Journal of Mathematical Sciences: Advances and Applications
Volume 15, Number 1, 2012, Pages 13-36

TANGENT LIFT OF HIGHER ORDER OF MULTIVECTOR FIELDS AND APPLICATIONS

P. M. KOUOTCHOP WAMBA, A. NTYAM and J. WOUAFO KAMGA

Department of Mathematics
The University of Yaoundé 1
P. O. BOX 812, Yaoundé
Cameroon
e-mail: wambapm@yahoo.fr

Abstract

Let M be a smooth manifold of dimension $m > 0$. In [3], the author's studied the tangent lift of multivector fields of M on TM and give some applications to the theory of Poisson structures and symplectic foliations. In this paper, we generalize this lifting to tangent functors of higher order.

1. Introduction

For two natural numbers p and r, we denote by T_p^r the bundle functor of p-dimensional velocities of order r. When $p = 1$, the functor T_1^r is called *tangent functor* of order r and is denoted by T^r.

Let M be a smooth manifold of dimension $m > 0$. The Lie bracket of vector fields can be extended to graded Lie bracket on the graded Lie

2010 Mathematics Subject Classification: Primary 53C15; Secondary 53C75, 53D05.

Keywords and phrases: generalized foliations, Poisson manifold, Lie algebroid, prolongation of vector fields, prolongation of differential forms, bundle functor, natural transformations.

Received March 29, 2012

algebra space $\mathfrak{X}^*(M)$ of multivector fields by Schouten bracket and for simple multivector fields, we have the following formula (see [13]):

$$[X_1 \wedge \ldots \wedge X_p, Y_1 \wedge \ldots \wedge Y_q]$$

$$= \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots Y_q, \quad (1)$$

where the hat over X_i and Y_j means that these terms are missing in the expression.

A Poisson structure on M is usually given by a bivector field Π such that $[\Pi, \Pi] = 0$. In [3], the authors have introduced the tangent lift of a multivector field P on the manifold M to a multivector field $P^{(c)}$ on the tangent manifold TM and they proved that this lifting commutes with the Schouten bracket. As application, they studied the tangent lift of a Poisson structure and the tangent lift of a generalized foliation. In this paper, we generalize these liftings to tangent bundle of higher order, so the main results are Theorems 1, 2, 3, 4 and Propositions 1, 2, 6.

All manifolds and maps are assumed to be infinitely differentiable. r is a natural number $(r \geq 1)$.

2. Prolongations of Vector Fields and Differential Forms Revisited

For any $\beta \in \{0, \ldots, r\}$, we denote by τ_β the canonical linear form on $J_0^r(\mathbb{R}, \mathbb{R})$ defined by

$$\tau_\beta(j_0^r g) = \frac{1}{\beta!} \frac{d^\beta}{dt^\beta}(g(t))\Big|_{t=0}.$$

Let M be a smooth manifold of dimension $m > 0$. For $f \in C^\infty(M)$, we set

$$f^{(\beta)} = \tau_\beta \circ T^r f.$$

The smooth function $f^{(\beta)} : T^r M \to \mathbb{R}$ is called the β-*prolongation of* f. For some properties of these prolongations, see [10].

We denote by $\kappa^r : T^r \circ T \to T \circ T^r$ the canonical flow-natural equivalence associated to T^r (see [4]) and $\alpha^r : T^* \circ T^r \to T^r \circ T^*$ the canonical isomorphism defined in [1]. Consider the natural transformation $\chi^{(\beta)} : T^r \to T^r$ defined for any vector bundle (E, M, π) by

$$\chi_E^{(\beta)}(j_0^r \varphi) = j_0^r(t^\beta \varphi).$$

This natural transformation induces, for a natural number $q \geq 1$, the natural transformation (see [12])

$$\chi^{(\beta), q} : T^r \circ \otimes_q^0 \to \otimes_q^0 \circ T^r,$$

such that for any $\tilde{x} \in T^r M$, $j_0^r \varphi \in T^r(\otimes_q^0 E)_{\tilde{x}}$, $j_0^r \eta_1, \ldots, j_0^r \eta_q \in (T^r E)_{\tilde{x}}$, we have

$$\chi_E^{(\beta), q}(j_0^r \varphi)(j_0^r \eta_1, \ldots, j_0^r \eta_q) = \tau_\beta(j_0^r(\varphi * (\eta_1, \ldots, \eta_q))),$$

where

$$\varphi * (\eta_1, \ldots, \eta_q) : \mathbb{R} \to \mathbb{R}, \quad z \mapsto \varphi(z)(\eta_1(z), \ldots, \eta_q(z)).$$

For a vector field X and a differential form ω of degree q on M, we denote by $X^{(\beta)}$ and $\omega^{(\beta)}$ the β-prolongations of X and ω on $T^r M$, respectively, i.e.,

$$X^{(\beta)} = \kappa_M^r \circ \chi_{TM}^{(\beta)} \circ T^r X,$$

$$\omega^{(\beta)} = \otimes_q^0 \kappa_M^r \circ \chi_{TM}^{(\beta), q} \circ T^r \omega.$$

The smooth sections, $X^{(0)}$ and $\omega^{(r)}$ are usually denoted by $X^{(c)}$ and $\omega^{(c)}$, respectively, and they are called the *complete lifts* of X and ω.

If locally, $X = a^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_{i_1 \ldots i_q} dx^{i_1} \wedge \ldots \wedge dx^{i_q}$, we have

$$\begin{cases} X^{(\beta)} = (a^i)^{(\gamma-\beta)} \dfrac{\partial}{\partial x^i_\gamma}, \\ \text{and} \\ \omega^{(\beta)} = \displaystyle\sum_{\beta_1 + \ldots + \beta_q + \gamma = \beta} \omega^{(\gamma)}_{i_1 \ldots i_q} dx^{i_1}_{\beta_1} \wedge \ldots \wedge dx^{i_q}_{\beta_q}. \end{cases}$$

For some properties of $X^{(\beta)}$ and $\omega^{(\beta)}$, see [10].

Remark 1. In the particular case $q = 1$, the β-prolongations of ω is given by the formula

$$\omega^{(\beta)} = \varepsilon^r_M \circ \chi^{(r-\beta)}_{T^*M} \circ T^r \omega,$$

where $\varepsilon^r := (\alpha^r)^{-1} : T^r \circ T^* \to T^* \circ T^r$.

3. Some Natural Transformations

3.1. The natural transformations $\alpha^{r,q} : \wedge^q T^* T^r \to T^r(\wedge^q T^*)$

Let $q \geq 2$ be a natural number. Consider the natural transformation $\wedge^q : \oplus^q T^* \to \wedge^q T^*$ defined for any smooth manifold M by

$$\wedge^q_M : \quad \overset{q}{\oplus} T^* M \to \wedge^q T^* M$$

$$u_1^* \oplus \ldots \oplus u_q^* \mapsto u_1^* \wedge \ldots \wedge u_q^*.$$

The bundle map

$$T^r(\wedge^q_M) \circ (\overset{q}{\oplus} \alpha^r_M) : \overset{q}{\oplus} T^* T^r M \to T^r(\wedge^q T^* M),$$

is a well-defined multilinear and skew-symmetric fibered morphism over $\mathrm{id}_{T^r M}$, so there is a unique vector bundle morphism

$$\alpha^{r,q}_M : \wedge^q T^* T^r M \to T^r(\wedge^q T^* M), \tag{2}$$

over $\mathrm{id}_{T^r M}$ such that the following diagram:

$$\begin{array}{ccc} \overset{q}{\bigoplus} T^* T^r M & \overset{\wedge^q_{T^r M}}{\rightarrow} & \wedge^q T^* T^r M \\ \overset{q}{\oplus}\alpha^r_M \downarrow & & \downarrow \alpha^{r,q}_M \\ \overset{q}{\bigoplus} T^r T^* M & \overset{T^r(\wedge^q_M)}{\rightarrow} & T^r(\wedge^q T^* M) \end{array} \tag{3}$$

commutes.

Remark 2. For any local diffeomorphism $f : M \to N$, the following diagram:

$$\begin{array}{ccc} \wedge^q T^* T^r M & \overset{\wedge^q T^* T^r f}{\rightarrow} & \wedge^q T^* T^r N \\ \alpha^{r,q}_M \downarrow & & \downarrow \alpha^{r,q}_N \\ T^r(\wedge^q T^* M) & \overset{T^r(\wedge^q T^* f)}{\rightarrow} & T^r(\wedge^q T^* N) \end{array}$$

commutes. So, $\alpha^{r,q} : \wedge^q T^* \circ T^r \to T^r \circ \wedge^q T^*$ is a natural transformation.

Let $\{x^1, \dots, x^m\}$ be a local coordinates system of M, we denote by (x^i, p_i) the corresponding local coordinates system of $T^* M$. Locally, we have

$$\wedge^q_M : (x^i, p_{1,i_1}, \dots, p_{q,i_q}) \mapsto (x^i, \omega_{i_1 \dots i_q}),$$

with $\omega_{i_1 \dots i_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma p_{1, i_{\sigma(1)}} \dots p_{q, i_{\sigma(q)}}$, and

$$T^r\left(\wedge^q_M\right) : \left(x^i_\beta, p^{\beta_1}_{1,i_1}, \dots, p^{\beta_q}_{q,i_q}\right) \mapsto \left(x^i_\beta, \omega_{i_1 \dots i_q}, \omega^{(\beta)}_{i_1 \dots i_q}\right);$$

but

$$\begin{cases} \omega^{(\beta)}_{i_1 \ldots i_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(\sum_{\beta_1 + \ldots + \beta_q = \beta} p^{\beta_1}_{1, i_{\sigma(1)}} \ldots p^{\beta_q}_{q, i_{\sigma(q)}} \right), \\ \text{and} \\ \left(\overset{q}{\oplus} \alpha^r_M \right) \left(x^i_\beta, p^{\beta_1}_{1, i_1}, \ldots, p^{\beta_q}_{q, i_q} \right) = \left(x^i_\beta, \Lambda^{\beta_1}_{1, i_1}, \ldots, \Lambda^{\beta_q}_{q, i_q} \right), \end{cases}$$

with $\Lambda^{\beta_1}_{1, i_1} = p^{r-\beta_1}_{1, i_1}, \ldots, \Lambda^{\beta_q}_{q, i_q} = p^{r-\beta_q}_{q, i_q}$, therefore,

$$T^r \left(\wedge^q_M \right) \circ \left(\overset{q}{\oplus} \alpha^r_M \right) \left(x^i_\beta, p^{\beta_1}_{1, i_1}, \ldots, p^{\beta_q}_{q, i_q} \right) = \left(x^i_\beta, \omega_{j_1 \ldots j_q}, \omega^{(\beta)}_{j_1 \ldots j_q} \right),$$

where

$$\begin{cases} \omega_{j_1 \ldots j_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma p^r_{1, j_{\sigma(1)}} \ldots p^r_{q, j_{\sigma(q)}}, \\ \text{and} \\ \omega^{(\beta)}_{j_1 \ldots j_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(\sum_{\beta_1 + \ldots + \beta_q = \beta} p^{r-\beta_1}_{1, j_{\sigma(1)}} \ldots p^{r-\beta_q}_{q, j_{\sigma(q)}} \right). \end{cases}$$

Remark 3. We deduce that

$$\alpha^{r,q}_M : \qquad \wedge^q T^* T^r M \to T^r (\wedge^q T^* M)$$

$$\left(x^i, x^i_\alpha, \omega_{i_{1,\alpha_1} \ldots i_{q,\alpha_q}} \right) \mapsto \left(x^i, x^i_\alpha, \Omega_{j_1 \ldots j_q}, \Omega^{(\beta)}_{j_1 \ldots j_q} \right),$$

where

$$\begin{cases} \Omega_{j_1 \ldots j_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \omega_{j_{\sigma(1), r} \ldots j_{\sigma(q), r}}, \\ \text{and} \\ \Omega^{(\beta)}_{j_1 \ldots j_q} = \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(\sum_{\beta_1 + \ldots + \beta_q = \beta} \omega_{j_{\sigma(1), r-\beta_1} \ldots j_{\sigma(q), r-\beta_q}} \right). \end{cases} \tag{4}$$

For a manifold M, we put $\alpha^{r,1}_M := \alpha^r_M$ and

$$\alpha^{r,0}_M : T^r M \times \mathbb{R} \to T^r M \times T^r \mathbb{R}, \quad (\tilde{x}, t) \mapsto (\tilde{x}, te_r).$$

3.2. The natural transformations $\epsilon^{r,q} : T^r \circ \wedge^q T \to \wedge^q T \circ T^r$

For a manifold M, the map $\alpha_M^{r,q} : \wedge^q \; T^* T^r M \to T^r(\wedge^q T^* M)$ is a vector bundle morphism over $\mathrm{id}_{T^r M}$ and

$$\begin{cases} <,>^q_{T^r M} : \;\; \wedge^q TT^r M \oplus \wedge^q T^* T^r M \to \mathbb{R} \\ \qquad\qquad\qquad\qquad k \oplus k^* \mapsto \langle k, k^* \rangle \\ <,>'^q_{T^r M} := \tau_r \circ T^r(<,>^q_M) : T^r(\wedge^q TM) \oplus T^r(\wedge^q T^* M) \to \mathbb{R}, \end{cases},$$

are two nondegenerate bilinear morphisms over $\mathrm{id}_{T^r M}$. There is a unique vector bundle morphism

$$\epsilon_M^{r,q} : T^r(\wedge^q TM) \to \wedge^q TT^r M, \tag{5}$$

such that

$$< u, \alpha_M^{r,q}(v) >'^q_{T^r M} = < \epsilon_M^{r,q}(u), v >^q_{T^r M},$$

holds for $u \oplus v \in T^r(\wedge^q TM) \oplus \wedge^q(T^* T^r M)$. $\epsilon_M^{r,q}$ is in fact the value on M of a well-defined $\epsilon^{r,q} : T^r \circ \wedge^q T \to \wedge^q T \circ T^r$. Locally,

$$\begin{aligned} \epsilon_M^{r,q} : \quad & T^r(\wedge^q TM) \to \wedge^q TT^r M \\ & \left(x^i_\beta, \prod\nolimits^{i_1 \dots i_q}_\beta\right) \mapsto \left(x^i_\beta, \widetilde{\prod}^{i_1, \beta_1 \dots i_q, \beta_q}\right), \end{aligned}$$

with

$$\widetilde{\prod}^{i_1, \beta_1 \dots i_q, \beta_q} = \sum_{\gamma_1 + \dots + \gamma_q + \mu = r} \delta^{r-\gamma_1}_{\beta_1} \dots \delta^{r-\gamma_q}_{\beta_q} \prod\nolimits^{i_1 \dots i_q}_\mu. \tag{6}$$

Remark 4. When $q = 1$, $\epsilon^{r,1} = \kappa^r$ and for $q = 0$, for all manifold M, we have

$$\epsilon_M^{r,0} : T^r M \times T^r \mathbb{R} \to T^r M \times \mathbb{R}, \quad (\tilde{x}, t, t^\alpha) \mapsto (\tilde{x}, t^r).$$

4. Complete Lift of Multivector Fields

In [3], Grabowski and Urbanski define the complete lift of multivector fields associated to the tangent bundle functor T by direct computations. We give here an intrinsic formula of these lifts associated to tangent bundle functors of higher order. Let $\prod$ be a multivector field of degree q on M. We put

$$\prod^{(c)} = \epsilon_M^{r,q} \circ T^r \prod : T^r M \to \wedge^q TT^r M. \tag{7}$$

This is a multivector field of degree q on $T^r M$. If locally

$$\prod = \sum_{1 \leq i_1 < \ldots < i_q \leq m} \prod^{i_1 \ldots i_q} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_q}},$$

then, we have

$$\prod^{(c)} = \sum_{\gamma_1 + \ldots + \gamma_q + \beta = r} (\prod^{i_1 \ldots i_q})^{(\beta)} \frac{\partial}{\partial x_{r-\gamma_1}^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{r-\gamma_q}^{i_q}}.$$

Example 1. In the particular case $r = 1$, we have (see [3])

$$\prod^{(c)} = (\prod^{i_1 \ldots i_q})^{(1)} \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial \dot{x}^{i_q}}$$

$$+ \sum_{k=1}^{q} \prod^{i_1 \ldots i_q} \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_k}} \wedge \ldots \wedge \frac{\partial}{\partial \dot{x}^{i_q}},$$

and for $q = 2$ and $\prod = \prod^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$,

$$\prod^{(c)} = \sum_{\alpha, \beta = 0}^{r} (\prod^{ij})^{(\alpha + \beta - r)} \frac{\partial}{\partial x_\alpha^i} \wedge \frac{\partial}{\partial x_\beta^j}. \tag{8}$$

Definition 1. The multivector field $\prod^{(c)}$ on $T^r M$ is called the *complete lift* on $T^r M$ of $\prod$.

Lemma 1. *For a simple multivector field* $P = X_1 \wedge \ldots \wedge X_p$, *we have*

$$P^{(c)} = \sum_{\gamma_1 + \ldots + \gamma_p = r} X_1^{(r-\gamma_1)} \wedge \ldots \wedge X_p^{(r-\gamma_p)}.$$

Proof. Let (U, x^i) be a local coordinates system of M over U; we have

$$P = \left(\sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma X_1^{i_{\sigma(1)}} \ldots X_q^{i_{\sigma(q)}} \right) \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_q}},$$

and

$$P^{(c)} = \sum_{\alpha_1 + \ldots + \alpha_q + \beta_1 + \ldots + \beta_p = r} \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(X_1^{i_{\sigma(1)}} \right)^{(\beta_1)}$$

$$\ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\beta_q)} \frac{\partial}{\partial x_{r-\alpha_1}^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{r-\alpha_q}^{i_q}},$$

thus

$$\left\langle P^{(c)}, dx_{\gamma_1}^{i_1} \wedge \ldots \wedge dx_{\gamma_q}^{i_q} \right\rangle$$

$$= \sum_{\alpha_1 + \ldots + \alpha_q + \beta_1 + \ldots + \beta_p = r} \left[\sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(X_1^{i_{\sigma(1)}} \right)^{(\beta_1)} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\beta_q)} \delta_{r-\alpha_1}^{\gamma_1} \ldots \delta_{r-\alpha_q}^{\gamma_q} \right]$$

$$= \sum_{\beta_1 + \ldots + \beta_p = (\sum \gamma_k) - r(q-1)} \left[\sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(X_1^{i_{\sigma(1)}} \right)^{(\beta_1)} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\beta_q)} \right]$$

$$= \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \sum_{\beta_1 + \ldots + \beta_p = (\sum \gamma_k) - r(q-1)} \left(X_1^{i_{\sigma(1)}} \right)^{(\beta_1)} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\beta_q)}$$

$$= \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left[\sum_{\beta_1 + \ldots + \beta_p = (\sum \gamma_k) - r(q-1)} \left(X_1^{i_{\sigma(1)}} \right)^{(\beta_{\sigma(1)})} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\beta_{\sigma(q)})} \right].$$

Moreover,

$$\begin{cases} X_1^{(r-\alpha_1)} = \sum_{\beta_1} \left(X_1^{i_1}\right)^{(\beta_1 - r + \alpha_1)} \frac{\partial}{\partial x_{\beta_1}^{i_1}} = \sum_{\beta_1} Y_1^{i_1, \beta_1} \frac{\partial}{\partial x_{\beta_1}^{i_1}}, \\ \vdots \\ X_q^{(r-\alpha_q)} = \sum_{\beta_q} \left(X_q^{i_q}\right)^{(\beta_q - r + \alpha_q)} \frac{\partial}{\partial x_{\beta_q}^{i_q}} = \sum_{\beta_q} Y_q^{i_q, \beta_q} \frac{\partial}{\partial x_{\beta_q}^{i_q}}, \end{cases}$$

and if we put $Q = \sum_{\alpha_1 + \ldots + \alpha_p = r} X_1^{(r-\alpha_1)} \wedge \ldots \wedge X_p^{(r-\alpha_p)}$, one has

$$\left\langle Q,\ dx_{\gamma_1}^{i_1} \wedge \ldots \wedge dx_{\gamma_q}^{i_q} \right\rangle$$

$$= \sum_{\alpha_1 + \ldots + \alpha_p = r} \det \left\langle X_k^{(r-\alpha_k)},\ dx_{\gamma_k}^{i_k} \right\rangle$$

$$= \sum_{\alpha_1 + \ldots + \alpha_p = r} \left[\sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma Y_1^{i_{\sigma(1)}, \gamma_{\sigma(1)}} \ldots Y_q^{i_{\sigma(q)}, \gamma_{\sigma(q)}} \right]$$

$$= \sum_{\alpha_1 + \ldots + \alpha_p = r} \left[\sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left(X_1^{i_{\sigma(1)}} \right)^{(\gamma_{\sigma(1)} - r + \alpha_1)} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\gamma_{\sigma(q)} - r + \alpha_q)} \right]$$

$$= \sum_{\sigma \in \mathfrak{S}_q} \varepsilon_\sigma \left[\sum_{\mu_1 + \ldots + \mu_p = (\sum \gamma_k) - r(q-1)} \left(X_1^{i_{\sigma(1)}} \right)^{(\mu_{\sigma(1)})} \ldots \left(X_q^{i_{\sigma(q)}} \right)^{(\mu_{\sigma(q)})} \right],$$

where $\mu_{\sigma(k)} = \gamma_{\sigma(k)} - r + \alpha_k$ and $1 \leq k \leq q$. □

Theorem 1. *For* $P \in \mathfrak{X}^p(M)$ *and* $Q \in \mathfrak{X}^q(M)$, *we have*

$$[P^{(c)},\ Q^{(c)}] = [P,\ Q]^{(c)}, \tag{9}$$

where $[.,\ .]$ *is the Schouten bracket.*

Proof. Let $P \in \mathfrak{X}^p(M)$ and $Q \in \mathfrak{X}^q(M)$; there is an open subset U of M such that

$$P|_U = X_1 \wedge \ldots \wedge X_p \quad \text{and} \quad Q|_U = Y_1 \wedge \ldots \wedge Y_q.$$

Let us denote $P|_U$ (resp., $Q|_U$) by P (resp., Q) for simplicity; we have

$$P^{(c)} = \sum_{\alpha_1 + \ldots + \alpha_p = r} X_1^{(r-\alpha_1)} \wedge \ldots \wedge X_p^{(r-\alpha_p)},$$

and

$$Q^{(c)} = \sum_{\beta_1 + \ldots + \beta_q = r} Y_1^{(r-\beta_1)} \wedge \ldots \wedge Y_q^{(r-\beta_q)}.$$

Computing $[P^{(c)}, Q^{(c)}]$, $[P, Q]^{(c)}$ and using

$$[X^{(\alpha)}, Y^{(\beta)}] = [X, Y]^{(\alpha+\beta)}, \qquad \text{(see [10])}$$

and the formula (1), the result follows. □

Remark 5. For any $\omega \in \Omega^p(M)$ and $\alpha \in \{0, 1, \ldots, r\}$, we have

$$< P^{(c)}, \omega^{(\alpha)} >^p_{T^r M} = \left(< P, \omega >^p_M\right)^{(\alpha-(p-1)r)}.$$

So, if $p \geq 3$, then $< P^{(c)}, \omega^{(\alpha)} >^p_{T^r M} = 0$.

5. First Application: Poisson Geometry

5.1. Tangent Poisson structure of higher order

Let (M, Π) be a Poisson manifold, i.e., M is a smooth manifold and $\Pi \in \mathfrak{X}^2(M)$ satisfies $[\Pi, \Pi] = 0$. It follows from Theorem 1, that $(T^r M, \Pi^{(c)})$ is also a Poisson manifold called the *tangent lifting* of order r of (M, Π).

Proposition 1. (i) *Let* (M, Π) *be a Poisson manifold, we have*

$$\sharp_{\Pi^{(c)}} = \kappa^r_M \circ T^r \sharp_\Pi \circ \alpha^r_M, \tag{10}$$

where $\sharp_\Pi$ *is the anchor map of* Π.

(ii) *If* $f : M \to N$ *is a Poisson map, then* $T^r f$ *is also Poisson map between the tangent lifting of Poisson structures of order r.*

(iii) *If* $(G, \mathfrak{m}, \prod)$ *is a Poisson Lie-group, then* $(T^r G, T^r \mathfrak{m}, \prod^{(c)})$ *is also a Poisson Lie-group.*

(iv) *If X is a Hamiltonian vector field for* $(M, \prod)$, *then* $X^{(c)}$ *is also a Hamiltonian vector field for* $(T^r M, \prod^{(c)})$.

(v) *If* $\prod$ *is a regular Poisson structure of rank* $2d$, *then* $\prod^{(c)}$ *is regular Poisson structure on* $T^r M$ *of rank* $2d(r+1)$.

Proof. Let $\{x^1, \ldots, x^m\}$ be a local coordinates system of M such that $\prod = \prod^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$; in this case, $\sharp_{\prod}(x^i, p_i) = (x^i, \prod^{ij} p_j)$. The first result comes from calculations in local coordinates. The second and third use the natural transformations κ_M^r and α_M^r. The fourth result uses the formula $Fl_t^{X^{(c)}} = T^r Fl_t^X$ and the last result comes from calculations in local coordinates. □

Corollary 1. *Let* $(M, \prod)$ *be a Poisson manifold, we denote by* $\{.\}_M$ *the bracket on* $C^\infty(M)$.

(i) $\forall f \in C^\infty(M)$, $\forall \alpha \in \{0, \ldots, r\}$, *we have*

$$X_{f^{(\alpha)}} = (X_f)^{(r-\alpha)}. \tag{11}$$

(ii) *For any* $\alpha, \beta \in \{0, 1, \ldots, r\}$, *we have* $\forall f, g \in C^\infty(M)$,

$$\{f^{(\alpha)}, g^{(\beta)}\}_{T^r M} = (\{f, g\}_M)^{(\alpha+\beta-r)}. \tag{12}$$

(iii) *For any* $\omega \in \Omega^1(M)$ *and* $\alpha \in \{0, 1, \ldots, r\}$,

$$i_{\prod^{(c)}} \omega^{(r-\alpha)} = (i_{\prod} \omega)^{(\alpha)}. \tag{13}$$

Proof. •

(i) Let $f \in C^{\infty}(M)$ and $\gamma \in \{0, 1, \ldots, r\}$, we have

$$\begin{aligned} X_{f^{(\gamma)}} &= \sharp_{\Pi^{(c)}}((df)^{(\gamma)}) \\ &= \kappa_M^r \circ T^r \sharp_{\Pi} \circ \alpha_M^r((df)^{(\gamma)}) \\ &= \kappa_M^r \circ T^r \sharp_{\Pi} \circ \chi_{T^*M}^{(r-\gamma)} \circ T^r(df) \\ &= \kappa_M^r \circ \chi_{TM}^{(r-\gamma)} \circ T^r(\sharp_{\Pi}(df)) \\ &= \kappa_M^r \circ \chi_{TM}^{(r-\gamma)} \circ T^r X_f. \end{aligned}$$

(ii) Let $f, g \in C^{\infty}(M)$ and $\alpha, \beta \in \{0, 1, \ldots, r\}$, we have

$$\begin{aligned} \{f^{(\alpha)}, g^{(\beta)}\}_{T^r M} &= X_{f^{(\alpha)}}(g^{(\beta)}) = (X_f)^{(r-\alpha)}(g^{(\beta)}) \\ &= (X_f(g))^{(\alpha+\beta-r)} = \{f, g\}_M^{(\alpha+\beta-r)}. \end{aligned}$$

(iii) The bundle $L = \{(i_{\Pi}\omega, \omega), \omega \in \Omega^1(M)\}$ is a Dirac structure on M, so the tangent Dirac structure of higher order L^r is a Dirac structure on $T^r M$. This Dirac structure is induced by the Poisson bivector $\Pi^{(c)}$ (see [7]). For $\omega \in \Omega^1(M)$ and $\alpha \in \{0, 1, \ldots, r\}$, $((i_{\Pi}\omega)^{(\alpha)}, \omega^{(r-\alpha)}) \in \Gamma(L^r)$, so $i_{\Pi^{(c)}}\omega^{(r-\alpha)} = (i_{\Pi}\omega)^{(\alpha)}$. □

Remark 6. If f is a Casimir function (i.e., $X_f = 0$), then for any α such that $0 \leq \alpha \leq r$, $f^{(\alpha)}$ is also a Casimir function.

5.2. Modular class of the tangent Poisson structures of order *r*

Let μ be a volume form on an orientable manifold M, the divergence of $X \in \mathfrak{X}(M)$ denoted by $div_{\mu}X$ is defined by the condition

$$\mathcal{L}_X \mu = (div_{\mu}X) \cdot \mu,$$

and one has

(i) For $X, Y \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$,

$$\begin{cases} div_{\mu}(f \cdot X) = f \cdot div_{\mu}X + X(f), \\ div_{\mu}([X, Y]) = X(div_{\mu}Y) - Y(div_{\mu}X). \end{cases}$$

(ii) For $X \in \mathfrak{X}(M)$ and $f \in C^{\infty}(M)$ such that $f > 0$,

$$div_{f \cdot \mu}(X) = div_{\mu}X + X(\ln \circ f).$$

Let (M, Π) be a Poisson manifold endowed with a volume form μ. The mapping

$$\Delta_{\mu} : C^{\infty}(M) \to C^{\infty}(M),$$

$$f \mapsto div_{\mu}(X_f),$$

is a derivation on $C^{\infty}(M)$, so there is a vector field on M, denoted Δ_{μ} and called the *modular vector field* of (M, Π, μ).

Let $\{x^1, \dots, x^m\}$ be a local coordinates system on M such that

$$\mu = a\, dx^1 \wedge \dots \wedge dx^m \quad \text{and} \quad \Pi = \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j};$$

we have

$$\Delta_{\mu} = \left(\frac{\partial \Pi^{ij}}{\partial x^j} + \Pi^{ij} \frac{\partial (\ln|a|)}{\partial x^j} \right) \frac{\partial}{\partial x^i}. \tag{14}$$

We recall that $\mathfrak{X}^k(M)$ denote the space of k-vector fields on M and $(\mathfrak{X}^*(M), \wedge)$ the contravariant Grassmann algebra on M. On a Poisson manifold (M, Π), one defines the Lichnerowicz-Poisson coboundary operator

$$\delta_{\Pi} : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M),$$

and one has the Lichnerowicz-Poisson cohomology space

$$H^k_{LP}(M, \Pi) = \frac{\ker(\delta_{\Pi} : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M))}{\operatorname{Im}(\delta_{\Pi} : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M))}.$$

For a modular vector field Δ_μ, one has $\delta_\Pi(\Delta_\mu) = 0$ and Δ_μ is a 1-cocycle. By the equality $\Delta_{f\cdot\mu} = \Delta_\mu - X_{\ln\circ f}$, we deduce that the LP-class $\mathcal{M} = [\Delta_\mu]$ does not depend on the volume form μ. It is called *modular class* of the Poisson manifold (M, Π). We say that (M, Π) is unimodular if $\mathcal{M} = 0$.

Let g be a Riemannian metric on an oriented manifold M. Then

$$\mu_g = \sqrt{\det(g)}dx^1 \wedge \ldots \wedge dx^m,$$

is a volume form of M. A local coordinates system of $T^r M$ associated at $\{x^1, \ldots, x^m\}$ is given by $\{x^i, x^i_\beta\}_{(i,\beta)\in\{1,\ldots,m\}\times\{1,\ldots,r\}}$, and we see easily that

$$\Phi^r_{\mu_g} = (\det(g))^{\frac{r+1}{2}} dx^1 \wedge \ldots \wedge dx^m \wedge \ldots \wedge dx^1_r \wedge \ldots \wedge dx^m_r, \tag{15}$$

is a volume form on $T^r M$, called *volume form* of Sasaki metric of order r associated to g.

Theorem 2. *The modular vector field* $\Delta_{\Phi^r_{\mu_g}}$ *of* $(T^r M, \Pi^{(c)}, \Phi^r_{\mu_g})$ *is such that*

$$\Delta_{\Phi^r_{\mu_g}} = (r+1)(\Delta_{\mu_g})^{(r)}, \tag{16}$$

where $(\Delta_{\mu_g})^{(r)}$ *is the r-prolongation of the vector field* Δ_{μ_g}.

Proof. The local expression of Poisson bivector $\Pi^{(c)}$ is given by the formula ([9]). By the formula (15), we have

$$\Delta_{\Phi^r_{\mu_g}} = \left(\sum_{\beta=0}^{r} \frac{\partial(\Pi^{ij})^{(\alpha+\beta-r)}}{\partial x^j_\beta} + \sum_{\beta=0}^{r}(\Pi^{ij})^{(\alpha+\beta-r)} \frac{\partial \ln(\det(g))^{\frac{r+1}{2}}}{\partial x^j_\beta}\right)\frac{\partial}{\partial x^i_\alpha}$$

$$= \left(\sum_{\beta=0}^{r}\left(\frac{\partial \Pi^{ij}}{\partial x^j}\right)^{(\alpha-r)} + (r+1)\sum_{\beta=0}^{r}(\Pi^{ij})^{(\alpha+\beta-r)} \frac{\partial \ln(\sqrt{\det(g)})}{\partial x^j_\beta}\right)\frac{\partial}{\partial x^i_\alpha}$$

$$= \left((r+1)\left(\frac{\partial \Pi^{ij}}{\partial x^j} \right)^{(\alpha-r)} + (r+1)\left(\Pi^{ij} \right)^{(\alpha-r)} \frac{\partial \ln(\sqrt{\det(g)})}{\partial x^j} \right) \frac{\partial}{\partial x_\alpha^i}$$

$$= (r+1)\left(\frac{\partial \Pi^{ij}}{\partial x^j} + \Pi^{ij} \frac{\partial \ln(\sqrt{\det(g)})}{\partial x^j} \right) \frac{\partial}{\partial x_r^i}.$$

Furthermore,

$$\Delta_{\mu_g} = \left(\frac{\partial \Pi^{ij}}{\partial x^j} + \Pi^{ij} \frac{\partial \ln(\sqrt{\det(g)})}{\partial x^j} \right) \frac{\partial}{\partial x^i},$$

we deduce that $\Delta_{\Phi^r_{\mu_g}} = (r+1)(\Delta_{\mu_g})^{(r)}$. □

Remark 7. This result generalizes the result established by Vaisman and Mitric in [14].

Corollary 2. *The modular class of the Poisson manifold* $(T^r M, \Pi^{(c)})$ *is represented by the vector field* $(r+1)(\Delta_\mu)^{(r)}$ *for any modular vector field* Δ_μ *of the Poisson manifold* (M, Π).

Corollary 3. *If the Poisson manifold* (M, Π) *is unimodular, then the tangent Poisson manifold of higher order* $(T^r M, \Pi^{(c)})$ *is also unimodular.*

6. Second Application: Lie Algebroids and Foliations

Let $\chi^{(\alpha)} : T^r \to T^r$ be the natural transformation defined in Section 2. Let $S : M \to E$ be a smooth section of a vector bundle (E, M, p), we define the section $S^{(\alpha)}$ of $(T^r E, T^r M, T^r \pi)$ by

$$S^{(\alpha)} = \chi_E^{(\alpha)} \circ T^r S, \quad 0 \leq \alpha \leq r.$$

$S^{(\alpha)}$ is called the α-*prolongation* of the section S (see [2] or [11]). In the same way, we denote by $(\varepsilon_1, \dots, \varepsilon_n)$ be a basis of local sections of E and $(\varepsilon^1, \dots, \varepsilon^n)$ be the dual basis of local section of $\pi^* : E^* \to M$. We have the induced adapted coordinate systems

$$\begin{cases}(x^i,\ y^j) \text{ in } E,\\ (x^i,\ \pi_j) \text{ in } E^*,\end{cases}$$

and

$$\begin{cases}(x^i,\ y^j,\ x^i_\alpha,\ y^j_\alpha) \text{ in } T^rE,\\ (x^i,\ \pi_j,\ x^i_\alpha,\ \pi^\alpha_j) \text{ in } T^rE^*,\\ (x^i,\ p_j,\ x^i_\alpha,\ p^\alpha_j) \text{ in } (T^rE)^*.\end{cases}$$

There is a natural bundle isomorphism

$$I^r_{E^*}: \quad (T^rE)^* \to T^rE^*,$$

such that locally,

$$I^r_{E^*}(x^i,\ p_j,\ x^i_\alpha,\ p^\alpha_j) = (x^i,\ \pi_j,\ x^i_\alpha,\ \pi^\alpha_j),$$

with

$$\begin{cases}\pi_j = p^r_j,\\ \pi^\alpha_j = p^{r-\alpha}_j.\end{cases}$$

6.1. The Lie algebroid T^rE

Let (E, M, π) be a vector bundle, in [9] is defined the natural submersion $r_E : T^*E \to E^*$ such that locally

$$r_E(x^i,\ y^j,\ p_i,\ \pi_j) = (x^i,\ \pi_j),$$

where $(x^i,\ y^j)$ are an adapted coordinates systems of E. The following commutative diagrams:

$$\begin{array}{ccccc} & T^rTE & \overset{\kappa^r_E}{\to} & TT^rE & \\ T^rT\pi & \downarrow & & \downarrow & TT^r\pi \\ & T^rTM & \underset{\kappa^r_M}{\to} & TT^rM & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & T^*T^rE & \overset{\alpha^r_E}{\to} & T^rT^*E & \\ r_{T^rE} & \downarrow & & \downarrow & T^rr_E \\ & (T^rE)^* & \underset{I^r_{E^*}}{\to} & T^rE^* & \end{array}$$

show that, the maps $(\kappa^r_E, \kappa^r_M, \mathrm{id}_{T^rE}, \mathrm{id}_{T^rM})$ and $(\alpha^r_E, I^r_{E^*}, \mathrm{id}_{T^rE}, \mathrm{id}_{T^rM})$ are the isomorphisms of double vector bundle of $(T^rTE, T^rTM, T^rE, T^rM)$, $(T^*T^rE, (T^rE)^*, T^rE, T^rM)$ on $(TT^rE, TT^rM, T^rE, T^rM)$, $(T^rT^*E, T^rE^*, T^rE, T^rM)$.

Theorem 3. *Let* $(E, [.], \rho)$ *be a Lie algebroid. There is one and only one Lie algebroid structure on* T^rE *such that*: *For all* $S_1, S_2 \in \Gamma(E)$ *and* $\alpha, \beta \in \{0, 1, \dots, r\}$

$$[S_1^{(\alpha)}, S_2^{(\beta)}] = [S_1, S_2]^{(\alpha+\beta)}, \tag{17}$$

the anchor map $\rho^{(r)}$ *is given by*

$$\rho^{(r)} = \kappa^r_M \circ T^r\rho. \tag{18}$$

This Lie algebroid structure is called the tangent lift of order r of the Lie algebroid $(E, [.], \rho)$.

Proof. Let $(E, [.], \rho)$ be a Lie algebroid, this structure induced linear Poisson structure on (E^*, Π). So, the map $\sharp_\Pi : T^*E^* \to TE^*$ is a morphism of double vector bundle. It follows that $\alpha^r_{E^*} \circ T^r(\sharp_\Pi) \circ \kappa^r_{E^*} = \sharp_{\Pi^{(c)}}$ is a morphism of double vector bundle. So that $\Pi^{(c)}$ is a Poisson bivector on T^rE^* linear on fibers. Consequently, it induces a Lie algebroid structure on $(T^rE^*)^*$. The Lie algebroid structure on T^rE is such that, the canonical isomorphism $I^r_E : (T^rE^*)^* \to T^rE$ is a Lie algebroid morphism. Let $S_1, S_2 \in \Gamma(E)$, $\widetilde{S_1}, \widetilde{S_2} : E^* \to \mathbb{R}$ be the linear maps on the fiber induced by S_1 and S_2, respectively. For all $\alpha, \beta \in \{0, 1, \dots, r\}$ and by the equalities

$$I^r_E\left(\widetilde{\widetilde{S_1}^{(r-\alpha)}}\right) = S_1^{(\alpha)} \text{ and } I^r_E\left(\widetilde{\widetilde{S_2}^{(r-\beta)}}\right) = S_2^{(\beta)};$$

we deduce that

$$[S_1^{(\alpha)}, S_2^{(\beta)}] = [S_1, S_2]^{(\alpha+\beta)}.$$

Let $S \in \Gamma(E)$ and $f \in C^\infty(M)$, we have

$$\begin{aligned}[\widetilde{S}^{(\alpha)}, f^{(\beta)} \circ T^r\pi^*] &= [\widetilde{S}^{(\alpha)}, (f \circ \pi^*)^{(\beta)}] \\ &= [\widetilde{S}, f \circ \pi^*]^{(\alpha+\beta-r)} \\ &= (\rho(S)(f))^{(\alpha+\beta-r)} \\ &= (\rho(S))^{(r-\alpha)}(f^{(\beta)}).\end{aligned}$$

So, the anchor map $\widetilde{\rho}$ of $(T^rE^*)^*$ is such that

$$\begin{aligned}\tilde{\rho}(\widetilde{\widetilde{S}^{(r-\alpha)}}) &= (\rho(S))^{(\alpha)} \\ &= \kappa_M^r \circ \chi_{TM}^{(\alpha)} \circ T^r\rho \circ T^rS \\ &= \kappa_M^r \circ T^r\rho(S^{(\alpha)}).\end{aligned}$$

By the equality $\rho^{(r)} \circ I_E^r = \widetilde{\rho}$, we deduce that $\rho^{(r)} = \kappa_M^r \circ T^r\rho$. □

Remark 8. (1) If locally, we have

$$[\varepsilon_i, \varepsilon_j] = c_{ij}^k\varepsilon_k, \quad 1 \le i < j \le n,$$

then

$$[\varepsilon_i^{(\alpha)}, \varepsilon_j^{(\beta)}] = \sum_{\nu=0}^{r-\alpha-\beta} (c_{ij}^k)^{(\nu)}\varepsilon_k^{(\alpha+\beta+\nu)}.$$

(2) Let $(E, [.], \rho)$ be a Lie algebroid and S be a smooth section of E. For any $\alpha \in \{0, 1, \dots, r\}$, we have

$$\rho^{(r)}(S^{(\alpha)}) = [\rho(S)]^{(\alpha)}. \tag{19}$$

6.2. Some properties of the tangent lift of higher order of a Lie algebroid

Proposition 2. *Let* $(M, \prod)$ *be a Poisson manifold. Then* $T^r T^* M$ *is a tangent Lie algebroid of order r. The canonical mapping* $\alpha_M^r : T^* T^r M \to T^r T^* M$ *is an isomorphism of Lie algebroids.*

Proof. By direct computation in local coordinates. □

Corollary 4. *Let* $(M, \prod)$ *be a Poisson manifold and* ω_1, ω_2 *be two* 1*-forms on M. For any* $\alpha, \beta \in \{0, 1, \dots, r\}$, *we have*

$$[\omega_1^{(\alpha)}, \omega_2^{(\beta)}] = [\omega_1, \omega_2]^{(\alpha+\beta-r)}.$$

In particular,

$$[\omega_1^{(c)}, \omega_2^{(c)}] = [\omega_1, \omega_2]^{(c)}.$$

Example 2. We consider the tangent Lie algebroid TM, $T^r TM$ is the tangent lift of order r and $TT^r M$ is the tangent Lie algebroid on $T^r M$. The vector bundle isomorphism $\kappa_M^r : T^r TM \to TT^r M$ over $\mathrm{id}_{T^r M}$ is a Lie algebroid isomorphism.

Example 3. A Lie algebra $\mathfrak{g}$ is a Lie algebroid. Let $\{e_1, \dots, e_m\}$ be a basis of $\mathfrak{g}$, we have

$$\forall i, j \in \{1, \dots, m\} \quad [e_i, e_j] = c_{ij}^k e_k,$$

where c_{ij}^k are constants functions, so that $(c_{ij}^k)^{(\nu)} = 0$ for all $\nu \geq 1$. The Lie algebroid structure of order r $T^r \mathfrak{g}$ is such that

$$\forall i, j \in \{1, \dots, m\}, \quad \forall \alpha, \beta \in \{0, \dots, r\}, \quad [e_i^\alpha, e_j^\beta] = c_{ij}^k e_k^{\alpha+\beta}.$$

So, the Lie algebroid structure $T^r \mathfrak{g}$ is the Lie algebra $T^r \mathfrak{g}$.

Proposition 3. *Let* $(A, [.], \rho)$ *be a Lie algebroid. The map*

$$\alpha_A^r : T^*T^rA \to T^rT^*A,$$

is an isomorphism of Lie algebroid over the canonical isomorphism $I^r_{A^*}$.

Proof. A^* is a Poisson structure, it follows that $\alpha^r_{A^*} : T^*T^rA^* \to T^rT^*A^*$ is an isomorphism of Lie algebroids. The rest of the proof comes from the commutative diagram

$$\begin{array}{ccccc} & T^*T^rA^* & \xrightarrow{\alpha^r_{A^*}} & T^rT^*A^* & \\ R_{T^rA} & \downarrow & & \downarrow & T^rR_A\, , \\ & T^*T^rA & \xrightarrow[\alpha^r_A]{} & T^rT^*A & \end{array}$$

where $R_A : T^*A^* \to T^*A$ is the natural bundle morphism defined in [9].

□

6.3. Tangent lifts of higher order of generalized foliations

Let M be a smooth manifold of dimension m and $X, Y \in \mathfrak{X}(M)$, we set

$$\begin{cases} \mathcal{L}_X^1 Y = \mathcal{L}_X Y = [X, Y], \\ \mathcal{L}_X^k Y = \mathcal{L}_X(\mathcal{L}_X^{k-1} Y), \qquad \forall k \geq 2. \end{cases}$$

In this case, for any $\alpha, \beta \in \{0, \dots, r\}$ and $k \in \mathbb{N}^*$, we have

$$\mathcal{L}^k_{X^{(\alpha)}} Y^{(\beta)} = (\mathcal{L}_X^k Y)^{(k\alpha+\beta)}.$$

Let E be a smooth generalized distribution on M, we denote by $\mathfrak{X}_E$ the set of all local vector fields such that: For all $x \in M$, $X(x) \in E_x$. Let us notice that for a completely integrable distribution E, the family $\mathfrak{X}_E$ is a Lie subalgebra of the Lie algebra of vector fields on M. Consequently,

$$\forall k \in \mathbb{N}^*, \quad \forall (X, Y) \in \mathfrak{X}_E \times \mathfrak{X}_E, \text{ we have } \mathcal{L}_X^k Y \in \mathfrak{X}_E.$$

Theorem 4. *Let E be a completely integrable generalized distribution on M. Then, the distribution* E^r *generated by the family* $\{X^{(\alpha)}, X \in \mathfrak{X}_E, 0 \leq \alpha \leq r\}$ *of vector fields on* $T^r M$ *is completely integrable.*

Proof. Let $X, Y \in \mathfrak{X}_E$ and $\alpha, \beta \in \{0, \ldots, r\}$ such that $\alpha \neq 0$; there is $q \in \mathbb{N}$ such that $(q+1)\alpha + \beta \geq r + 1$. So,

$$(Fl_t^{X^{(\alpha)}})^* Y^{(\beta)} = Y^{(\beta)} + \sum_{k=1}^{q} \frac{t^k}{k!} (\mathcal{L}_X^k Y)^{(k\alpha+\beta)},$$

and when $\alpha = 0$,

$$(Fl_t^{X^{(c)}})^* Y^{(\beta)} = [(Fl_t^X)^* Y]^{(\beta)}.$$

$\mathcal{L}_X Y \in \mathfrak{X}_E$ and $(Fl_t^X)^* Y \in \mathfrak{X}_E$, it follows that E^r is invariant with respect to flow of $X^{(\alpha)}$ for all α. So, E^r is integrable. □

Definition 2. The tangent foliation of order r, $T^r\mathcal{F}$ of a generalized foliation $\mathcal{F}$ defined by E is the foliation defined by E^r.

In the particular case where $r = 1$, we have the same results of Grabowski and Urbanski in [3].

Proposition 4. *If a submanifold* $N \subset M$ *is the union of leaves of the foliation* $\mathcal{F}$, *then* $(\pi_M^r)^{-1}(N)$ *is the union of leaves of the foliation* $T^r\mathcal{F}$.

Proof. Let F^r be a leaf of $T^r\mathcal{F}$, by this equalities,

$$\begin{cases} T\pi_M^r(X^{(\alpha)}) = 0, & \text{if } \alpha > 0, \\ T\pi_M^r(X^{(c)}) = X, & \text{if } \alpha = 0, \end{cases}$$

and $T_\xi F^r$ is generated by $\{X^{(\alpha)}(\xi)\}$, the set $\pi_M^r(F^r)$ is a subset of a leaf $\mathcal{F}$ of the foliation $\mathcal{F}$. For all $\xi \in (\pi_M^r)^{-1}(F)$, the tangent projection $\pi_M^r(T^r\mathcal{F}(\xi))$ of $T^r\mathcal{F}(\xi)$ is contained in F. In this case, $(\pi_M^r)^{-1}(F)$ is the union of leaves $T^r\mathcal{F}$. □

Proposition 5. *If a submanifold* $F \subset M$ *is a leaf of a generalized foliation* $\mathcal{F}$, *then* $T^r F$ *is a leaf of the generalized foliation* $T^r \mathcal{F}$.

Proof. Since F is maximal, $T_x F = E_x$. We set $E(F) = \bigcup_{x \in F} E_x$; $E(F)$ is generated by the vector fields tangent to F. $i : F \to M$ is the natural immersion. By the equalities

$$(i^* Z)^{(\alpha)} = (T^r i)^* Z^{(\alpha)}, \quad Z \in \mathfrak{X}(M),$$

we deduce that $X^{(\alpha)}$ with $0 \leq \alpha \leq r$ are tangent to $T^r F$. Since E^r is generated by the family $\{X^{(\alpha)}, X \in \mathfrak{X}_E\}$, it follows that $E^r(T^r F) = TT^r F$. Clearly, $T^r F$ is a maximal integral submanifold of E^r. □

All the leaves of the foliation $T^r \mathcal{F}$ are not necessarily of the form $T^r F$.

Proposition 6. *Let* $(E, [.], \rho)$ *be a Lie algebroid. The generalized foliation induced by the tangent Lie algebroid of higher order* $(T^r E, [.], \rho^{(r)})$ *coincide with the tangent lifting of higher order of the generalized foliation induced by* $(E, [.], \rho)$.

Proof. This proof comes from the equalities $(\rho(S))^{(\alpha)} = \rho^{(r)}(S^{(\alpha)})$ for all $S \in \Gamma(E)$ and $\alpha = 0, 1, \ldots, r$. □

Let $(M, \prod)$ be a Poisson manifold. We denote by D the characteristic distribution generated by the Hamiltonian vector fields of $(M, \prod)$.

Corollary 5. D^r *is the characteristic distribution induced by* $\sharp_{\prod^{(c)}}$.

Proof. Comes from Proposition 6. □

References

[1] F. Cantrijn, M. Crampin, W. Sarlet and D. Saunders, The canonical isomorphism between T^kT^* and T^*T^k, C. R. Acad. Sci. Paris, t. 309, Série II, (1989), 1509-1514.

[2] J. Gancarzewicz, W. Mikulski and Z. Pogoda, Lifts of some tensor fields and connections to product preserving functors, Nagoya Math. J. 135 (1994).

[3] J. Grabowski and P. Urbanski, Tangent lifts of Poisson and related structures, J. Phys. A: Math. Gen. 28 (1995), 6743-6777, Printed in the UK.

[4] I. Kolar, P. Michor and J. Slovak, Natural Operations in Differential Geometry, Springer-Verlag, 1993.

[5] I. Kolar, Functorial prolongations of Lie algebroids, Proceedings Conf. Prague 2004, Charles University, Prague, (2005), 301-309.

[6] P. M. Kouotchop Wamba, A. Ntyam and J. Wouafo Kamga, Tangent Dirac structures of higher order, Archivum Mathematicum (BRNO), Tomus 47 (2011), 17-22.

[7] P. M. Kouotchop Wamba, A. Ntyam and J. Wouafo Kamga, Some properties of tangent Dirac structure of higher order, to appear.

[8] J. Kurek, Natural affinors in higher order cotangent bundle, Archivum Mathematicum (BRNO), Tomus 28 (1992), 175-180.

[9] K. C. H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, London Mathematical Society; Lecture Note Series 213, Cambridge University Press.

[10] A. Morimoto, Lifting of some type of tensors fields and connections to tangent bundles of p^r-velocities, Nagoya Math. J. 40 (1970), 13-31.

[11] A. Ntyam and J. Wouafo Kamga, New versions of curvature and torsion formulas for the complete lifting of a linear connection to Weil bundles, Annales Polonici Mathematici 82.3 (2003).

[12] A. Ntyam and A. Mba, On natural vector bundle morphisms $T^A \circ \otimes^q_s \to \otimes^q_s \circ T^A$ over id_{T^A}, Annales Polonici Mathematici 96.3 (2009), 233-240.

[13] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Birkhäuser, 1994 (Progress in Mathematics; vol. 118).

[14] I. Vaisman and G. Mitric, Poisson structures on tangent bundle, arXiv: math/0108130v1 [math. DG] 20 Aug 2001.

[15] J. Wouafo Kamga, Global Prolongation of Geometric Objects to Some Jet Spaces, International Centre for Theoretical Physics, Trieste, Italy, 1997.

[16] J. Wouafo Kamga, On the Tangential Linearization of Hamiltonian Systems, International Centre for Theoretical Physics, Trieste, Italy, 1997.

■