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A NOTE ON TOPOLOGICAL LOCAL GROUPS

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Abstract

In this paper, we investigate the basic properties of topological local groups and some related theorems.

1. Introduction

A topological local group X is an object, which satisfies the usual axioms of topological group, expect that the product xy may not be defined for some $x, y \in X$; the existence of identity and inverses are assumed.

Cartan [2] defined a local Lie group in 1936, and Pontryagin expressed that local Lie groups are basis for Lie groups [9]. Many authors investigated local Lie groups [4, 8, 10]. Olver tried to extend local Lie groups to the Lie groups [7]. Recently, Goldbring verified Hilbert's fifth problem for local Lie groups by methods from nonstandard analysis [5].

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In this paper, we investigate the basic properties of topological local group. Section 2 contains the main definitions of local groups, topological local groups, and some examples. We study some properties of topological local groups such as productivity and inverse limit.

In Section 3, the quotient topological local group is defined and we prove some results for topological local groups similar to topological groups, such as the first and second isomorphism theorems.

We use the following notations:

- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Q}^c$ are the set of integers, the set of real numbers, the set of natural numbers, and irrational numbers, respectively.
- "$\prod$" : direct product.
- "$\leq$" : $G \leq H$, G a sublocal or subgroup of local group or group H.
- "$\simeq$" : homeomorphism between topological local groups.
- $D = \{(x, y) \in X \times X;\ xy \in X\}$, where X is a local group.

2. Topological Local Groups

In this section, we define and give examples of topological local groups, topological sublocal groups and morphisms between them.

Definition 2.1. Let X be a set, $*$ be an action on X, and $D = \{(x, y) \in X \times X : x * y \in X\}$. A *local group* is a binary $(X, *)$ with the following properties:

(1) There exists $e \in X$ such that $e * x$ and $x * e$ exist for every $x \in X$ and $x * e = e * x = x$.

(2) For every $x \in X$, there exists a unique $x^{-1} \in X$ such that $x * x^{-1}$ and $x^{-1} * x$ exist and $x * x^{-1} = x^{-1} * x = e$; (globally inversion).

(3) If $x * y$ exists, then $y^{-1} * x^{-1}$ exists and $(x * y)^{-1} = y^{-1} * x^{-1}$.

(4) If $x * y$ and $y * z$ exist, then either both $(x * y) * z$ and $x * (y * z)$ exist and $(x * y) * z = x * (y * z)$ or both $(x * y) * z$ and $x * (y * z)$ do not exist.

It is clear that every group is a local group.

Example 2.2. Let $X = \{1, 2, \ldots, n\}$ and define the action $*$ on X as

$$x * x = 1 \text{ and } x * 1 = 1 * x = x.$$

Then X is a local group, since 1 is the identity and $x = x^{-1}$.

Definition 2.3. A *sublocal group* of X is a subset $Y \subseteq X$ such that $e \in Y$, $Y = Y^{-1}$ and if $x, y \in Y$ and $x * y^{-1} \in X$, then $x * y^{-1} \in Y$.

A *subgroup* of a local group X is a subset $H \subseteq X$ such that $e \in H$, $H \times H \subseteq D$ and for all $x, y \in H$, $x * y \in H$.

Definition 2.4. A *topological local group* is a triple $(X, *, \mathfrak{J})$, where $(X, *)$ is a local group and $\mathfrak{J}$ is a topology on X such that the maps $X \to X$, $x \mapsto x^{-1}$ and $\varphi : D \to X$, $(x, y) \to x * y$ are continuous.

Note: By xy we mean $x * y$.

Example 2.5. Let X be a Hausdorff topological space and Δ_X be the diagonal of X, $a \in X$ and $D = (\{a\} \times X) \cup (X \times \{a\}) \cup \Delta_X$. Define $\varphi : D \to X$ by

$$\varphi(x, y) = \begin{cases} x, & y = a, \\ y, & x = a, \\ a, & x = y. \end{cases}$$

Now X by the action of φ is a local group.

If $x \in X$, $x \neq a$, we have $\varphi(x, a) = x$. If U is a neighbourhood of x, then $\varphi^{-1}(U) = U \times \{a\}$. There are two cases

(1) $a \in U$: Since X is Hausdorff, there are disjoint neighbourhood U_1, U_2 containing a, x, respectively. Then $x \in U_2 \cap U$ and $a \notin U_2 \cap U = V$ and $\varphi^{-1}(V) = V \times \{a\}$. Hence, $\varphi(V \times \{a\}) \subset U$. So φ is continuous.

(2) $a \notin U$: $\varphi^{-1}(U) = U \times \{a\}$.

If $x = a$ and W is a closed neighbourhood of a in X, then $\varphi^{-1}(W) = \Delta_X \cup (W \times \{a\}) \cup (\{a\} \times W)$. Hence φ is continuous. Therefore, $\varphi : D \to X$, $(x, y) \mapsto xy$ and $X \to X$, $x \mapsto x^{-1}$ are continuous. So, X is a topological local group.

Definition 2.6. Let X be a topological local group and Y be a sublocal group of X. Then Y, endowed with the topology induced by X, is called a *topological sublocal group*.

Note 2.7. Let X be a local group and a topological space. If X satisfies Pontryagin conditions [9], then X is a topological local group [13].

A neighbourhood base $\mathfrak{U}$ at e in a topological local group X is a collection of topological sublocal group such that

(1) for any $V_1, V_2 \in \mathfrak{U}$, there exists a $V_3 \in \mathfrak{U}$ with $V_3 \subset V_1 \cap V_2$;

(2) for any $x \in X$ and $V \in \mathfrak{U}$, there is a $W \in \mathfrak{U}$ such that xW, Wx^{-1}, $(xW)x^{-1}$ are defined and $xWx^{-1} \subset V$;

(3) for any $V \in \mathfrak{U}$, there is a $W \in \mathfrak{U}$ such that $W^2 \subset V$.

There is a unique topology in X in which $\mathfrak{U}$ is a neighbourhood base at e, and so that the map $(x, y) \mapsto xy^{-1}$ defined on an open domain of $D \subseteq X \times X$, is continuous. Conversely, if X is a topological local group in this sense, then the neighbourhood base at e in X has the above described properties.

Lemma 2.8. *Let X be a topological local group, A be a subset of X, and e be the identity element. Then*

(1) *A is a local group, if and only if A is a topological local group*;

(2) *A is a local group, if and only if A is symmetric* (*i.e.*, $A = A^{-1}$);

(3) *if A is a local group, then $\overline{A}$ (closure A) is a topological local group*;

(4) *if A is a local group and $e \in A^o$, then A^o (interior of A) is a topological local group.*

Proof. (1) It is clear by the definition of a local group and a topological local group.

(2) It is clear.

(3) Let A' be the set of limit point of A, if $x \in A'$, then it is enough to show that $x^{-1} \in A'$. There exists a net $\{x_\alpha\}_{\alpha \in \Gamma}$ in A such that $x_\alpha \to x$ [6, p.187]. Then $x_\alpha^{-1} \in A$ for every $\alpha \in \Gamma$. Since A is a topological local group and the map $X \to X$, $x \mapsto x^{-1}$ is a continuous homomorphism of topological local groups, then x_α^{-1} converges to x^{-1}. By (1) and (2), $\overline{A}$ is a topological local group.

(4) The map $\nu : X \to X$, $x \mapsto x^{-1}$ is onto and continuous. Since the inverse element in X is unique, so ν is injective. Hence ν is a homeomorphism. If $x \in A^o$, then there exists an open neighbourhood U of x in A so that $\nu(x) \in \nu(U) \subset \nu(A)$. Since ν is a homeomorphism, then $x^{-1} \in U \subset A$ and $x^{-1} \in A^o$. Now by (1) and (2), A^o is a topological local group.

□

Remark. The converse of (3) and (4) may not hold.

Example 2.9. (a) $X = (\mathbb{R}, +)$, $A = \mathbb{R} - \{0\}$.

$\overline{A}$ is a topological local group, but A is not a topological local group.

(b) $X = (\mathbb{R}, +)$ and $A = [-1, 1] \cup \{4\}$.

Since $0 \in A^o$ is the identity, then by Lemma 2.8, A^o is a topological local group. But A is not a topological local group, since $\{4\}$ does not have an inverse.

Definition 2.10. A continuous map of topological local groups $f : (X, ., \mathfrak{J}) \to (X', *, \mathfrak{J}')$, is called a *homomorphism of topological local groups*, if

(1) $(f \times f)(D) \subseteq D'$, where $D' = \{(x', y') \in X' \times X', x' * y' \in X'\}$;

(2) $f(e) = e'$ and $f(x^{-1}) = (f(x))^{-1}$;

(3) if $x \cdot y \in X$, then $f(x) * f(y)$ exists in X' and $f(x \cdot y) = f(x) * f(y)$.

With these morphisms topological local groups form a category, which contains the subcategory of topological groups.

Definition 2.11. A homomorphism of topological local groups $f : X \to X'$ is called *strong* if for every $x, y \in X$, the existence of $f(x)f(y)$ implies the existence of $xy \in X$. A morphism is called a monomorphism (epimorphism) if it is injective (surjective).

Corollary 2.12. *Let* $\theta : X \to Y$ *be a continuous strong homomorphism of topological local groups. Then* $\theta(X)$ *is a topological local group.*

Proof. By Lemma 2.8, it suffices to show that $\theta(X)$ is a local group. Let "$\diamond$" be the action on X and "$\bullet$" be the action on Y. The first three properties of topological local groups hold (see Definition 2.1).

Let $x_1 \diamond x_2$, $x_2 \diamond x_3$ and $x_1 \diamond (x_2 \diamond x_3) \in X$. So $\theta(x_1) \bullet \theta(x_2) \in Y$, $\theta(x_2) \bullet \theta(x_3) \in Y$, and $\theta(x_1) \bullet (\theta(x_2) \bullet \theta(x_3)) \in Y$. Since θ is a strong continuous homomorphism, then

$$\begin{aligned}\theta(x_1) \bullet (\theta(x_2) \bullet \theta(x_3)) &= \theta(x_1) \bullet \theta(x_2 \diamond x_3) \\ &= \theta(x_1 \diamond (x_2 \diamond x_3)) \\ &= \theta((x_1 \diamond x_2) \diamond x_3) \\ &= (\theta(x_1) \bullet \theta(x_2)) \bullet \theta(x_3). \qquad \square\end{aligned}$$

Note 2.13. Let F be a closed topological sublocal group, P be an open topological sublocal group, A be any topological sublocal group of a topological local group X, and $\{x\} \times X \subset D$, $\{x^{-1}\} \times X \subset D$. Suppose $A \times X \subset D$. Then xF is closed and xP, AP are open in X.

Let X be a topological local group. Suppose U_1 and U_2 are open neighbourhoods of e in X and $f_1 : U_1 \to X'$ and $f_2 : U_2 \to X'$ are both morphisms. We say that f_1 and f_2 are *equivalent*, if there exists an open neighbourhood U_3 of e in X such that $U_3 \subseteq U_1 \cap U_2$ and $f_1|_{U_3} = f_2|_{U_3}$.

We will call the equivalence classes $[f]$ the *local morphisms* from X to X'.

In [11], enlargement of a local group X and monodrome were introduced. Now, we define them in the topological context.

Definition 2.14. We say that a topological local group X is *enlargeable*, if there exists a topological group G and a morphism $\phi : X \to G$ such that $\phi : X \to \phi(X)$ is a homeomorphism related to the equivalent class $[\theta]$.

If a topological local group is not globally associative, then it may not be extended to a topological group.

Example 2.15. Let $X = \{1, a, b, c, d, ab, bc, cd, de, (ab)c, a(bc), (cd)e, c(de), b(cd)e, h = ((a(bc))d)e, k = a(b(cd)e) | (ab)c = a(bc), (cd)e = c(de)\}$ with discrete topology. Now X is a topological local group, with the action $1 * x = x * 1 = x$, $x * x^{-1} = 1$, and $x = x^{-1}$ for every $x \in X$, but X can not be a topological local subgroup of a topological group, since $h \neq k$.

If a topological group G is an enlargement of X, then X is homeomorphic to $\phi(X)$ (ϕ as in Definition 2.14). Since $h, k \in X$, $h \neq k$, then $\phi(h) \neq \phi(k)$ in G, that is, G is not a group and this is a contradiction.

So, a topological local group X may not be extended to a topological group, but there exists a topological sublocal group of X in which, it is enlargeable to a topological group.

Example 2.16. Let $X' = \{1, a, b, ab\}$ with the action as in the Example 2.15, such that $a^{-1} = a$, $b^{-1} = b$, $((ab)^{-1} = b^{-1}a^{-1})$. Then X' is a topological sublocal group of X, which embeds in S_3 (permutation group of order 3) with discrete topology: $1 \mapsto I$, $a \mapsto (231)$, $b \mapsto (321)$. We see that X' is a non-abelian topological local group and is enlargeable to a topological group. So, there exists a topological sublocal group X' of a topological local group X, which is enlargeable, but X is not enlargeable to a topological group.

Definition 2.17. Let X be a topological local group, G be a topological group, and $X \subset G$. Then G is called an *X-monodrome*, if

(1) X generates G topologically (i.e, X is the smallest closed sublocal group in G, which generate G);

(2) for a topological group H and every continuous homomorphism $\psi : X \to H$, there exists a continuous homomorphism $\nu : G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\phi"] \arrow[dr, "\psi"] & \\ G \arrow[r, "\nu"'] & H \end{tikzcd}$$

Note 2.18. Let G be an enlargement of X. Note that $\phi : X \to \phi(X)$ should be a strong homeomorphism of topological local groups.

We consider $X = \{-1, 0, 1\}$ with the following actions:

$$1 * -1 = -1 * 1 = 0 \quad \text{and} \quad x * 0 = 0 * x = x, \quad \forall x \in X.$$

Then X is a local group. It can not be extended to $\mathbb{Z}_3$. For if, $\phi : X \to Z_3$ is the identity map, we have $1 * 1 = 2$ in $\mathbb{Z}_3$. On the other hand, we know that $1 * 1 \notin X$ and $2 = 1 * 1 = \phi(1) * \phi(1)$, which is a contradictions. But, X can be extend to $\mathbb{Z}_5$ and $\mathbb{Z}$ such that $\mathbb{Z}$ is an X-monodrome.

Example 2.19. Let $\{G_\alpha : \alpha \in \Gamma\}$ be a family of Hausdorff topological group such that e is the identity of each G_α, and

$$G_\alpha \cap G_\beta = \{e\}, \quad \forall \alpha \neq \beta.$$

We take $X = \bigcup_{\alpha \in \Gamma} G_\alpha$ with the topology generated by basis

$$\{U \subseteq X : \forall \alpha \in \Gamma, U \cap G_\alpha \text{ is an open subset of } G_\alpha\}.$$

Suppose $D_\alpha \subseteq G_\alpha \times G_\alpha$, where D_α consists of elements of G_α, which can be multiplied. Now $D = \bigcup_{\alpha \in \Gamma} D_\alpha$. It is clear that $D_\alpha \cap D_\beta = (e, e)$. Then, X is a topological local group.

If $xy \in X$, then there exists $\alpha \in \Gamma$ such that $xy \in G_\alpha$. On the other hand, every G_α is a topological group, so $\phi_\alpha : G_\alpha \times G_\alpha \to G_\alpha$, $(x_\alpha, y_\alpha) \mapsto x_\alpha y_\alpha$ is continuous. If V is an open neighbourhood of xy, then there exist open neighbourhoods U_1 of x and U_2 of y such that $U_1 U_2 \subset V$. We have $\phi|_{D_\alpha} = \phi_\alpha$. Hence, ϕ is continuous.

It is easy to show that X is enlargeable to $\prod_{\alpha \in \Gamma} G_\alpha$, which is an X-monodrome.

Example 2.20. Suppose $X = \{0\} \cup \{(-1)^t(1 - \frac{1}{n}) : n \in \mathbb{N}, t = 1, 2\}$ with, the following action for $n, m \in \mathbb{N}$, $t = 1, 2$:

$$(-1 + \frac{1}{n}) * (-1 + \frac{1}{m}) = (-1 + \frac{1}{n+m});$$

$$(1 - \frac{1}{n}) * (1 - \frac{1}{m}) = (1 - \frac{1}{n+m});$$

$$(-1)^t * (-1)^t = (-1)^t;$$

$$x * 0 = 0 * x = x, \quad \forall x \in X.$$

Then X is a topological local group, which is enlargeable to Torus, T^2, by $\psi : X \to T^2$, $\psi(x) = (e^{i\alpha x}, e^{i\beta x})$, where $\frac{\alpha}{\beta} \in \mathbb{Q}^c$ [3].

Now, let $H = X \times X$ with the action

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2).$$

All points are isolated points except (1,1),(–1,1),(–1, –1), and (1, –1). The maps $H \to H$, $x \mapsto x^{-1}$ and $\varphi : D \to H$, $D \subseteq H \times H$ are continuous on isolated points.

Now, we suppose V is an open neighbourhood such that $(1, 1) \in V$ and $(0, 0) \notin V$. Then $\varphi^{-1}(V) = V \times V$ is open in D. Hence, H is a topological local group and is enlargeable to $T^2 \times T^2$, by $\xi : H \to T^2 \times T^2$, $\xi(x, y) = (e^{i\alpha x}, e^{i\beta x}, e^{i\alpha y}, e^{i\beta y})$ for $\frac{\alpha}{\beta} \in \mathbb{Q}^c$.

Note 2.21. Example 2.20 shows that every point in a topological local group is not necessarily a limit point.

2.1. Some properties of topological local groups

In this part, we study some properties of topological local groups such as productivity, inverse limit, and semi-direct product. Also, we will prove that each T_0 topological local group is completely regular.

Convention. We define the binary action on the product space $X \times Y$ by $(x, y) \cdot (x', y') = (x \diamond x', y \bullet y')$, where $x \diamond x'$ and $y \bullet y'$ are define in X, Y, respectively. Consequently, if $\{X_\alpha\}_{\alpha \in \Gamma}$ is a collection of local groups, $\prod_{\alpha \in \Gamma} X_\alpha$ is a local group, where $(x_\alpha)_{\alpha \in \Gamma}(y_\alpha)_{\alpha \in \Gamma} = (x_\alpha y_\alpha)_{\alpha \in \Gamma}$, whenever $x_\alpha y_\alpha$ is defined in X_α.

Proposition 2.22. *Let* $(X_\alpha, *_\alpha, \mathfrak{J}_\alpha)_{\alpha \in \Gamma}$ *be a family of topological local groups.*

(1) *Let* X_α *be a topological sublocal group of topological local group X. Then* $\bigcap_{\alpha \in \Gamma} X_\alpha$ *is a topological local group.*

(2) $\prod_{\alpha \in \Gamma} X_\alpha$ *is a local group with the product topology.*

(3) $\prod_{\alpha \in \Gamma} X_\alpha$ *is a local group with the box topology.*

Proof. (1) It is clear by Lemma 2.8.

(2) Suppose $(x_\alpha)_{\alpha \in \Gamma}, (y_\alpha)_{\alpha \in \Gamma} \in \prod_{\alpha \in \Gamma} X_\alpha$ and $(x_\alpha^\lambda)_{\lambda \in \Lambda}$ and $(y_\alpha^\lambda)_{\lambda \in \Lambda}$ are nets converging to x_α and y_α on X_α, respectively.

Let $(x_\alpha^\lambda \cdot y_\alpha^\lambda)_{\lambda \in \Lambda}$, $x_\alpha \cdot y_\alpha$, and $x_\alpha \cdot y_\alpha^{-1}$ be defined for every topological local group X_α. We have $(x_\alpha^\lambda, y_\alpha^\lambda)_{\lambda \in \Lambda} \mapsto (x_\alpha^\lambda \cdot y_\alpha^{\lambda^{-1}})_{\lambda \in \Lambda}$ on X_α, since X_α is a topological local group. Hence, $((x_\alpha^\lambda)_{\alpha \in \Gamma}, (y_\alpha^\lambda)_{\alpha \in \Gamma})_{\lambda \in \Lambda}$ converges to $(x_\alpha, y_\alpha)_{\alpha \in \Gamma}$ on $\prod_{\alpha \in \Gamma} D_\alpha$. Now $(x_\alpha, y_\alpha) \mapsto (x_\alpha \cdot y_\alpha^{-1})$ for every $\alpha \in \Gamma$ in X_α.

Therefore, $((x_\alpha^\lambda, y_\alpha^\lambda)_{\alpha \in \Gamma})_{\lambda \in \Lambda} \mapsto ((x_\alpha^\lambda \cdot y_\alpha^{\lambda^{-1}})_{\alpha \in \Gamma})_{\lambda \in \Lambda}$ and the right hand side converges to $(x_\alpha \cdot y_\alpha^{-1})_{\alpha \in \Gamma}$. So $\prod_{\alpha \in \Gamma} D_\alpha \to \prod_{\alpha \in \Gamma} X_\alpha$ is a continuous map.

(3) It is clear. □

Note 2.23. Let X, Y be local groups. If $X \times Y$ has a topological structure under which $X \times Y$ is a topological local group, then X and Y admit a topological structures under which, they are topological local groups and their product topology is the same as the topology on $X \times Y$.

Because we know that $X \times \{e'\}$ is a sublocal group of $X \times Y$, where e' is the identity on Y. By Lemma 2.8, $X \times \{e'\}$ is a topological sublocal group of $X \times Y$. It is clear that $X \times \{e'\}$ is topologically isomorphic with X. So X and similarly Y are topological local groups.

Lemma 2.24. *Let* $\{X_\alpha\}_{\alpha \in \Gamma}$ *be a family of topological local groups and topologically enlargeable for every* $\alpha \in \Gamma$. *Then* $\prod_{\alpha \in \Gamma} X_\alpha$ *is topologically enlargeable with the product topology.*

Proof. Let $\phi_\alpha : X_\alpha \to G_\alpha$ be an enlargeable map (see Definition 2.14). Now $< \phi_\alpha(X_\alpha) >= G_\alpha$ for every $\alpha \in \Gamma$. Let $\phi : \prod X_\alpha \to \prod G_\alpha$, $(x_\alpha)_{\alpha \in \Gamma} \mapsto (\phi_\alpha(x_\alpha))_{\alpha \in \Gamma}$. Then, $\prod X_\alpha \simeq \phi(\prod X_\alpha)$ is a sublocal group of $\prod G_\alpha$, since ϕ is a strong homomorphism.

Let $T :=< \phi(\prod X_\alpha) >$. We know that $T \leq \prod G_\alpha$, so T is a topological group. Now $< \prod \phi_\alpha(X_\alpha) >$ is a topological group. □

Inverse limit. Let $\{X_\alpha, \phi_{\alpha\beta}\}_{\alpha, \beta \in \Gamma, \alpha \leq \beta}$ be the inverse system of topological local groups and continuous strong homomorphisms. There exists an inverse limit $\{X, \phi_\alpha\}$.

Let $x_\alpha, y_\alpha \in X_\alpha$ and $x_\alpha y_\alpha$ be defined in X_α. Then $x_\alpha y_\alpha = \phi_{\alpha\beta}(x_\beta)\phi_{\alpha\beta}(y_\beta) = \phi_{\alpha\beta}(x_\beta y_\beta)$, since $\phi_{\alpha\beta}$ is a strong homomorphism and the following diagram commutes:

$$\begin{tikzcd} X_\beta \arrow[rr, "\phi_{\alpha\beta}"] & & X_\alpha \\ & X \arrow[ul, "\phi_\beta"] \arrow[ur, "\phi_\alpha"'] & \end{tikzcd}$$

Theorem 2.25 (Inverse limit). *Let* $\{X_\alpha, \phi_{\alpha\beta}\}_{\alpha,\beta\in\Gamma,\alpha\leq\beta}$ *be an inverse system of family of topological local groups and continuous strong homomorphism of topological local groups. Suppose* X_α *is enlargeable for each* $\alpha \in \Gamma$. *Then, the inverse limit* $\{X, \phi_\alpha\}$ *is a topological local group and is enlargeable to the topological group* $< \prod_{\alpha\in\Gamma} \phi_\alpha(X_\alpha) >$.

Proof. By Definition 2.1, it is clear that X is a local group.

By Proposition 2.22, $\prod X_\alpha$ is a topological local group and X is a sublocal group of $\prod X_\alpha$. By Lemma 2.8, X is a topological local group.

Now, we show that X is enlargeable to a topological group. Each X_α is enlargeable to a topological group G_α for every $\alpha \in \Gamma$. By Lemma 2.24, $\prod X_\alpha$ is enlargeable to the topological group $< \phi(\prod X_\alpha) >$ and X is a subspace of $\prod X_\alpha$ with the product topology. Then, $< \prod_{\alpha\in\Gamma} \phi_\alpha(X_\alpha) >$ is the enlargement of X. □

Semi-direct product

Definition 2.26. Let X be a topological space and G be a topological local group. Suppose a neighbourhood G_1 of identity element of G such that all products are defined in G_1. We define the G-action as a map $\psi : G \times X \to X$ such that

(1) gx is continuous on $g \in G$ and $x \in X$;

(2) $\psi(e, x) = x$ for each $x \in X$;

(3) $\psi(g_2, \psi(g_1, x)) = \psi(g_2 g_1, x)$, for every $g_1, g_2 \in G_1$ and $x \in X$.

The triple (X, G, ψ) is called a *topological transformation local group* (G_1 exists by [5]).

Suppose X and G are topological local groups in Definition 2.26 and G acts on X, then we define semi-direct product by the following:

The set of all automorphism of X is denote by $Aut(X)$. It is a group under composition and carries the compact-open topology.

Definition 2.27. Let X and Y be topological local groups and suppose neighbourhoods X_1 and Y_1 of identity elements e_X and e_Y, respectively, on X and Y such that all products are defined on them. $\theta : X \to Aut(Y)$, $\theta(x)(y) = \theta_x(y) = xyx^{-1}$ a continuous strong homomorphism, which means X acts on Y. We define $\mu : (X \times Y) \times (X \times Y) \to X \times Y$ by

$$\mu((x, y), (x', y')) = (xx', y'\theta(x')y),$$

for every $x, x' \in X_1$ and $y, y' \in Y_1$. The space $(X \times Y, \mu)$ is called the *semi-direct product* of topological local groups X and Y with respect to θ, denoted by $X \underset{\theta}{\ltimes} Y$.

Theorem 2.28 (Semi-direct product)**.** *Let X and Y be local groups, and let $\theta : X \to Aut(Y)$ be a strong homomorphism. Then, the following statements hold*:

(1) *The semi-direct product $X \underset{\theta}{\ltimes} Y$ is a local group. The set $X \times \{e\}$ is a sublocal group, and $\{e\} \times Y$ is a normal sublocal group of $X \underset{\theta}{\ltimes} Y$.*

(2) *If X and Y are topological local groups and the action $\omega_\theta : Y \times X \to Y$, $\omega_\theta(y, x) = \theta(x)(y)$ is continuous, then $X \underset{\theta}{\ltimes} Y$ is a topological local group.*

Proof. (1) If for all product defined $x_1, x_2, x_3 \in X_1$ and $y_1, y_2, y_3 \in Y_1$. Let e_X, e_Y be the identity elements of X and Y, respectively. Then (e_X, e_Y) is the identity element of $X \underset{\theta}{\ltimes} Y$ and the inverse of (x, y) is

$(x, y)^{-1} = (x^{-1}, \theta(x^{-1})(y^{-1}))$. Now $\mu : (X \times Y)^2 \to X \times Y$, $\mu((x, y), (x', y')) = (xx', y'\theta(x')y)$ has the fourth condition of Definition 2.1. On one hand;

$$\begin{aligned}\mu(\mu((x_1, y_1), (x_2, y_2)), (x_3, y_3)) &= \mu((x_1x_2, y_2\theta(x_2)y_1), (x_3, y_3)) \\ &= (x_1x_2x_3, y_3\theta(x_3)y_2\theta(x_2)y_1) \\ &= (x_1x_2x_3, y_3(\theta(x_3)y_2)\theta(x_3)(\theta(x_2)y_1)).\end{aligned}$$

On the other hand;

$$\begin{aligned}\mu((x_1, y_1), \mu((x_2, y_2), (x_3, y_3))) &= \mu((x_1, y_1), (x_2x_3, y_3\theta(x_3)y_2)) \\ &= (x_1x_2x_3, y_3\theta(x_3)y_2\theta(x_2x_3)y_1).\end{aligned}$$

Hence

$$(x_1x_2x_3, y_3(\theta(x_3)y_2)\theta(x_3)(\theta(x_2)y_1)) = (x_1x_2x_3, y_3\theta(x_3)y_2\theta(x_2x_3)y_1),$$

the rest of assertion (1) is clear.

(2) Let

$$\mu : (X \underset{\theta}{\ltimes} Y) \times (X \underset{\theta}{\ltimes} Y) \to X \underset{\theta}{\ltimes} Y;$$

$$\mu((x_1, y_1), (x_2, y_2)) = (x_1x_2, y_2\theta(x_2)(y_1)) \text{ for } x_1, x_2 \in X, y_1, y_2 \in Y.$$

Let $x_1x_2 \in X$, $y_2\theta(x_2)(y_1) \in Y$ be defined. Then μ is continuous, since X and Y are topological local groups and θ is a continuous map. Similarly, $\beta : (x, y) \mapsto (x, y)^{-1}$, with semi-direct product property, is continuous. Thus $X \underset{\theta}{\ltimes} Y$ becomes a topological local group. □

Theorem 2.29. *Every T_0 topological local group X is completely regular.*

Proof. Let X be a T_0 topological local group and $D = \{(x, y) \in X \times X, xy \in X\}$. By [5, Lemma 2.5], there exist symmetric neighbourhoods $U_1, U_2, \ldots$ of identity $e \in X$, where $\{U_k\}_{k=1}^{\infty}$ is a symmetric open basis at e, which has the following properties:

(1) $U_i = U_i^{-1}$;

(2) $\varphi((U_{i+1}, U_{i+1}) \cap D) \subset U_i, \quad \varphi : (U_{i+1}, U_{i+1}) \cap D \to X$;

(3) $U_i \subset U_0$.

The rest of the proof is similar to the case of topological groups [6, page. 213]. □

Note 2.30. If a Hausdorff topological local group X is enlargeable to a Hausdorff topological group G, then X is completely regular. We have the embedding homomorphism $\phi : X \to G$, $X \simeq \phi(X)$. Since G is a Hausdorff topological group, it is completely regular and every subspace of a completely regular is completely regular [6].

3. Basic Theorems in Local Groups and Topological Local Groups

This section is divided into two parts. The first one is about quotient of local group (algebraically) and we prove some theorems for local groups, such as the first isomorphism theorem. In the second part, we will show the same results for topological local groups.

Definition 3.1. Let $(X, *)$ be a local group and $H = \{x \in X | xy, yx \in X, \forall y \in X\}$.

$$N \leq H, \text{ and } xN = Nx \text{ for every } x \in X. \qquad (\star)$$

Let $\frac{X}{N} = \{xN | x \in X\} = \{[x] | x \in X\}$.

The costs $\{xN : x \in X\}$ form a local group called the *quotient local group*.

Suppose $[x], [y] \in \frac{X}{N}$ and $[x] \cap [y] \neq \emptyset$, then $[x] = [y]$. We define $[x]^{-1} = [x^{-1}] = x^{-1}N$. The action on the quotient local group is given by $[x] * [y] = x * yN$, where $x * y$ is defined in X. This action is well define. Because, if $xN = x'N$ and $yN = y'N$ for $x, x', y, y' \in X$, we will have

$$(xy)^{-1}x'y' = y^{-1}x^{-1}x'y' \in y^{-1}Ny' = y^{-1}y'N \in N.$$

So $xyN = x'y'N$. It is easy to show that the quotient local group is a local group.

Example 3.2. Let $X = \mathbb{Z}$ and $2k_1 * k_2 = 2k_1 + k_2$ and $k_1 * (-k_1) = 0$, for every $k_1, k_2 \in \mathbb{Z}$. Hence X is an abelian local group. $2\mathbb{Z}$ is a group in X and $x(2\mathbb{Z}) = (2\mathbb{Z})x$ for every $x \in \mathbb{Z}$. So, the quotient local group $\frac{\mathbb{Z}}{2\mathbb{Z}}$ exists.

We omit the proofs of Theorems 3.3-3.7, since they are similar to groups. Note that we consider productable elements in $X, (X')$, and strong homomorphisms of local groups.

Theorem 3.3. *Let X and X' be local groups, $f : X \to X'$ be a strong homomorphism with kernel K, and e, e' be the identity of X, X', respectively. Then*

(1) *K is a normal subgroup of X;*

(2) *f is one to one if and only if $K = \{e\}$.*

So, the quotient local group $\frac{X}{\ker f}$ is defined. Hence, by Definition 3.1, $N \neq \emptyset$.

Theorem 3.4. *Let X and X' be local groups, $f : X \to X'$ be a strong homomorphism with kernel K, and e, e' be the identity of X, X', respectively. Then*

(1) *if H is a subgroup of X, then $f(H)$ is a subgroup of X';*

(2) *if H is a sublocal group of X, then $f(H)$ is a sublocal group of X';*

(3) *if f is a surjective and H is a normal sublocal group of X, then $f(H)$ is a normal sublocal group;*

(4) *if* H' *is a normal sublocal group of* X', *then* $f^{-1}(H')$ *is a normal sublocal group of* X *such that* $K \subset f^{-1}(H')$.

Theorem 3.5. *Let* X, X' *be local groups and* $f : X \to X'$ *be a strong homomorphism, then*

(1) $f(X)$ *is a sublocal group of* X' *and* $\frac{X}{\ker f} \simeq f(X)$.

(2) *If* f *is surjective, then* $\frac{X}{\ker f} \simeq X'$.

Theorem 3.6. *Let* X *be a local group and* H, N *be normal subgroups of* X *such that* $H \subset N$. *Then* $\frac{N}{H}$ *is a normal subgroup of* $\frac{X}{H}$ *and* $\frac{X}{N} \simeq \frac{X/H}{N/H}$.

Theorem 3.7. *Let* X *be a local group and* H, N *be subgroups of* X *such that* N *normal in* X, *then* $\frac{NH}{N} \simeq \frac{H}{H \cap N}$.

In the rest of this section, we discuss the quotient and we will obtain some results for topological local groups.

Definition 3.8. Let X be a topological local group and $N \leq H$, with $(\star)$ property (see Definition 3.1). Suppose N is a topological normal subgroup of X. Let π be the canonical mapping of X onto the local group $\frac{X}{N}$. The *quotient topological local group* on $\frac{X}{N}$ is defined as follows: A symmetric subset A in $\frac{X}{N}$ is open, if and only if $\pi^{-1}(A)$ is a symmetric open subset of X.

Remark. Definition 3.8 is a generalization of the definition given by [9].

Definition 3.9. Let X be a topological local group and N be a normal sublocal group. Then $\frac{X}{N}$ with the following properties:

(1) $\frac{X}{N} = \{xN \cap U,\ x \in U\}$ such that U is open in X;

(2) if there exists product xy^{-1} in U for $x,\ y \in U$; then $xy^{-1} \in N$ is called *locally factor group*.

Remark. If $U = X$ and N is a normal subgroup of X, then Definitions 3.8 and 3.9 are the same.

Example 3.10. Let $X = \{A \in GL_n(\mathbb{R})\}$. We defined productable elements of X as follows:

- (i) if $\det A = 1$, then $A \cdot M \in X,\ \forall M \in X$;
- (ii) if $\det B = -1$ and $\det C = -1$, then $B \cdot C \in X$;
- (iii) for every $M \in X,\ M \cdot M^{-1} \in X$.

For $A,\ B,\ C \in X$, we know that

(1) $I \in X$ and $\det I = 1$, the identity matrix;

(2) $I \cdot M = M \cdot I,\ \forall M \in X$;

(3) $M \cdot M^{-1} = M^{-1} \cdot M = I$;

(4) let $A \cdot B \in X$ and $B \cdot C \in X$. Then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

Now X is a local group of topological group $GL_n(\mathbb{R})$. Then, by Note 2.13 and Lemma 2.8, X is a topological local group.

We have $SL_n(\mathbb{R}) = \{A \in X | \det A = 1\}$, which is a compact subgroup and the identity component of X. Then $\frac{X}{SL_n(\mathbb{R})}$ is the quotient topological local group.

Theorem 3.11. *Let X be a topological local group and N be a topological normal subgroup of X. Let $\frac{X}{N}$ be the quotient space, endowed with the quotient topological local group, and π be the strong homomorphism mapping of X into $\frac{X}{N}$. Then*

(1) π *is onto*;

(2) π *is continuous*;

(3) π *is open*.

Proof. (1) and (2) are obvious by the definition of the quotient topological local group.

(3) Let U be a symmetric open set in X. We show that $\pi(U)$ is a symmetric open set in $\frac{X}{N}$, i.e., $\pi^{-1}(\pi(U))$ is a symmetric open set in X. But

$$\pi^{-1}(\pi(U)) = \{x : x \in uN, \text{ for some } u \in U\} = UN,$$

which is open by Note 2.13. □

Theorem 3.12. *Let X be a topological local group and N be a topological normal subgroup of X, with $(\star)$ property* (*see Definition* 3.1). *Then $\frac{X}{N}$ is a topological local group.*

Proof. If $\phi : X \to X$, $x \to x^{-1}$ is a continuous strong homomorphism of topological local group, then ϕ induce, $\tilde{\phi} : \frac{X}{N} \to \frac{X}{N}$, $[x] \to [x^{-1}]$, which is a continuous strong homomorphism of topological local groups [1, I.3.10].

Since X is a topological local group, we have $\psi : D \to X$ is a continuous. Then ψ induce, $\tilde{\psi} : \frac{D}{N \times N} \to \frac{X}{N}$, which is continuous and $\frac{D}{N \times N} \subseteq \frac{X}{N} \times \frac{X}{N}$.

So, the quotient topological local group is a topological local group. □

We recall the following definitions from [7].

(1) Suppose X and $\widetilde{X}$ are local Lie groups. We will say that $\widetilde{X}$ is a covering group of X, if there is a generalized covering map $f : \widetilde{X} \to X$, which is a homomorphism of local Lie groups.

(2) A topological local group X is *strongly connected*, if

(i) the domains of definition of multiplication and inversion maps are connected;

(ii) if $V \subset X$ is any symmetric neighbourhood of identity, then V generates X.

V generates X means that $X = \bigcup_{n=1}^{\infty} V^n$ such that $V^n \subset X$ denotes the subset consisting of all topological local group elements $x \in X$, which can be written as an n-tuple product of elements $x_1, \ldots, x_n \in V$.

Theorem 3.13. *Let X be a strongly connected local Lie group. Then there exists a local covering group $\widetilde{X} \to X$, which is also a local covering group $\widetilde{X} \to M$ of an open subset $M \subset G$ of a global Lie group G* [7, Theorem 21].

Remark. By Theorem 3.13, the covering space of local Lie groups is enlargeable to a Lie group.

Corollary 3.14. *Let X be a strongly connected local Lie group. Then there exists an exact sequence*

$$1 \longrightarrow \ker f \xrightarrow{\ i\ } \widetilde{X} \xrightarrow{\ f\ } X \longrightarrow 1,$$

where $\widetilde{X}$ is a generalized covering space and $f : \widetilde{X} \to X$ is the generalized covering map, which is a strong homomorphism.

Proof. By Theorem 3.13, for every strongly connected local Lie group, there exists a covering space, which is a local Lie groups. Since f is a strong homomorphism, then ker f is a normal Lie subgroup of local Lie group $\widetilde{X}$. □

Therefore, $\frac{\widetilde{X}}{\ker f}$ is a quotient local Lie group.

Theorem 3.15. *Let f be a strong homomorphism of a topological local group X into another topological local group* X'. *Let* $K = f^{-1}(e')$, *where* e' *is the identity of* X' *and* $\pi : X \to \frac{X}{K}$ *is the canonical map and* $f = f_0 \circ \pi$ *for some* $f_0 : \frac{X}{K} \to X'$. *Then*

(1) *f is continuous if and only if* f_0 *is continuous*;

(2) *f is open if and only if* f_0 *is open*;

(3) f_0 *is a one to one strong homomorphism of* $\frac{X}{K}$ *into* X'.

Proof. Since K is an invariant subgroup of X, $\frac{X}{K}$ is a quotient topological local group and by Theorem 3.11, π is continuous and an open strong homomorphism of X onto the topological local group $\frac{X}{K}$.

(1) If f_0 is continuous and $f = f_0 \circ \pi$, then f, being the composition of two continuous strong homomorphism, is continuous.

Conversely, if f is continuous, then for each symmetric neighbourhood V of $e' \in X'$, $f^{-1}(V)$ is symmetric [12]. Now $[e] \in \phi \circ f^{-1}(V)$ is open in $\frac{X}{K}$, since π is open. So $\pi \circ f^{-1}(V) = f_0^{-1}(V)$ and $\pi \circ f^{-1}(V) = \{f^{-1}(V)K, V$ is open in $X'\}$ is an open set $\frac{X}{K}$. Therefore, f_0 is continuous.

(2) For each open symmetric neighbourhood V of $e \in X$, $f(V) = f_0 \circ \pi(V)$. Since π is open, then (2) holds.

(3) The result is clear, since π and f are strong homomorphism. □

Theorem 3.16. *Let X and X' be topological local groups, and $f : X \to X'$ a continuous open strong homomorphism. Suppose $K = f^{-1}(e')$, where e' is the identity of X'. Then X' is topologically isomorphic to $\frac{X}{K}$.*

Proof. Since $K = \ker f$ is a normal subgroup of X, then by Theorem 3.12, $\frac{X}{K}$ is a quotient topological local group. By Theorem 3.5, $\frac{X}{K}$ and X' are algebraically isomorphic. By Theorem 3.15, $f_0 : \frac{X}{K} \to X'$ is continuous, open, 1-1 and onto. So f_0 is a topological isomorphism. □

Proposition 3.17. *Let X be a topological local group, H be a normal subgroup of X, M be a subgroup of X, and $\phi : X \to \frac{X}{H}$ be the quotient map. Then $\phi(M)$ is a topological subgroup of $\frac{X}{H}$ and is homeomorphic with $\frac{MH}{H}$.*

Proof. Since H is a normal subgroup of X, by Definition 3.1, $\{x\} \times H$, $H \times \{x\} \subset D$ for every $x \in X$. So $H \times M \subset D$ and MH is definable and clearly MH is a subgroup of X. Since H is a normal subgroup of X, then H is a normal subgroup of MH and by Theorem 3.7, $\iota : \frac{MH}{H} \to \phi(M)$, $\{[x] : x \in MH\} \mapsto \{[x] : x \in M\}$ is a topological local group isomorphism.

Now, we show that ι is a homeomorphism.

Let $S \subset M$ and the set $\{xH, x \in S\}$ be an open subset in $\phi(M)$ (see Note 2.13). So, there is an open symmetric neighbourhood V of X such that

$$\{xH,\ x \in S\} = \phi(M) \cap \{vH,\ v \in V\}.$$

We have $\phi(M) \cap \{vH,\ v \in V\} = \{vH,\ v \in V \cap MH\}$. Now

$$SH = (V \cap MH)H = VH \cap MH,$$

where VH is open in X by Note 2.13. Then, SH is open in $\frac{MH}{H}$. Hence, ι is continuous.

Let $T \subset MH$ and TH is open in MH. Then, there exists an open symmetric neighbourhood U of X such that $TH = U \cap MH$ and

$$\{xH\ :\ x \in U \cap MH,\ U \text{ is open in } X\},$$

is open in $\frac{MH}{H}$.

$$\begin{aligned} TH &= \{[x] \in \frac{MH}{H} : x \in U \cap MH\} \\ &= \{[x] : x \in U\} \cap \{[x] : x \in MH\} \\ &= \phi(UH) \cap \phi(MH). \end{aligned}$$

Since H is a normal subgroup of X and by Note 2.13, UH is an open set in X, then TH is open in $\phi(M)$. Hence, ι is an open map. □

Example 3.18. Let $\mathbb{R}$ be the real numbers, with the usual topology. We define an action on $\mathbb{R}$. Let $\alpha \in \mathbb{Q}^c$ be fixed.

$$x + 0 = 0 + x = x, \quad x + x^{-1} = x^{-1} + x = 0, \quad \forall x \in \mathbb{R};$$

$$x + x' \in \mathbb{Z}, \quad \forall x,\ x' \in \mathbb{Z};$$

$$x + x' \in \alpha\mathbb{Z}, \quad \forall x,\ x' \in \alpha\mathbb{Z};$$

$$x + y = y + x \in \mathbb{R}, \quad \forall x \in \mathbb{R},\ \forall y \in \mathbb{Z};$$

$$x + z = z + x \in \mathbb{R}, \quad \forall x \in \mathbb{R},\ \forall z \in \alpha\mathbb{Z}.$$

Therefore, $\mathbb{R}$ is a local group. Now $\alpha\mathbb{Z}$ and $\mathbb{Z}$ with the subspace topology are normal topological subgroups of $\mathbb{R}$. On the other hand, $\alpha\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$ and $\mathbb{Z}$ is a normal topological subgroup of $\mathbb{R}$. Then $\frac{\alpha\mathbb{Z} + \mathbb{Z}}{\mathbb{Z}}$ is dense in $\frac{\mathbb{R}}{\mathbb{Z}}$.

By Theorem 3.7, $\frac{\alpha\mathbb{Z} + \mathbb{Z}}{\mathbb{Z}}$ and $\frac{\alpha\mathbb{Z}}{\alpha\mathbb{Z} \cap \mathbb{Z}}$ are algebraically isomorphic. But it is not a homeomorphism.

Proposition 3.19. *Let X be a locally compact and σ-compact topological local group. Suppose $f : X \to X'$ is a continuous onto strong homomorphism, where X' is a strongly connected locally compact T_0 topological local group. Then f is open.*

Proof. The proof is similar to topological groups [1]. □

Theorem 3.20. *Let X be a topological local group, M be a subgroup of X and strongly connected, locally compact and σ-compact, H be a normal subgroup of X. Let MH be locally compact and ψ the isomorphism $\psi : \frac{MH}{H} \to \frac{M}{M \cap H}, \psi(mH) = m(A \cap H)(m \in M)$. Then ψ is a homeomorphism.*

Proof. Let X be a local group and M, H be subgroups of X. Since H is normal in X, so by Definition 3.1, MH is definable and $H \cap M$ is a normal subgroup of M. So, the groups $\frac{MH}{H}$ and $\frac{M}{M \cap H}$ are algebraically isomorphic.

We know that $\{xH : x \in U \cap MH, U$ is an open symmetric in $X\}$ is a symmetric open set of $\frac{MH}{H}$. So $\psi(\{xH : x \in U \cap M\}) = \{x(M \cap H) : x \in U \cap M\}$, where $\{x(M \cap H) : x \in U \cap M\}$ is a symmetric open set of $\frac{M}{M \cap H}$. Then ψ carries an open symmetric set of $\frac{MH}{H}$ to symmetric open set of $\frac{M}{M \cap H}$.

Now, we show that ψ^{-1} carries a symmetric open set of $\frac{M}{M \cap H}$ to open symmetric set of $\frac{MH}{H}$.

Note that MH is strongly connected, locally compact and H is a normal subgroup of MH. Then $\frac{MH}{H}$ is strongly connected, locally compact and T_0 [6]. Now $\Psi : X \to \frac{X}{H}$ is a natural continuous strong homomorphism, where its restriction to M will be $\widetilde{\Psi} : M \to \frac{MH}{H}$, which is a homomorphism of topological groups. By Propositions 3.17 and 3.19, $\widetilde{\Psi}$ is open.

But $y \in M$ and H is a normal subgroup. So yM and yH are definable. Let $\{y(M \cap H) : y \in Y, Y \subset M\}$ be a symmetric open set of $\frac{M}{M \cap H}$. Thus, there exists a symmetric open set U in X such that $\{y(M \cap H) : y \in Y \subset M\} = \{y(M \cap H) : y \in Y \cap U\}$. Since Ψ is an open map on M, then

$$\widetilde{\Psi}(\{y(M \cap H) : y \in Y \cap U\}) = \{yH, y \in U \cap Y\},$$

which $\{yH, y \in U \cap Y\}$ is an open set of $\frac{MH}{H}$. By definition of ψ

$$\{yH : y \in Y \cap U\} = \psi^{-1}(\{x(M \cap H) : x \in U \cap Y\}).$$

This completes the proof. □

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