.2.1. Notations

Let $\lambda \geq 0$ denote Lamé's first parameter and $\mu > 0$ denote the shear modulus of a body occupying a polyhedral region Ω, whose boundary Γ is partitioned into disjoint sets Γ_u and Γ_t on which the displacement and tractions are prescribed, respectively.

We define the spaces

$$L_2(\Omega) = \left\{ v : \int_\Omega v^2 dx < \infty \right\},$$

$$H^1(\Omega) = \{ v \in L_2(\Omega) : \operatorname{grad} v \in L_2(\Omega)^2 \},$$

and

$$H^1_{D,\bar{v}}(\Omega) = \{ v \in H^1(\Omega) : v = \bar{v}, \text{ on } \Gamma_u \}.$$

For $u = (u_1, u_2) \in H^1(\Omega)^2$, we define the standard differential operators

$$\operatorname{div} u = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}, \quad \operatorname{grad} u = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix},$$

and for a matrix $\tau = (\tau_{ij})$, we define

$$\operatorname{div} \tau = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \\ \frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \end{pmatrix}$$

For $v \in H^1(\Omega)^2$, let

$$\epsilon(v) = \frac{1}{2}(\operatorname{grad} v + (\operatorname{grad} v)^T),$$

and

$$\sigma(v) = \mathcal{C}\varepsilon(v) = 2\mu\varepsilon(v) + \lambda \operatorname{tr}(\varepsilon(v))I,$$

denote the linearized strain and stress associated with a displacement v, where I is the identity tensor. The compliance tensor is given in terms of the inverse $\mathcal{C}^{-1}$ defined by

$$\mathcal{C}^{-1}w = \frac{1}{2\mu}\left(w - \frac{\lambda}{2\mu + 3\lambda}\operatorname{tr}(w)I\right).$$

2.2.2. Weak formulation

Consider the following two-dimensional problem of finding the linearly elastic displacement u such that:

$$\operatorname{div} \sigma(u) + f = 0, \quad \text{in } \Omega, \tag{9}$$

where σ is the stress tensor, which corresponds to the displacement field u and f is the body force. The corresponding boundary conditions are given as

$$\sigma(u) \cdot n = \bar{t}, \quad \text{on } \Gamma_t; \quad u = \overline{u}, \quad \text{on } \Gamma_u, \tag{10}$$

where $\overline{u}$ and $\bar{t}$ are the prescribed displacements and tractions, respectively, on the displacement boundary Γ_u and on the traction boundary Γ_t, and n is the unit outward normal to the boundary Γ.

The weak formulation of the partial differential equation (9) and boundary condition (10) over Ω is to find $u \in H^1(\Omega)^2$ such that $u = \overline{u}$, on Γ_u and

$$\int_\Omega \sigma(u) : \varepsilon(v) d\Omega - \int_\Omega f \cdot v d\Omega - \int_{\Gamma_t} \bar{t} \cdot v d\Gamma = 0, \quad \forall v \in H^1_{D,\overline{u}}(\Omega)^2, \tag{11}$$

where $w : v = \sum_{i=1}^{2} \sum_{j=1}^{2} w_{ij} v_{ij}$.

The data are supposed to satisfy $f \in L_2(\Omega)^2$ and $\bar{t} \in L_2(\Gamma_t)^2$. Likewise, we assume that the data $\overline{u}$ is smooth, and, in addition, is compatible in the sense that problem (9) with the boundary condition (10) admits a solution. We shall also assume that the boundary Γ_u is closed and is such that a unique solution $u \in H^1(\Omega)^2$ to the problem (11) exists.

In the finite element method, the solution u is sought in a finite dimensional subspace V of $H^1_{D, \overline{u}}(\Omega)^2$ and the variational equation is checked for a test function v in a subspace S of $H^1_{D, \overline{u}}(\Omega)^2$. If $V = S$ than the numerical method is called *Galerkin method*. In this case, we have

$$u(x) = \sum_{i=1}^{N} q_i \phi_i(x) \quad \text{and} \quad v(x) = \sum_{i=1}^{N} d_i \phi_i(x).$$

These approximations are introduced in the variational form (11) to obtain the usual linear system that allows us to determine the unknowns $\{q_i\}_i$.

In the meshless method, we use the shape functions obtained by the MLS approximation and the equations that describe the finite element method are still valid to describe this method. This choice of shape functions leads to two different points; one in the imposition of boundary conditions and one for the numerical integration.

2.2.3. Boundary conditions

The enforcement of essential boundary conditions remains an area of ongoing research, since the shape functions in meshless methods are not strict interpolates, i.e., they do not satisfy the Kronecker delta condition

$$\phi_i(x_j) \neq \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\phi_i(x_j)$ is the shape function of node A_i vaulted at node A_j and δ_{ij} is the Kronecker delta. A consequence of this is that the approximated value on the boundary depends on interior nodes as well as boundary nodes. Boundary conditions are in terms of linear combinations of nodal values. Various ways of implementing boundary conditions for meshless methods have been suggested, such as combining the element free Galerkin (EFG) method with finite element shape functions close to the boundary, the use of a modified variational principle [18], the use of window or correction functions that vanish on the boundary [6], and the use of Lagrange multipliers [2]. In this paper, the essential boundary condition is imposed by using the Lagrange multipliers method

$$\int_\Omega \sigma(u) : \epsilon(v) d\Omega - \int_\Omega f \cdot v d\Omega - \int_{\Gamma_t} \bar{t} \cdot v d\Gamma - \int_{\Gamma_u} \delta\lambda \cdot (u - \bar{u}) d\Gamma$$

$$- \int_{\Gamma_u} v \cdot \lambda d\Gamma = 0. \tag{12}$$

The Lagrange multiplier λ is approximated by $\lambda = \sum_i \lambda_i \varphi_i(x)$, where $\varphi_i(x)$ are the finite element interpolations on the boundary Γ_u.

Using this method, the system of equations of the linear elasticity problem is given by

$$\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ q \end{bmatrix}, \tag{13}$$

with

$$K_{ij} = \int_\Omega B_i^T C B_j d\Omega,$$

$$G_{ij} = -\int_{\Gamma_u} \phi_i \varphi_j S d\Gamma,$$

$$F_i = \int_\Omega f \phi_i d\Omega - \int_{\Gamma_t} \bar{t} \phi_i d\Gamma,$$

$$q_j = -\int_{\Gamma_u} \varphi_j S\bar{u}\, d\Gamma,$$

where S is a diagonal matrix of size 2×2 in two dimensions, with S_{ii} equal to one if the displacement is imposed on x_i and zero otherwise and

$$B_i = \begin{pmatrix} \frac{\partial \phi_i}{\partial x} & 0 \\ 0 & \frac{\partial \phi_i}{\partial y} \\ \frac{\partial \phi_i}{\partial y} & \frac{\partial \phi_i}{\partial x} \end{pmatrix}.$$

2.2.4. Numerical intégration

In the MLS method, the concept of element does not exist, and the shape functions are not polynomial, so we can not evaluate the integrals as for the finite element method. However, we can use a direct integration nodes or an underlying grid that serves only in the numerical integration, and does not interfere in the approximation scheme. In this work, we used the second method [2].

3. The Implementation of a 2D Unilateral Contact Problems by Penalty/MLS

3.1. Notations

We consider two elastics bodies Ω_1 and Ω_2 (Figure 1). We denote $\Omega = \Omega_1 \cup \Omega_2$, and n_i is the unit normal outward to the domain boundary Γ_i of Ω_i.

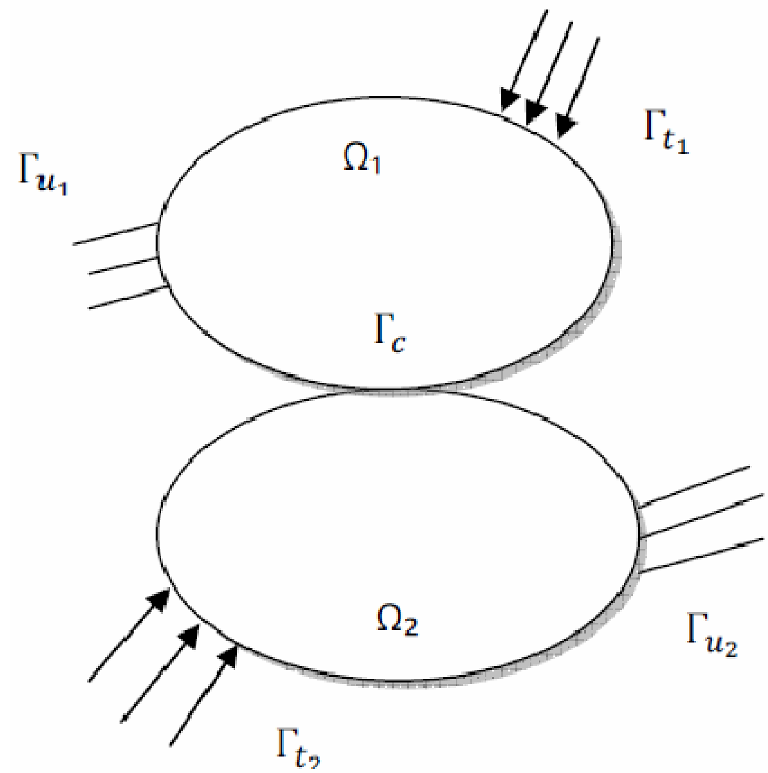

Figure 1. Elastic bodies in contact.

The classical equations of elasticity problem are

$$\begin{cases} \operatorname{div} \sigma + f = 0 & \text{in} \quad \Omega, \\ \sigma = \mathcal{C}\varepsilon & \text{in} \quad \Omega, \\ \sigma n = \bar{t} & \text{on } \Gamma_t, \\ u = \bar{u} & \text{on } \Gamma_u, \end{cases}$$

where $\bar{u}$ is the prescribed displacement on the essential boundary, $\Gamma_u = \Gamma_{u_1} \cup \Gamma_{u_2}$ and $\bar{t}$ is the prescribed traction on the natural boundary, $\Gamma_t = \Gamma_{t_1} \cup \Gamma_{t_2}$. The contact interface consists of the intersection of the surfaces of the two bodies, and is shown by Γ_c with $\Gamma_c = \Gamma_{c_1} \cap \Gamma_{c_2}$. In Γ_c, we set $n = n_1 = -n_2$.

We decompose the displacement u into u_1 and u_2 onto Ω_1 and Ω_2, respectively, and it is the same for the stress tensor σ and the strain ε. The n-normal force is denoted by

$$\bar{t}_{i_n} = n_i \sigma_i n_i, \quad i = 1, 2, \tag{14}$$

and the tangential force is denoted by

$$\bar{t}_{i_t} = \sigma_i n_i - \bar{t}_{i_n} n_i, \quad i = 1, 2. \tag{15}$$

The contact conditions on Γ_c are

$$(u_2 - u_1)n \geq 0, \tag{16}$$

$$\bar{t}_n = \bar{t}_{1_n} = -\bar{t}_{2_n} \leq 0, \tag{17}$$

$$\bar{t}_t = \bar{t}_{1_t} = -\bar{t}_{2_t} = 0, \tag{18}$$

$$(u_2 - u_1) \cdot n\bar{t}_n = 0. \tag{19}$$

Under these conditions, the classical formulation of our problem is as follows:

$$\begin{cases} \operatorname{div} \sigma + f = 0 & \text{in } \Omega, \\ \sigma = \mathcal{C}\varepsilon & \text{in } \Omega, \\ \sigma n = \bar{t} & \text{on } \Gamma_t, \\ u = \bar{u} & \text{on } \Gamma_u, \\ (u_2 - u_1)n \geq 0 & \text{on } \Gamma_c, \\ (u_2 - u_1) \cdot n\bar{t}_n = 0 & \text{on } \Gamma_c, \\ \bar{t}_n = \bar{t}_{1_n} = -\bar{t}_{2_n} \leq 0 & \text{on } \Gamma_c, \\ \bar{t}_t = \bar{t}_{1_t} = -\bar{t}_{2_t} = 0 & \text{on } \Gamma_c. \end{cases} \tag{20}$$

3.2. Weak formulation

We introduce the space

$$K = \{v \in H^1_{D,\overline{u}}(\Omega) / (u_2 - u_1)n \geq 0 \text{ on } \Gamma_c\}.$$

The weak formulation of the problem (20) is as follows: Find a displacement field $u \in K$ such as, for all $v \in K$, we have

$$\int_{\Omega_1} \sigma_1 : \varepsilon(v_1 - u_1) d\Omega + \int_{\Omega_2} \sigma_2 : \varepsilon(v_2 - u_2) d\Omega$$

$$= \int_{\Omega_1} f(v_1 - u_1) d\Omega + \int_{\Omega_2} f(v_2 - u_2) d\Omega + \int_{\Gamma_{t_1}} \bar{t}_1 (v_1 - u_1) d\Gamma$$

$$+ \int_{\Gamma_{t_2}} \bar{t}_2 (v_2 - u_2) d\Gamma + \int_{\Gamma_{c_1}} \sigma_1 n_1 (v_1 - u_1) d\Gamma + \int_{\Gamma_{c_2}} \sigma_2 n_2 (v_2 - u_2) d\Gamma.$$

Then

$$\int_{\Omega} \sigma : \varepsilon(v - u) d\Omega = \int_{\Omega} f(v - u) d\Omega + \int_{\Gamma_t} \bar{t}(v - u) d\Gamma$$

$$+ \int_{\Gamma_{c_1}} \sigma_1 n_1 (v_1 - u_1) d\Gamma + \int_{\Gamma_{c_2}} \sigma_2 n_2 (v_2 - u_2) d\Gamma.$$

We define

$$a(u, v - u) = \int_{\Omega} \sigma : \varepsilon(v - u) d\Omega,$$

$$l(v - u) = \int_{\Omega} f(v - u) d\Omega + \int_{\Gamma_t} \bar{t}(v - u) d\Gamma,$$

then, we obtain

$$a(u, v - u) - l(v - u) = \int_{\Gamma_{c_1}} \sigma_1 n_1 (v_1 - u_1) d\Gamma + \int_{\Gamma_{c_2}} \sigma_2 n_2 (v_2 - u_2) d\Gamma.$$

Using Equations (17) and (18), we get

$$a(u,\, v-u) - l(v-u) = \int_{\Gamma_{c1}} \bar{t}_n (v_1 - u_1) n d\Gamma - \int_{\Gamma_{c2}} \bar{t}_n (v_2 - u_2) n d\Gamma.$$

By Equations (17) and (19), we have

$$\int_{\Gamma_c} \bar{t}_n (v_2 - v_1) n d\Gamma \leq 0.$$

The weak formulation becomes: Find a displacement field $u \in K$ such as, for all $v \in K$, we have

$$a(u,\, v-u) - l(v-u) \geq 0. \tag{21}$$

The nonlinearity due to the unilateral constraints makes it difficult to solve the contact problem directly. For this reason, an equivalent minimization problem is formulated, which is particularly suitable for numerical implementation. It can be shown [7] that the optimization problem: Find $u \in K$, which minimizes the potential energy

$$E_c(v) = \frac{1}{2} a(v,\, v) - l(v) \tag{22}$$

is equivalent to the contact problem (21). In this paper, we propose a numerical resolution strategies by using the penalty method.

3.3. The implementation of the penalty method

The penalty methods avoid the need for additional variables by introducing an approximation of the constraint condition. An additional term enters in the weak form of the governing equations, which penalizes the dissatisfaction of the constraint condition by a positive penalty parameter.

Physically, this regulation is interpreted as follows: We accept that in the contact zone there is a slight penetration; the condition (16) is abandoned and we assume that $g = (u_2 - u_1)n$ can take negative values and sufficiently small by relative to the dimensions of solids (see Figure 2).

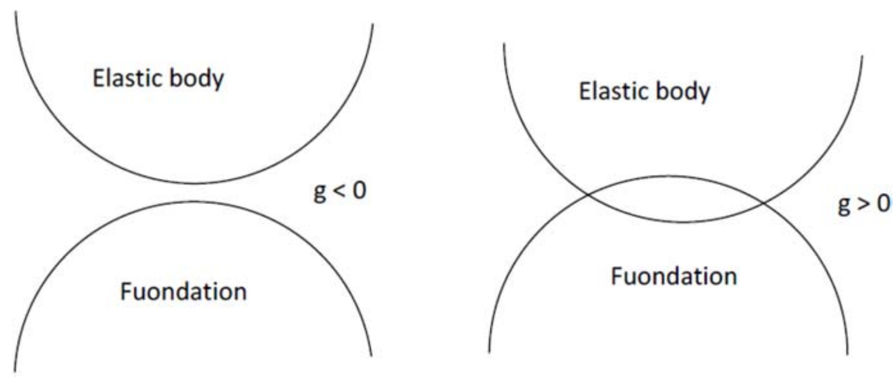

Figure 2. Definition of interpenetration in two bodies before and after interpenetrating.

The penalty term is introduced as follows:

$$\overline{E}_c(u) = E_c(u) + \frac{\alpha}{2} g^t g. \tag{23}$$

According to the minimum potential energy principle, the first variation of the total potential energy is equal to zero. Therefore, the following linear system is obtained

$$\delta E_c(u) + \alpha g^t \delta g = 0. \tag{24}$$

By enforcing the essential boundary by using Lagrange multiplier as in (12), the potential energy in the entire solution domain can be given by

$$\int_\Omega \sigma(u) : \varepsilon(v) d\Omega - \int_\Omega f \cdot v d\Omega - \int_{\Gamma_t} \bar{t} \cdot v d\Gamma - \int_{\Gamma_u} \delta\lambda \cdot (u - \bar{u}) d\Gamma$$

$$- \int_{\Gamma_u} v \cdot \lambda d\Gamma + \alpha \int_{\Gamma_C} g \delta g H(g) d\Gamma = 0, \tag{25}$$

where α is the penalty parameter and H is the heaviside step function and is defined as

$$H(g) = \begin{cases} 1, & \text{if } g \leq 0, \\ 0, & \text{if } g > 0. \end{cases}$$

To discretize the contact interface, the interpenetration g can be discretized as follows:

$$g = \phi u \quad \text{and} \quad \delta g = \phi \delta u.$$

We have

$$\alpha \int_{\Gamma_C} g \delta g H(g) d\Gamma = \alpha \int_{\Gamma_C} \phi^T \delta u \phi u H(u) d\Gamma = P_c \delta u, \quad \forall \delta u,$$

with

$$P_c = \alpha \int_{\Gamma_C} \phi^T \phi H(u) d\Gamma u = K_c u,$$

where K_c is the contact matrix for small-displacement elastostatic and is defined as

$$K_c = \alpha \int_{\Gamma_C} \phi^T \phi H(u) d\Gamma. \tag{26}$$

Using (26) and the system (13), we get

$$\begin{bmatrix} K_T & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ q \end{bmatrix}, \tag{27}$$

where

$$K_T = K + K_c.$$

To implement the contact in our meshless code, the following algorithm is considered:

○ For a well chosen α (penalty parameter) and TOL (tolerance), we have

○ Compute K, G, F and q.

○ iter = 0 and a given u_0.

● Begin

. Find contact nodes of the bodie

. Compute K_c

. Solve (27)

. Compute $\text{RES} = \dfrac{\|u - u_0\|}{\|u_0\|}$

End

○ While $(\text{RES} \geq \text{TOL})$

iter = iter + 1

○ End.

4. Numerical Examples

4.1. Example 1

In this first example, we consider a rectangular plate of dimension $L \times D$, which is in frictionless contact with a rigid foundation (Figure 3), where $L = 40\text{mm}$, $D = 10\text{mm}$, the material behaviour will be isotropic linear elastic with $E = 3.10^4\,\text{MPa}$, $\upsilon = 0.3$ and a load $P = 10\text{MPa}$ and the boundary conditions $u_x(0, .) = 0$ and $u_x(L, .) = 0$.

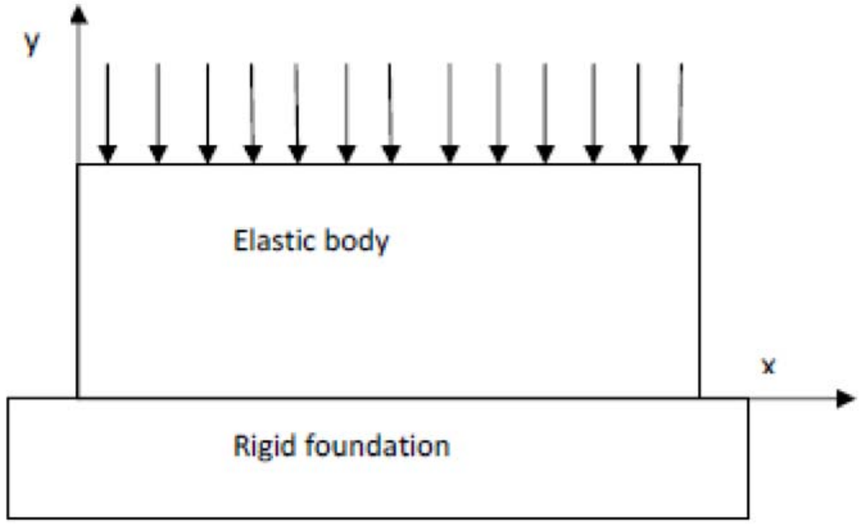

Figure 3. Elastic body in contact with a rigid foundation.

For a nodal distribution 25×12 nodes, we obtain the following results:

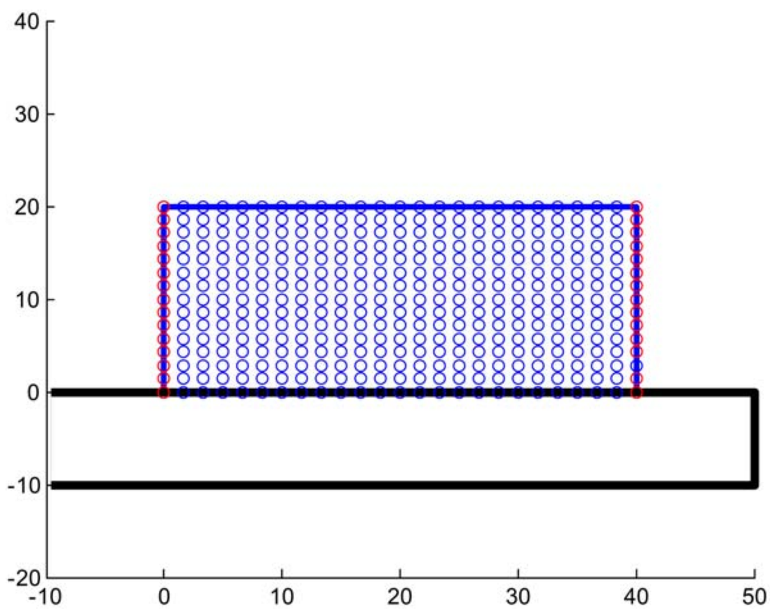

Figure 4. Distribution of nodal 25×15 nodes.

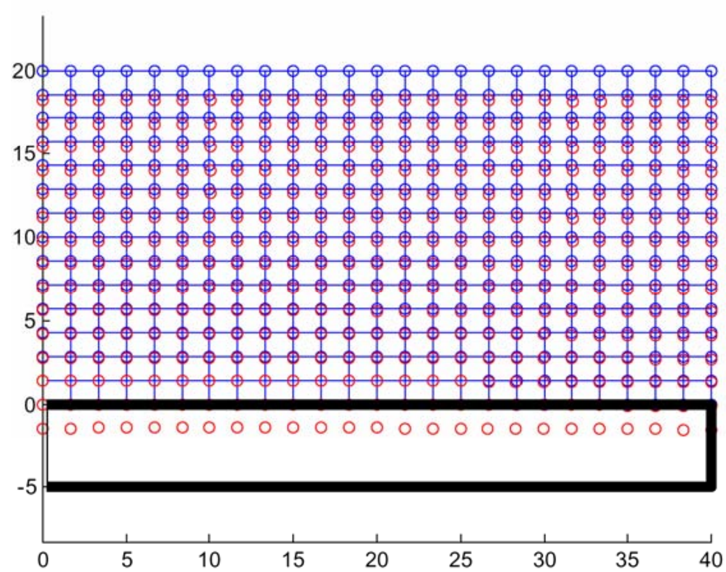

Figure 5. Deformation of the structure for $\alpha = 10^2$, the deformation is in red color.

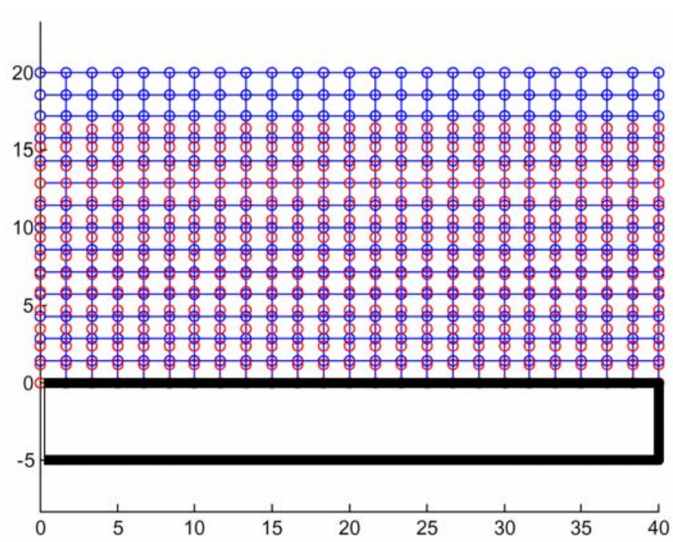

Figure 6. Deformation of the structure for $\alpha = 10^9$, the deformation is in red color.

The Figure 6 shows that the body not interpenetrated.

To study the effect of the contact penalty parameter, the vertical displacement is shown in Figure 7.

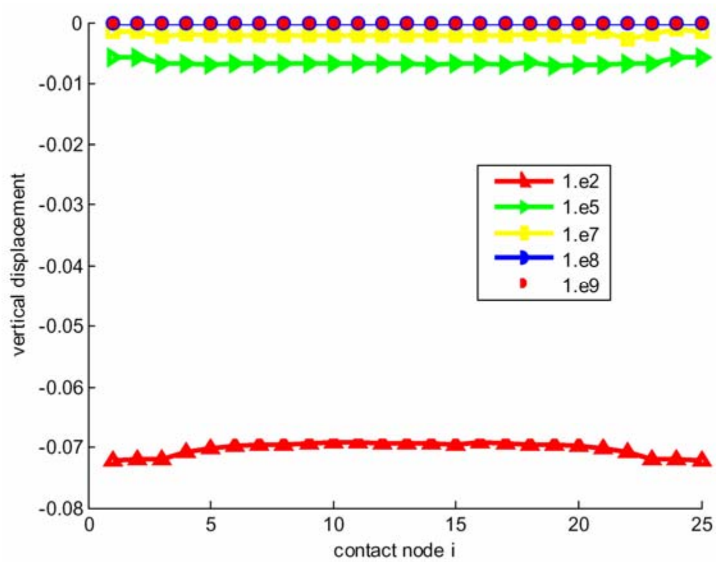

Figure 7. Distribution of vertical displacement in the contact boundary for different values of contact penalty parameter.

Figure 7 shows that a little change in the value of penalty parameter has a considerable effect on the results.

4.2. Example 2

In this second example, we consider a rectangular plate of dimension $L \times D$ fixed at both ends and that is in frictionless contact with a rigid foundation, where $L = 60\text{mm}$, $D = 30\text{mm}$, the material behaviour will be isotropic linear elastic with $E = 21.10^3\text{MPa}, \upsilon = 0.3$ and a load $P = 10\text{MPa}$ and the boundary conditions $u_x(0, .) = u_y(0, .) = 0$ and $u_x(L, .) = u_y(L, .) = 0.$

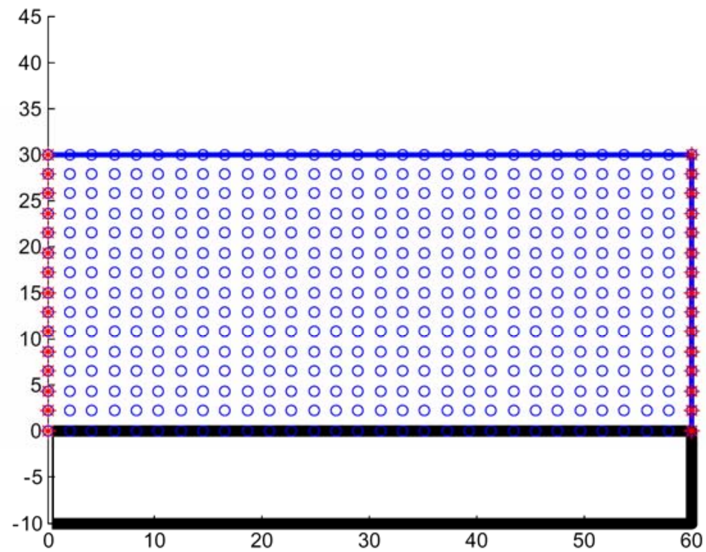

Figure 8. Distribution of nodal 30×15 nodes.

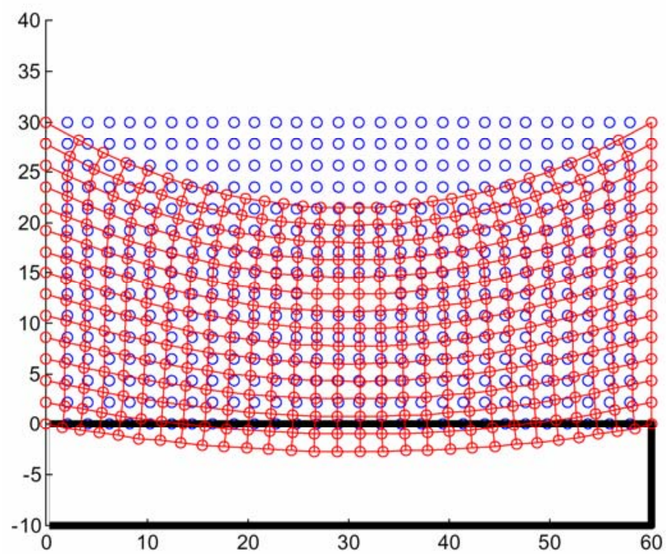

Figure 9. Deformation of the structure for $\alpha = 10^2$, the deformation is in red color.

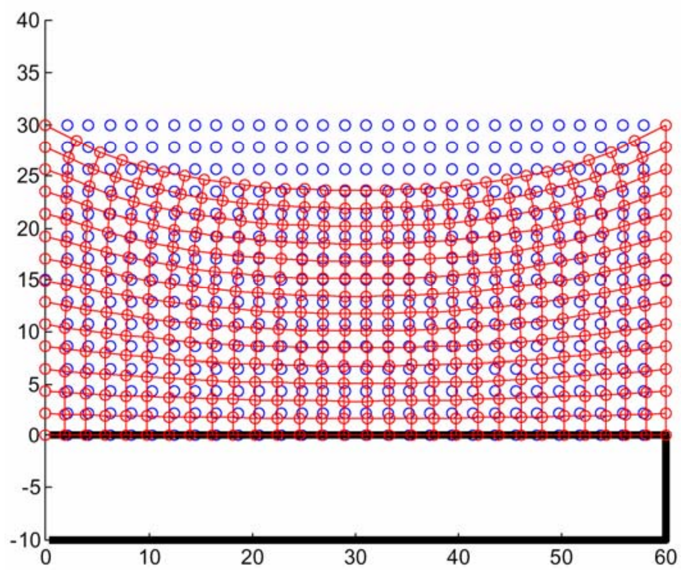

Figure 10. Deformation of the structure for $\alpha = 10^8$, the deformation is in red color.

To study the effect of the contact penalty parameter, the vertical displacement is shown in Figure 11.

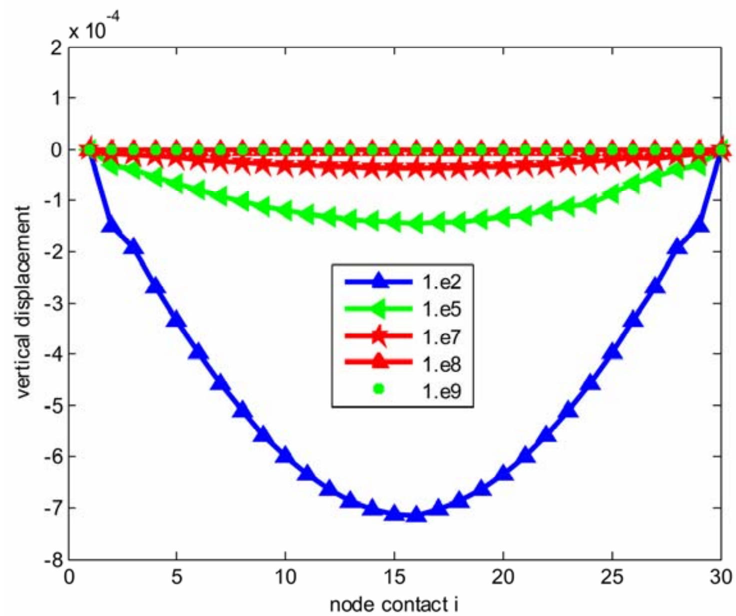

Figure 11. Vertical displacement for different values of contact penalty parameter.

Figure 11 shows that a little change in the value of penalty parameter has a considerable effect on the results.

5. Conclusion

In this paper, the EFG-MLS meshfree approach was applied to solve the two dimensional frictionless contact problem. The penalty method was used for the imposition of the contact constraint and the Lagrange multiplier method was employed to impose essential boundary conditions. The method was evaluated by using several tests. The results showed a good performance of the proposed method. Also, the final formulation of the method proves its simplicity and feasibility. The future development is promising for more complex problems.

References

[1] T. Belytschko, Y. Y. Lu and L. Gu, Element-free Galerkin methods, Inter. J. Numer. Methods Engg. 37 (1994), 229-256.

[2] T. Belytschko, Y. Y. Lu, L. Gu and M. Tabbara, Element-free Galerkin methods for static and dynamic fracture, Inter. J. Solids Struct. 32 (1995), 2547-2570.

[3] T. Belytschko, Y. Krongauz, D. Organ, M. Fleming and P. Krysl, Meshless methods: An overview and recent developments, Comput. Methods Appl. Mech. Engg. 139 (1996), 3-47.

[4] M. Chehel Amirani and N. Nemati, Simulation of two dimensional unilateral contact using a coupled FE/EFG method, Engineering Analysis with Boundary Elements.

[5] M. Dehghan and D. Mirzaei, Meshless local Petrov-Galerkin (MLPG) method for the unsteady magnetohydrodynamic (MHD) flow through pipe with arbitrary wall conductivity, Appl. Numer. Math. 59 (2009), 1043-1058.

[6] C. A. Duarte, The HP-Cloud Method, PhD. Thesis, The University of Texas at Austin, 1996.

[7] G. Duvaut and J. L. Lions, Les Inégalités Variationnelles en Mécanique et en Physique, Dunod, Paris, 1972.

[8] G. Duvaut, Equilibre d'un solide élastique avec contact unilatéral et frottement, C. R. Acad. Sci., Paris I, Math. 290 (1980), 1075-1093.

[9] T.-P. Fries and H.-G. Matthies, Classification and Overview of Meshfree Methods, Institute of Scientific Computing, Technical University Braunschweig: Brunswick, Germany, 2004.

[10] P. Hild and P. Laborde, Quadratic finite element methods for unilateral contact problems, Appl. Numer. Math. 41 (2002), 401-421.

[11] D. A. Hu, S. Y. Long, X. Han and G. Y. Li, A meshless local Petrov-Galerkin method for large deformation contact analysis of elastomers, Engg. Anal. Bound. Elem. 31 (2007), 657-666.

[12] R. El Jid and H. F. Fassi, A meshless (EFG) approach for linear elasticity and numerical comparison with the finite element method, Inter. J. Engg. Sci. Tech. (2011), 6593-6599.

[13] W. Ju and R. L. Taylor, A perturbed Lagrange formulation for the finite element solution of nonlinear frictional contact problems, J. Theo. Appl. Mech. 7 (1988), 1-14.

[14] N. Kikuchi and J. T. Oden, Contact Problems in Elasticity, SIAM Philadelphia, 1988.

[15] K. C. Kwon and S. K. Youn, The least-squares meshfree method for rigid-plasticity with frictional contact, Int. J. Solids Struct. 43 (2006), 7450-7481.

[16] P. Lancaster and K. Salkauskas, Surface generated by moving least squares method, Math. Comput. 37 (1981), 141-158.

[17] Y. Li, G. R. Liu, M. T. Luan, K. Y. Dai, Z. H. Zhong and G. Y. Li et al., Contact analysis for solids based on linearly conforming radial point interpolation method, Comp. Mech. 39 (2007), 537-554.

[18] Y. Y. Lu, T. Belytschko and L. Gu, A new implementation of the element free Galerkin, Comput. Methods Appl. Mech. Engg. 113 (1994), 397-414.

[19] D. Mirzaei and M. Dehghan, Implementation of meshless LBIE method to the 2D nonlinear SG problem, Inter. J. Numer. Methods Engg. 79 (2009), 1662-1682.

[20] Y. X. Mukherjee and S. Mukherjee, The boundary node method for potential problems, Inter. J. Numer. Methods Engg. 40 (1997), 797-815.

[21] D. Peric and D. R. J. Owen, Computational model for contact problems with friction based on the penalty method, Inter. J. Numer. Methods Engg. 35 (1992), 1289-1309.

[22] D. Peric and D. R. J. Owen, Computational model for contact problems with friction based on the penalty method, Inter. J. Numer. Methods Engg. 36 (1992), 1289-309.

[23] N. Vinh Phu, T. Rabczuk, S. Bordas and M. Duflot, Meshless methods: A review and computer implementation aspects, Math. Comp. Simul. (2008).

[24] D. Shepard, A two-dimensional interpolation function for irregularly spaced points, Proc. 23rd Nat. Conf. ACM, ACM Press, New York, (1968), 517-524.

[25] J. C. Simo and T. A. Laursen, An augmented Lagrangian treatment of contact problems involving friction, Comput. and Struct. 42 (1992), 97-116.

[26] J. Sladek, V. Sladek and S. N. Atluri, Local boundary integral equation (LBIE) method for solving problem of elasticity with nonhomogeneous material properties, Comput. Mech. 24 (2000), 456-462.

[27] C. Tiago and P. M. Pimenta, An EFG method for the nonlinear analysis of plates undergoing arbitrarily large deformations, Engineering Analysis with Boundary Elements (2008).

[28] L. Wang, On the duality methods for the contact problem in elasticity, Comput. Methods Appl. Mech. Engg. 167 (1998), 275-282.

[29] P. Wriggers, Computational Contact Mechanics, 2nd Edition, Springer, 2006.

[30] T. Zhu, J. D. Zhang and S. N. Atluri, A local boundary integral equation (LBIE) method in computational mechanics and a meshless discretization approach, Comput. Mech. 21 (1998), 223-235.

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