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STRONG DIFFERENTIAL SUPERORDINATION DEFINED BY FRACTIONAL DERIVATIVE OPERATOR

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Abstract

In the present paper, we derive some results for strong differential superordination of p-valent functions involving certain fractional derivative operator.

1. Introduction and Preliminaries

Let $\mathcal{U}$ denote the unit disk of the complex plane

$$\mathcal{U} = \{z : |z| < 1\},$$

and

$$\overline{\mathcal{U}} = \{z : |z| \leq 1\},$$

the closed unit disk of the complex plane.

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Let $\mathcal{H}(\mathcal{U} \times \overline{\mathcal{U}})$ denote the class of analytic functions in $\mathcal{U} \times \overline{\mathcal{U}}$ and the classes [7, 8]

$$\mathcal{H}^*[a, n, \zeta] = \{f \in \mathcal{H}(\mathcal{U} \times \overline{\mathcal{U}}) : f(z, \zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \ldots, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}\},$$

with $a_k(\zeta)$ are holomorphic functions in $\overline{\mathcal{U}}$, $k \geq n$, $n \in \mathbf{N}$, and $a \in \mathbb{C}$,

$$\mathcal{H}_u(\mathcal{U}) = \{f \in \mathcal{H}^*[a, n, \zeta] : f(z, \zeta) \text{ univalent in } \mathcal{U}, \text{ for all } \zeta \in \overline{\mathcal{U}}\},$$

$$K^* = \left\{f \in \mathcal{H}^*[a, n, \zeta] : \operatorname{Re}\left\{1 + \frac{zf''(z, \zeta)}{f'(z, \zeta)}\right\} > 0, z \in \mathcal{U}, \text{ for all } \zeta \in \overline{\mathcal{U}}\right\},$$

the class of convex functions, and

$$S^* = \left\{f \in \mathcal{H}^*[a, n, \zeta] : \operatorname{Re}\left\{\frac{zf'(z, \zeta)}{f(z, \zeta)}\right\} > 0, z \in \mathcal{U}, \text{ for all } \zeta \in \overline{\mathcal{U}}\right\},$$

the class of starlike functions.

Let $f(z, \zeta)$, $H(z, \zeta)$ analytic in $\mathcal{U} \times \overline{\mathcal{U}}$. The function $f(z, \zeta)$ is said to be strongly subordinate to $H(z, \zeta)$, or $H(z, \zeta)$ is said to be strongly superordinate to $f(z, \zeta)$, if there exists a function w analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z, \zeta) = H(w(z), \zeta)$ for all $\zeta \in \overline{\mathcal{U}}$. In such a case, we write $f(z, \zeta) \prec\prec H(z, \zeta)$, $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$. If $f(z, \zeta)$ is analytic in $\mathcal{U} \times \overline{\mathcal{U}}$ and univalent in $\mathcal{U}$, for all $\zeta \in \overline{\mathcal{U}}$, then $f(0, \zeta) = H(0, \zeta)$, for all $\zeta \in \overline{\mathcal{U}}$ and $f(\mathcal{U} \times \overline{\mathcal{U}}) \subset H(\mathcal{U} \times \overline{\mathcal{U}})$. If $H(z, \zeta) \equiv H(z)$ and $f(z, \zeta) \equiv f(z)$, then strong superordination becomes the usual notion of superordination [7, 8].

Let $\mathcal{A}^*_\zeta(p)$ be the class of functions $f \in \mathcal{H}(\mathcal{U} \times \overline{\mathcal{U}})$ of the form

$$f(z, \zeta) = z^p + \sum_{n=1}^{\infty} a_{p+n}(\zeta)z^{p+n}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, p \in \mathbf{N}, \tag{1.1}$$

and set $\mathcal{A}^*_\zeta(1) \equiv \mathcal{A}^*_\zeta$.

The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined for $z \in \mathcal{U}$ by [11],

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{1.2} $$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$ (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{when } n = 0, \\ \lambda(\lambda + 1)(\lambda + 2)\ldots(\lambda + n - 1), & \text{when } n \in \mathbf{N}. \end{cases} \tag{1.3} $$

for $\lambda \neq 0, -1, -2, \ldots$.

Next, we recall the following definitions of fractional derivative operators, which were used by Owa [9], (see also [10]) as follows:

Definition 1.1. The fractional derivative operator of order λ is defined by

$$ D_z^{\lambda} f(z) = \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z - \xi)^{\lambda}} d\xi, \tag{1.4} $$

where $0 \leq \lambda < 1$, $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 1.2. Let $0 \leq \lambda < 1$, and $\mu, \eta \in \mathbf{R}$. Then, in terms of the familiar Gauss's hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$ J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz} \left(\frac{z^{\lambda-\mu}}{\Gamma(1 - \lambda)} \int_0^z (z - \xi)^{-\lambda} f(\xi) {}_2F_1\left(\mu - \lambda, 1 - \eta; 1 - \lambda; 1 - \frac{\xi}{z} \right) d\xi \right), \tag{1.5} $$

where $f(z)$ is analytic function in a simply-connected region of the z-plane containing the origin, with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, where $\varepsilon > \max\{0, \mu - \eta\} - 1$ and the multiplicity of $(z - \xi)^{-\lambda}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

Definition 1.3. Under the hypotheses of Definition 1.2, the fractional derivative operator $J_{0,z}^{\lambda+m,\mu+m,\eta+m}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda+m,\mu+m,\eta+m} f(z) = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\mu,\eta} f(z). \tag{1.6}$$

Notice that

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = D_z^{\lambda} f(z), \quad 0 \le \lambda < 1. \tag{1.7}$$

Now, for a function f given by (1.1), we define the modification of the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ by

$$M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta) = \frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\mu)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)} z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z, \zeta),$$

$$z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}. \tag{1.8}$$

It is observed that, the operator $M_{0,z}^{\lambda,\mu,\eta} : \mathcal{A}_{\zeta}^{*}(p) \to \mathcal{A}_{\zeta}^{*}(p)$ is defined by the following infinite series:

$$M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta) = z^p + \sum_{n=1}^{\infty} \delta_n(\lambda, \mu, \eta, p) a_{p+n}(\zeta) z^{p+n}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{1.9}$$

$$(\lambda \ge 0; \mu < p+1; \eta > \max(\lambda, \mu) - p - 1; p \in \mathbf{N}),$$

where

$$\delta_n(\lambda, \mu, \eta, p) = \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n}. \tag{1.10}$$

It is easily verified from (1.9) that

$$z\left(M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta)\right)_z' = (p-\mu)M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z,\zeta) + \mu M_{0,z}^{\lambda,\mu,\eta}f(z,\zeta). \quad (1.11)$$

Notice that

$$M_{0,z}^{0,0,\eta}f(z,\zeta) = f(z,\zeta),$$

and

$$M_{0,z}^{1,1,\eta}f(z,\zeta) = \frac{zf_z'(z,\zeta)}{p}.$$

In order to prove our main strong superordination results, we shall make use of the following known results:

Lemma 1.4 ([5]). *Let* $h(z,\zeta)$ *be a convex with* $h(0,\zeta) = a$, *for all* $\zeta \in \overline{\mathcal{U}}$ *and let* $\gamma \in \mathbb{C}^*$ *be a complex number with* $\operatorname{Re}\gamma \geq 0$ *and* $p(z,\zeta) \in \mathcal{H}^*[a,1,\zeta] \cap \mathcal{Q}$. *If* $p(z,\zeta) + \frac{1}{\gamma}zp'(z,\zeta)$ *is univalent in* $\mathcal{U}$, *for all* $\zeta \in \overline{\mathcal{U}}$, *and*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma}zp'(z,\zeta),$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

where

$$q(z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z h(t,\zeta)t^{\gamma-1}dt.$$

The function q is convex and it is the best subordinant.

Lemma 1.5 ([5]). *Let* $q(z,\zeta)$ *be convex function in* $\mathcal{U}$ *for all* $\zeta \in \overline{\mathcal{U}}$, *and let* $h(z,\zeta)$ *be defined by*

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma}zq'(z,\zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where $\operatorname{Re}\gamma \geq 0$. *If* $p(z, \zeta) \in \mathcal{H}^*[a, 1, \zeta] \cap Q$, $p(z, \zeta) + \frac{1}{\gamma} zp'(z, \zeta)$ *is univalent in* $\mathcal{U}$, *for all* $\zeta \in \overline{\mathcal{U}}$, *and satisfy*

$$h(z, \zeta) \prec\prec p(z, \zeta) + \frac{1}{\gamma} zp'(z, \zeta),$$

then

$$q(z, \zeta) \prec\prec p(z, \zeta),$$

where

$$q(z, \zeta) = \frac{\gamma}{z^\gamma} \int_0^z h(t, \zeta) t^{\gamma-1} dt.$$

The function q is the best subordinant.

Miller and Mocanu [5] introduced the notion of differential superordination as a dual concept of differential subordination [2] and was developed in [4]. The notation of strong differential subordination was introduced by Antonino and Romaguera [1]. In this paper, we obtain some strong differential superordination results of p-valent functions by using the fractional derivative operator $M_{0,z}^{\lambda,\mu,\eta}$ defined above by (1.9).

2. Main Results

Theorem 2.1. *Let* $h(z, \zeta)$ *be a convex function with* $h(0, \zeta) = 1$. *If* $f \in \mathcal{A}_\zeta^*(p)$ *and suppose that* $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}} \right)_z'$ *is univalent and* $\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p} \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. *If the strong differential superordination*

$$h(z, \zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}} \right)_z', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{2.1}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z, \zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$$

where

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider $p(z, \zeta) = \dfrac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p} = 1 + p_1(\zeta)z + p_2(\zeta)z^2 + \ldots,$

$z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$ we obtain

$$\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)' = p(z, \zeta) + zp_z'(z, \zeta), \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}}.$$

Then (2.1) becomes

$$h(z, \zeta) \prec\prec p(z, \zeta) + zp_z'(z, \zeta), \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1,$ we have

$$q(z, \zeta) \prec\prec p(z, \zeta) \text{ or } q(z, \zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$$

where

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt.$$

The function q is convex and it is the best subordinant.

□

Theorem 2.2. *Let* $q(z, \zeta)$ *be a convex function and* h *be the function defined by*

$$h(z, \zeta) = q(z, \zeta) + zq'_z(z, \zeta).$$

If $f \in \mathcal{A}^*_\zeta(p)$ *and suppose that* $\left(\frac{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}{z^{p-1}}\right)'_z$ *is univalent and* $\frac{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}{z^p} \in \mathcal{H}^*[1, 1, \zeta] \cap Q$ *and the strong differential superordination*

$$h(z, \zeta) \prec\prec \left(\frac{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}{z^{p-1}}\right)'_z, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{2.2}$$

$$(\lambda \geq 0;\ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z, \zeta) \prec\prec \frac{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}{z^p}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt.$$

The function q *is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z, \zeta) = \frac{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}{z^p}$, the strong differential superordination (2.2) becomes

$$q(z, \zeta) + zq'_z(z, \zeta) \prec\prec p(z, \zeta) + zp'_z(z, \zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.5, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta) \text{ or } q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p},$$

$$z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

The function q is the best subordinant.

□

Theorem 2.3. *Let* $h(z, \zeta) = \dfrac{\zeta + (2\beta - \zeta)z}{1 + z}$ *be a convex function in* $\mathcal{U} \times \overline{\mathcal{U}}$, $0 \le \beta < 1$. *If* $f \in \mathcal{A}_\zeta^*(p)$ *and suppose that* $\left(\dfrac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)_z'$ *is univalent and* $\dfrac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p} \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. *If the strong differential superordination*

$$h(z, \zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)_z', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{2.3}$$

$$(\lambda \ge 0;\ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z, \zeta) \prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where q *is given by* $q(z, \zeta) = 2\beta - \zeta + \dfrac{2(\zeta - \beta)}{z}\ln(1 + z)$, $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z, \zeta) = \dfrac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}$, the strong differential superordination (2.3) becomes

$$h(z, \zeta) = \frac{\zeta + (2\beta - \zeta)z}{1 + z} \prec\prec p(z, \zeta) + zp'_z(z, \zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta),$$

or

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt$$

$$= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta - \zeta)t}{1 + t}\, dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(1 + z)$$

$$\prec\prec \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

The function q is convex and it is the best subordinant.

□

Theorem 2.4. *Let $h(z, \zeta)$ be a convex function with $h(0, \zeta) = 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and suppose that $\left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}\right)'_z$ is univalent and $\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)} \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. If the strong differential superordination*

$$h(z, \zeta) \prec\prec \left(\frac{zM_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}\right)'_z, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{2.4}$$

$$(\lambda \geq 0;\ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z,\,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1} f(z,\,\zeta)}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)}, \quad z \in \mathcal{U},\, \zeta \in \overline{\mathcal{U}},$$

where

$$q(z,\,\zeta) = \frac{1}{z}\int_0^z h(t,\,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\,\zeta) = \dfrac{M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1} f(z,\,\zeta)}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)}$, $z \in \mathcal{U},\, \zeta \in \overline{\mathcal{U}},$ we have

$$p_z'(z,\,\zeta) = \frac{\left(M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1} f(z,\,\zeta)\right)_z'}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)} - p(z,\,\zeta)\frac{\left(M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)\right)_z'}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)},$$

and we obtain

$$p(z,\,\zeta) + zp_z'(z,\,\zeta) = \left(\frac{zM_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1} f(z,\,\zeta)}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)}\right)_z'.$$

Then (2.4) becomes

$$h(z,\,\zeta) \prec\prec p(z,\,\zeta) + zp_z'(z,\,\zeta), \quad z \in \mathcal{U},\, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1,$ we have

$$q(z,\,\zeta) \prec\prec p(z,\,\zeta) \text{ or } q(z,\,\zeta) \prec\prec \frac{M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1} f(z,\,\zeta)}{M_{0,z}^{\lambda,\,\mu,\,\eta} f(z,\,\zeta)}, \quad z \in \mathcal{U},\, \zeta \in \overline{\mathcal{U}},$$

where

$$q(z,\,\zeta) = \frac{1}{z}\int_0^z h(t,\,\zeta)dt.$$

The function q is convex and it is the best subordinant.

□

Theorem 2.5. *Let* $q(z, \zeta)$ *be a convex function and* h *be the function defined by* $h(z, \zeta) = q(z, \zeta) + zq'_z(z, \zeta)$. *If* $f \in \mathcal{A}^*_\zeta(p)$ *and suppose that* $\left(\frac{zM^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}\right)'_z$ *is univalent and* $\frac{M^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)} \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. *If the strong differential superordination*

$$h(z, \zeta) \prec\prec \left(\frac{zM^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}\right)'_z, \quad z \in \mathcal{U}, \zeta \in \overline{U}, \tag{2.5}$$

$$(\lambda \geq 0;\ \mu < p + 1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z, \zeta) \prec\prec \frac{M^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}, \quad z \in \mathcal{U}, \zeta \in \overline{U},$$

where

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt.$$

The function q *is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z, \zeta) = \frac{M^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)}$, the strong differential superordination (2.5) becomes

$$q(z, \zeta) + zq'_z(z, \zeta) \prec\prec p(z, \zeta) + zp'_z(z, \zeta), \quad z \in \mathcal{U}, \zeta \in \overline{U}.$$

By using Lemma 1.5, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta) \text{ or } q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt \prec\prec \frac{M^{\lambda+1,\mu+1,\eta+1}_{0,z} f(z, \zeta)}{M^{\lambda,\mu,\eta}_{0,z} f(z, \zeta)},$$

$$z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$$

The function q is the best subordinant.

□

Theorem 2.6. *Let* $h(z, \zeta) = \dfrac{\zeta + (2\beta - \zeta)z}{1 + z}$ *be a convex function in* $\mathcal{U} \times \overline{\mathcal{U}}$, $0 \le \beta < 1$. *If* $f \in \mathcal{A}^*_\zeta(p)$ *and suppose that* $\left(\dfrac{zM^{\lambda+1,\,\mu+1,\,\eta+1}_{0,\,z} f(z,\,\zeta)}{M^{\lambda,\,\mu,\,\eta}_{0,\,z} f(z,\,\zeta)}\right)'_z$ *is univalent and* $\dfrac{M^{\lambda+1,\,\mu+1,\,\eta+1}_{0,\,z} f(z,\,\zeta)}{M^{\lambda,\,\mu,\,\eta}_{0,\,z} f(z,\,\zeta)} \in \mathcal{H}^*[1,\,1,\,\zeta] \cap Q$. *If the strong differential superordination*

$$h(z,\,\zeta) \prec\prec \left(\frac{zM^{\lambda+1,\,\mu+1,\,\eta+1}_{0,\,z} f(z,\,\zeta)}{M^{\lambda,\,\mu,\,\eta}_{0,\,z} f(z,\,\zeta)}\right)'_z, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}}, \tag{2.6}$$

$$(\lambda \ge 0;\ \mu < p + 1;\ \eta > \max(\lambda,\,\mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z,\,\zeta) \prec\prec \frac{M^{\lambda+1,\,\mu+1,\,\eta+1}_{0,\,z} f(z,\,\zeta)}{M^{\lambda,\,\mu,\,\eta}_{0,\,z} f(z,\,\zeta)}, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$$

where q *is given by* $q(z,\,\zeta) = 2\beta - \zeta + \dfrac{2(\zeta - \beta)}{z}\ln(1 + z)$, $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\,\zeta) = \dfrac{M^{\lambda+1,\,\mu+1,\,\eta+1}_{0,\,z} f(z,\,\zeta)}{M^{\lambda,\,\mu,\,\eta}_{0,\,z} f(z,\,\zeta)}$, the strong differential superordination (2.6) becomes

$$h(z,\,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1 + z} \prec\prec p(z,\,\zeta) + zp'_z(z,\,\zeta), \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta),$$

or

$$\begin{aligned} q(z, \zeta) &= \frac{1}{z}\int_0^z h(t, \zeta)dt \\ &= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta - \zeta)t}{1+t}\, dt \\ &= 2\beta - \zeta + \frac{2(\zeta - \beta)}{z}\ln(1+z) \\ &\prec\prec \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}. \end{aligned}$$

The function q is convex and it is the best subordinant.

□

Theorem 2.7. *Let $h(z, \zeta)$ be a convex function such that $h(0, \zeta) = 1$. If $f \in \mathcal{A}_\zeta^*(p)$ and*

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z, \zeta)}{z^p} + (p-\mu)(3 + 2\mu - 2p) \times \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{z^p} + (\mu - p + 1)^2 \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}$$

is univalent and $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)' \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. If the strong differential superordination

$$h(z, \zeta) \prec\prec (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z, \zeta)}{z^p} + (p-\mu)(3 + 2\mu - 2p)$$

$$\times \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{z^p} + (\mu - p + 1)^2 \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \tag{2.7}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}} \right)_z', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where

$$q(z,\zeta) = \frac{1}{z} \int_0^z h(t,\zeta)dt.$$

The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}} \right)_z' = 1 + p_1(\zeta)z + p_2(\zeta)z^2 +$ ..., $z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}$, we have

$$p(z,\zeta) + zp_z'(z,\zeta) = (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z,\zeta)}{z^p}$$

$$+ (p-\mu)(3+2\mu-2p)\frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z,\zeta)}{z^p}$$

$$+ (\mu - p + 1)^2 \frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^p}.$$

Then (2.7) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp_z'(z,\zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta) \text{ or } q(z,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}} \right)_z', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where

$$q(z,\,\zeta) = \frac{1}{z}\int_0^z h(t,\,\zeta)dt.$$

The function q is convex and it is the best subordinant.

□

Theorem 2.8. *Let $q(z,\,\zeta)$ be a convex function and h be the function defined by $h(z,\,\zeta) = q(z,\,\zeta) + zq'_z(z,\,\zeta)$, $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$. If $f \in \mathcal{A}^*_\zeta(p)$ and suppose that*

$$(p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\,\mu+2,\,\eta+2}f(z,\,\zeta)}{z^p} + (p-\mu)(3+2\mu-2p)$$

$$\times \frac{M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1}f(z,\,\zeta)}{z^p} + (\mu-p+1)^2\,\frac{M_{0,z}^{\lambda,\,\mu,\,\eta}f(z,\,\zeta)}{z^p}$$

is univalent and $\left(\dfrac{M_{0,z}^{\lambda,\,\mu,\,\eta}f(z,\,\zeta)}{z^{p-1}}\right)'_z \in \mathcal{H}^*[1,\,1,\,\zeta] \cap Q$. If the strong differential superordination

$$h(z,\,\zeta) \prec\prec (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\,\mu+2,\,\eta+2}f(z,\,\zeta)}{z^p} + (p-\mu)(3+2\mu-2p)$$

$$\times \frac{M_{0,z}^{\lambda+1,\,\mu+1,\,\eta+1}f(z,\,\zeta)}{z^p} + (\mu-p+1)^2\,\frac{M_{0,z}^{\lambda,\,\mu,\,\eta}f(z,\,\zeta)}{z^p}, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}}, \tag{2.8}$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda,\,\mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z,\,\zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\,\mu,\,\eta}f(z,\,\zeta)}{z^{p-1}}\right)'_z, \quad z \in \mathcal{U},\ \zeta \in \overline{\mathcal{U}},$$

where

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt.$$

The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z, \zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)'_z$, the strong differential superordination (2.8) becomes

$$q(z, \zeta) + zq'_z(z, \zeta) \prec\prec p(z, \zeta) + zp'_z(z, \zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.5, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta) \text{ or } q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)'_z,$$

$$z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

The function q is the best subordinant.

□

Theorem 2.9. *Let* $h(z, \zeta) = \frac{\zeta + (2\beta - \zeta)z}{1 + z}$ *be a convex function in* $\mathcal{U} \times \overline{\mathcal{U}}$, $0 \le \beta < 1$. *If* $f \in A_\zeta^*(p)$ *and suppose that*

$$(p - \mu)(p - \mu - 1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2} f(z, \zeta)}{z^p} + (p - \mu)(3 + 2\mu - 2p) \times \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1} f(z, \zeta)}{z^p} + (\mu - p + 1)^2 \frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^p}$$

is univalent and $\left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z, \zeta)}{z^{p-1}}\right)'_z \in \mathcal{H}^*[1, 1, \zeta] \cap Q$. *If the strong superordination*

$$h(z, \zeta) \prec\prec (p-\mu)(p-\mu-1)\frac{M_{0,z}^{\lambda+2,\mu+2,\eta+2}f(z, \zeta)}{z^p} + (p-\mu)(3+2\mu-2p)$$

$$\times \frac{M_{0,z}^{\lambda+1,\mu+1,\eta+1}f(z, \zeta)}{z^p} + (\mu-p+1)^2 \frac{M_{0,z}^{\lambda,\mu,\eta}f(z, \zeta)}{z^p}, \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}, \quad (2.9)$$

$$(\lambda \geq 0;\ \mu < p+1;\ \eta > \max(\lambda, \mu) - p - 1;\ p \in \mathbf{N})$$

holds, then

$$q(z, \zeta) \prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z, \zeta)}{z^{p-1}}\right)', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}},$$

where q is given by $q(z, \zeta) = 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z)$, $z \in \mathcal{U}$, $\zeta \in \overline{\mathcal{U}}$. *The function q is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z, \zeta) = \left(\frac{M_{0,z}^{\lambda,\mu,\eta}f(z, \zeta)}{z^{p-1}}\right)'$, the strong differential superordination (2.9) becomes

$$h(z, \zeta) = \frac{\zeta + (2\beta-\zeta)z}{1+z} \prec\prec p(z, \zeta) + zp'_z(z, \zeta), \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

By using Lemma 1.4, for $\gamma = 1$, we have

$$q(z, \zeta) \prec\prec p(z, \zeta),$$

or

$$q(z, \zeta) = \frac{1}{z}\int_0^z h(t, \zeta)dt$$

$$= \frac{1}{z}\int_0^z \frac{\zeta + (2\beta-\zeta)t}{1+t}\,dt$$

$$= 2\beta - \zeta + \frac{2(\zeta-\beta)}{z}\ln(1+z)$$

$$\prec\prec \left(\frac{M_{0,z}^{\lambda,\mu,\eta} f(z,\zeta)}{z^{p-1}} \right)', \quad z \in \mathcal{U}, \zeta \in \overline{\mathcal{U}}.$$

The function q is convex and it is the best subordinant.

□

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