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A STUDY OF A CLASS OF FUNCTIONS DEFINED BY SUBORDINATION, CONVOLUTION AND INTEGRAL OPERATORS

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Abstract

Inclusion relations for certain classes of analytic functions defined by subordination, convolution, and integral operators are investigated. This interesting family of analytic functions provides a continuous passage from convexity to starlikeness in the open unit disk $|z| < 1$.

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1. Introduction

For functions f and g analytic in the open unit disk $U = \{z : |z| < 1\}$, we say that f is subordinate to g, written $f \prec g$, if there exists a Schwartz function $w(z)$ such that $f(z) = g(w(z))$, where $w(0) = 0$ and $|w(z)| < 1$ in U.

For real numbers A and B, $(-1 \leq B < A \leq 1)$ and for functions p analytic in U and normalized by $p(0) = 1$, we write $p \in P[A, B]$, if and only if

$$p(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in U.$$

For $p(z) = zf'(z)/f(z)$, two classes that have been studied extensively are $S[A, B] \equiv P[A, B]$ and $K[A, B] = \{f : zf' \in S[A, B]\}$. For $0 \leq \alpha < 1$, we observe that $S[1 - 2\alpha, -1]$ is the class of starlike functions of order α and $K[1 - 2\alpha, -1]$ is the class of convex functions of order α. The class $P[A, B]$ was studied by Janowski [4] and the relationships between $S[A, B]$ and $K[A, B]$ were studied in [3, 7].

Let $A(m)$ denote the class of functions f, which are analytic in U and normalized by $f(0) = f^{(m)}(0) - m! = 0$, $(m \in \mathbb{N} = \{1, 2, 3, \ldots\})$. These functions may be represented by the power series

$$f(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k, \quad m \in \mathbb{N} = \{1, 2, 3, \ldots\}.$$

For $0 \leq \lambda \leq 1$, $n > -m$, $m \in \mathbb{N}$ and $-1 \leq B < A \leq 1$, we define $H_{m,n}(\lambda, A, B)$ to be the class of functions $f \in A(m)$ so that

$$\frac{1}{m}\left[(1 - \lambda)\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)} + \lambda\left(1 + \frac{z(I_{m,n}f(z))''}{(I_{m,n}f(z))'}\right)\right] \prec \frac{1 + Az}{1 + Bz},$$

where

$$I_{m,n}f(z) = z^m + \sum_{k=m+1}^{\infty} \frac{(m+1)(m+2)\ldots k}{(n+m)(n+m+1)\ldots(n+k-1)} a_k z^k.$$

The integral operator $I_{m,n} : A(m) \to A(m)$ was defined by Liu and Noor [5] using the Goel and Sohi [2] operator $D^{m+n-1} : A(m) \to A(m)$, where

$$D^{m+n-1}f(z) = \frac{z^m}{(1-z)^{m+n}} * f(z) = z^m + \sum_{k=m+1}^{\infty} \binom{m+n-1}{k-m} a_k z^k,$$

and

$$I_{m,n}f(z) = f_{m,n}^{(-1)}(z) * f(z) = \left(\frac{z^m}{(1-z)^{m+n}}\right)^{(-1)} * f(z).$$

The operator $D^n : A(1) \to A(1)$ was first defined by Ruscheweyh [6] and it is easy to verify that $z(I_{m,n+1}f(z))' = (m+n)I_{m,n}f(z) - nI_{m,n+1}f(z)$, $I_{m,0}f(z) = zf'(z)/m$, and $I_{m,1}f(z) = f(z)$.

We note that for $f \in H_{m,n}(0, 1-2\alpha, -1)$; $0 \le \alpha < 1$, the class $I_{m,n} : A(m) \to A(m)$ consists of functions starlike of order α and for $f \in H_{m,n}(1, 1-2\alpha, -1)$; $0 \le \alpha < 1$, the class $I_{m,n} : A(m) \to A(m)$ consists of functions convex of order α. Moreover, the family $H_{m,n}(\lambda, A, B)$ provides a continuous passage from convexity to starlikeness for the class $I_{m,n} : A(m) \to A(m)$ as λ descends in value from $\lambda = 1$ to $\lambda = 0$. The aim of this paper is to investigate the inclusion relations between the various classes of $H_{m,n}(\lambda, A, B)$ for different values of n, λ, A, and B.

2. Main Results

To prove our theorems in this section, we shall need the following lemma, which can be found in [1, Theorem 1, page 342]:

Lemma 2.1. *Let* $h(z)$ *be convex and univalent in* U *with* $h(0) = 1$ *and* $\operatorname{Re}(n + mh(z)) > 0$; $m, n \in \mathbb{C}$. *If* p *is analytic in* U *with* $p(0) = 1$, *then*

$$p(z) + \frac{zp'(z)}{n + mp(z)} \prec h(z) \Rightarrow p(z) \prec h(z); \quad z \in U.$$

Our first two theorems establish containment relations between the two classes $H_{m,n}(\lambda, A, B)$ and $H_{m,n+1}(\lambda, A, B)$ for the fixed values of $\lambda : (\lambda = 0; \lambda = 1)$.

Theorem 2.2. *If* $n > -m$, $m \in \mathbb{N}$ *and* $-1 \leq B < A \leq 1$, *then* $H_{m,n}(0, A, B) \subset H_{m,n+1}(0, A, B)$.

Proof. Let $f \in H_{m,n}(0, A, B)$. Setting $p(z) = \frac{1}{m}\left(\frac{z(I_{m,n+1}f(z))'}{I_{m,n+1}f(z)}\right)$, where p is analytic in U and $p(0) = 1$, we observe that

$$(n + m)\frac{I_{m,n}f(z)}{I_{m,n+1}f(z)} = n + m.p(z).$$

Differentiating the above equation logarithmically, we obtain

$$\frac{m + n}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n+1}f(z)} - \frac{I_{m,n}f(z)}{I_{m,n+1}f(z)} \cdot \frac{z(I_{m,n+1}f(z))'}{I_{m,n+1}f(z)}\right) = z.p'(z).$$

A simple algebraic manipulation yields

$$p(z) + \frac{z.p'(z)}{n + m.p(z)} = \frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right) \prec \frac{1 + Az}{1 + Bz}.$$

Now, by definition, for $f \in H_{m,n}(0, A, B)$, we have

$$\frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right) \prec \frac{1+Az}{1+Bz}.$$

Therefore,

$$p(z) + \frac{z.p'(z)}{n+m.p(z)} \prec \frac{1+Az}{1+Bz}.$$

From this and Lemma 2.1, we conclude that

$$p(z) = \frac{1}{m}\left(\frac{z(I_{m,n+1}f(z))'}{I_{m,n+1}f(z)}\right) \prec \frac{1+Az}{1+Bz}.$$

Thus $f \in H_{m,n+1}(0, A, B)$.

From the above Theorem 2.2 and definition of $H_{m,n}(\lambda, A, B)$, it follows that

$$f \in H_{m,n}(1, A, B) \Leftrightarrow zf' \in H_{m,n}(0, A, B) \Rightarrow zf' \in H_{m,n+1}(0, A, B)$$

$$\Leftrightarrow f \in H_{m,n+1}(1, A, B).$$

This yields the following:

Theorem 2.3. *If* $n > -m$, $m \in \mathbb{N}$ *and* $-1 \leq B < A \leq 1$, *then* $H_{m,n}(1, A, B) \subset H_{m,n+1}(1, A, B)$.

Our next theorem proves that the classes $H_{m,n}(\lambda, A, B)$ for different values of $0 < \lambda \leq 1$ are subsets of $H_{m,n}(0, A, B)$.

Theorem 2.4. *If* $0 < \lambda \leq 1$, $n > -m$, $m \in \mathbb{N}$ *and* $-1 \leq B < A \leq 1$, *then* $H_{m,n}(\lambda, A, B) \subset H_{m,n}(0, A, B)$.

Proof. Let $f \in H_{m,n}(\lambda, A, B)$. Setting $p(z) = \frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right)$, where p is analytic in U and $p(0) = 1$ and differentiating logarithmically, we obtain

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z(I_{m,n}f(z))''}{(I_{m,n}f(z))'} - \frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)},$$

or

$$p(z) + \frac{\lambda}{m}\frac{zp'(z)}{p(z)} = \frac{1}{m}\left[(1-\lambda)\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)} + \lambda\left(1 + \frac{z(I_{m,n}f(z))''}{(I_{m,n}f(z))'}\right)\right].$$

Since $f \in H_{m,n}(\lambda, A, B)$, by definition, the right hand side of the above equation is subordinate to $(1 + Az)/(1 + Bz)$, and so we can write

$$p(z) + \frac{\lambda}{m}\frac{zp'(z)}{p(z)} \prec \frac{1 + Az}{1 + Bz}.$$

Now an application of Lemma 2.1 yields

$$p(z) = \frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right) \prec \frac{1 + Az}{1 + Bz}.$$

Therefore, $f \in H_{m,n}(0, A, B)$.

Finally, we examine the containment relations between the classes of $H_{m,n}(\lambda_i, A, B)$ for different values of λ_i : $(0 \le \lambda_i \le 1, i \in \mathbb{N})$.

Theorem 2.5. *If* $0 \le \lambda_1 \le \lambda_2 \le 1$, $n > -m$, $m \in \mathbb{N}$ *and* $-1 \le B < A \le 1$, *then* $H_{m,n}(\lambda_2, A, B) \subseteq H_{m,n}(\lambda_1, A, B)$.

Proof. Let $f \in H_{m,n}(\lambda_2, A, B)$. Then by Theorem 2.4, we have $f \in H_{m,n}(0, A, B)$, or

$$\frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right) \prec \frac{1 + Az}{1 + Bz}.$$

Now consider $0 \le \lambda_3 = \dfrac{\lambda_1}{\lambda_2} \le 1$. So, we can write

$$\frac{1}{m}\left[(1-\lambda_1)\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}+\lambda_1\left(1+\frac{z(I_{m,n}f(z))''}{(I_{m,n}f(z))'}\right)\right]$$

$$=(1-\lambda_3)\frac{1}{m}\left(\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}\right)+\lambda_3\frac{1}{m}\left[(1-\lambda_2)\frac{z(I_{m,n}f(z))'}{I_{m,n}f(z)}+\lambda_2\left(1+\frac{z(I_{m,n}f(z))''}{(I_{m,n}f(z))'}\right)\right]$$

$$\prec(1-\lambda_3)\frac{1+Az}{1+Bz}+\lambda_3\frac{1+Az}{1+Bz}=\frac{1+Az}{1+Bz}.$$

Therefore $f\in H_{m,n}(\lambda_1, A, B)$.

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