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RECONSTRUCTION FORMULAS BY LINEAR WAVELET PACKETS ON THE LAGUERRE GENERALIZED HYPERGROUP

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Abstract

The goal of this work is to generalize the important subject of linear wavelet packets to the case of harmonic analysis on the Laguerre generalized hypergroup. For that, we consider the family of Laguerre functions defined on $\mathbb{K} = [0, +\infty[$, which are eigenfunctions of a given differential operator. These functions satisfy a product formula, which permits to define a convolution structure on $\mathbb{K}$ leading to obtain a commutative hypergroup called the generalized Laguerre hypergroup. Using some harmonic analysis results on $\mathbb{K}$, we present a construction of a linear wavelet packets and of the corresponding linear wavelet packet transform, and we prove for this transform a reconstruction formula. Finally, using the corresponding scale discrete linear scaling function, we establish new reconstruction formulas on the Laguerre generalized hypergroup.

1. Introduction

The notion of generalized hypergroup were introduced in [1, 9]. In this work, we study an important example of generalized hypergroup. More precisely, we consider the family of functions φ_n^α, $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$, defined on $\mathbb{K} = [0, +\infty[$ by

$$\varphi_n^\alpha(x) = e^{-x^2/2} \frac{L_n^\alpha(x^2)}{L_n^\alpha(0)}, \tag{1.1}$$

where L_n^α is the Laguerre polynomial of degree n and index α.

The functions φ_n^α, $n \in \mathbb{N}$, are the eigenfunctions of the differential operator

$$\Delta = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx} - x^2. \tag{1.2}$$

The functions φ_n^α, $n \in \mathbb{N}$, satisfy a product formula, which permits to define a generalized convolution $\star$ on $\mathbb{K}$ and to show that $\mathbb{K}$ provided with this generalized convolution is a generalized hypergroup called the *Laguerre generalized hypergroup*.

The first part of this work has been studied by many authors [4, 6, 7, 9], it concerns the harmonic analysis on the Laguerre generalized hypergroup.

The objective of the second part of this work, is to generalize the important subject of linear wavelets to the case of the harmonic analysis on the Laguerre generalized hypergroup $\mathbb{K}$. It is believed that these linear wavelet packets will be a very useful tool in many areas of mathematics. Moreover, numerous wavelet theory based techniques have evolved independently in different real life applications, for example, signal processing applications, image compression, speech recognition, geomathematical problems, etc.

The goal of this work is to study the continuous linear wavelet analysis on the generalized hypergroup $\mathbb{K}$. According to [3], we present a construction of linear wavelet packets and of the corresponding linear wavelet packet transform, on the Laguerre generalized hypergroup $\mathbb{K}$.

This paper is organized as follows:

In the second section, we give some preliminaries on harmonic analysis properties of the Laguerre generalized hypergroup.

The third section is devoted to the generalized Fourier transform, which will be useful for our purpose. We prove in this section Plancherel, inversion, and Paley-Wiener theorems for the generalized Fourier transform on $\mathbb{K}$.

Finally, in the fourth section, we define and study linear wavelets and the continuous linear wavelet transform on the generalized hypergroup $\mathbb{K}$, and we prove for this transform a reconstruction formula. Next using these results, we study linear wavelet packets and the corresponding linear wavelet packet transform on the generalized hypergroup $\mathbb{K}$, and we prove also for this transform a reconstruction formula. We conclude this section by proving reconstruction formulas associated to the scale discrete linear scaling function corresponding to linear wavelet packets on the generalized hypergroup $\mathbb{K}$.

2. Preliminaries

We consider the sequence of functions φ_n^α, $n \in \mathbb{N}$, $\alpha > -\frac{1}{2}$, defined on $\mathbb{K} = [0, +\infty[$ by relation (1.1). One knows that the function φ_n^α is the unique solution of the following differential equation:

$$\begin{cases} \Delta u(x) = -4(n + \frac{\alpha+1}{2})u(x), & n \in \mathbb{N}, \\ u(0) = 1, \quad u'(0) = 0, \end{cases} \tag{2.1}$$

where Δ is the differential operator defined on $\mathbb{K}^*$ by relation (1.2). We recall that the family φ_n^α, $n \in \mathbb{N}$, form an orthogonal basis in the space square integrable functions on $\mathbb{K}$ with respect to the measure $x^{2\alpha+1}dx$.

The functions φ_n^α, $n \in \mathbb{N}$, satisfy the following product formula:

$$\forall x, y \in \mathbb{K}, \quad \varphi_n^\alpha(x)\varphi_n^\alpha(y) =$$

$$\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \varphi_n^\alpha\left(\sqrt{x^2+y^2+2xy\cos\theta}\right) j_{\alpha-\frac{1}{2}}(xy\sin\theta)(\sin\theta)^{2\alpha}d\theta, \tag{2.2}$$

where $j_{\alpha-1/2}(u)$ is the Bessel function given by

$$j_{\alpha-1/2}(u) = \begin{cases} 2^{\alpha-1/2}\Gamma(\alpha+\frac{1}{2})\frac{J_{\alpha-1/2}(u)}{u^{\alpha-1/2}}, & u \neq 0, \\ 1, & u = 0, \end{cases} \tag{2.3}$$

and $J_{\alpha-1/2}$ is the Bessel function of first kind and index $\alpha - 1/2$.

φ_n^α verifies the property

$$\|\varphi_n^\alpha\|_\infty = \sup_{x\in\mathbb{K}} |\varphi_n^\alpha(x)| = 1. \tag{2.4}$$

We note that the relation (2.2) can also be written in the form

$$\forall x, y \in \mathbb{K}, \quad \varphi_n^\alpha(x)\varphi_n^\alpha(y) =$$

$$\int_{|x-y|}^{x+y} \varphi_n^\alpha(t) j_{\alpha-1/2}\left(\frac{\left((x+y)^2-t^2\right)^{1/2}\left(t^2-(x-y)^2\right)^{1/2}}{2}\right) W_\alpha(x, y, t)t^{2\alpha+1}dt, \tag{2.5}$$

with

$$W_\alpha(x,\ y,\ t) =$$

$$\begin{cases} \dfrac{2\Gamma(\alpha+1)}{2^{2\alpha+1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\dfrac{\left((x+y)^2-t^2\right)^{\alpha-1/2}\left(t^2-(x-y)^2\right)^{\alpha-1/2}}{(txy)^{2\alpha}}, & |x-y|<t<x+y, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.6}$$

Proposition 2.1. *The following properties hold*:

(1) $\forall x,\ y,\ t \in \mathbb{K}^* = \mathbb{K} \setminus \{0\}$, $W_\alpha(x,\ y,\ t) = W_\alpha(y,\ x,\ t) = W_\alpha(x,\ t,\ y)$.

(2) *The function* $W_\alpha(x,\ y,\ t)$ *is positive*, *continuous on* $]|x-y|,\ x+y[$ *with support in* $[|x-y|,\ x+y]$ *and verifies*

$$\int_{|x-y|}^{x+y} W_\alpha(x,\ y,\ t)t^{2\alpha+1}dt = 1. \tag{2.7}$$

Notations. Throughout this paper, we fix $\alpha > -\frac{1}{2}$, and we denote by

- $\mathcal{C}_*(\mathbb{K})$ the space of even continuous functions on $\mathbb{R}$.
- $\mathcal{C}_*^p(\mathbb{K})$, $p \in \mathbb{N}^*$, the space of even functions on $\mathbb{R}$, which are of class $\mathcal{C}^p$ on $\mathbb{R}$.
- $\mathcal{C}_{*,c}(\mathbb{K})$ (resp., $\mathcal{D}_*(\mathbb{K})$) the space of even continuous functions on $\mathbb{R}$, (resp., $\mathcal{C}^\infty$ functions on $\mathbb{R}$) with compact support.
- $L_\alpha^p(\mathbb{K},\ dm)$, $p \in [1,\ +\infty]$, the space of measurable functions on $\mathbb{K}$ such that

$$\|f\|_{\alpha,\,p} = \left(\int_{\mathbb{K}} |f(x)|^p\, dm(x)\right)^{1/p} < +\infty, \qquad 1 \le p < +\infty,$$

$$\|f\|_{\alpha,\,\infty} = ess\sup_{x\in\mathbb{K}} |f(x)| < +\infty,$$

where

$$dm(x) = x^{2\alpha+1}dx. \tag{2.8}$$

We provide these spaces with the classical topology.

Definition 2.2. (1) The generalized translation operators T_x^α, on the Laguerre generalized hypergroup $\mathbb{K}$ are defined for $f \in \mathcal{C}_*(\mathbb{K})$ by the relation

$$\forall y \in \mathbb{K}, \quad T_x^\alpha(f)(y) =$$

$$\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right) j_{\alpha-1/2}(xy\sin\theta)(\sin\theta)^{2\alpha}d\theta. \tag{2.9}$$

(2) The generalized convolution product $\star$ on the Laguerre generalized hypergroup $\mathbb{K}$ is defined for a pair of functions f and g in $\mathcal{C}_{*,c}(\mathbb{K})$ by

$$\forall x \in \mathbb{K}, \quad f \star g(x) = \int_{\mathbb{K}} T_x^\alpha(f)(y)g(y)dm(y). \tag{2.10}$$

Proposition 2.3. *The following properties hold*:

(1) *The operators* T_x^α, $x \in \mathbb{K}^*$, *can also be written in the form*

$$\forall y \in \mathbb{K}^*, \quad T_x^\alpha(f)(y) =$$

$$\int_{|x-y|}^{x+y} f(t)j_{\alpha-1/2}\left(\frac{\left((x+y)^2-t^2\right)^{1/2}\left(t^2-(x-y)^2\right)^{1/2}}{2}\right) W_\alpha(x, y, t)dm(t). \tag{2.11}$$

(2) *For all* $x, y, z \in \mathbb{K}$ *and* $f \in \mathcal{C}_*(\mathbb{K})$, *we have*

- $T_0^\alpha(f)(y) = f(y)$.
- $T_x^\alpha(f)(y) = T_y^\alpha(f)(x)$.
- $T_x^\alpha \circ T_y^\alpha(f)(z) = T_y^\alpha \circ T_x^\alpha(f)(z)$.

(3) $\forall x,\ y \in \mathbb{K},$ *we have*

$$T_x^\alpha(\varphi_n^\alpha)(y) = \varphi_n^\alpha(x)\varphi_n^\alpha(y). \tag{2.12}$$

(4) *For all* $x \in \mathbb{K}$ *and* $f \in \mathcal{D}_*(\mathbb{K})$ *such that* $supp(f) \subset [-a,\ a]$, $a \geq 0$, *the function* $f \to T_x^\alpha(f)(y)$ *belongs to* $\mathcal{D}_*(\mathbb{K})$ *and we have*

$$Supp(T_x^\alpha(f)) \subset [-x - a,\ x + a].$$

Proposition 2.4. *For all* $x \in \mathbb{K},$ *the function* $f \to T_x^\alpha(f)$ *is linear and continuous from* $L_\alpha^p(\mathbb{K},\ dm),\ p \in [1,\ +\infty],$ *into itself and we have*

$$\|T_x^\alpha(f)\|_{\alpha,\,p} \leq \|f\|_{\alpha,\,p}. \tag{2.13}$$

Proof.

• If $p = +\infty,$ then by using relation (2.11) and the fact that

$$\forall x,\ y \in \mathbb{K}, \quad \forall \theta \in [0,\ \pi], \quad |j_{\alpha-1/2}(xy\sin\theta)| \leq 1,$$

we obtain for all $x \in \mathbb{K}^*,$

$$\forall y \in \mathbb{K}^*, \quad |T_x^\alpha(f)(y)| \leq \int_{|x-y|}^{x+y} |f(t)|W_\alpha(x,\ y,\ t)dm(t). \tag{2.14}$$

Using relation (2.7), we deduce that

$$\forall y \in \mathbb{K}^*, \quad |T_x^\alpha(f)(y)| \leq \|f\|_{\alpha,\,\infty}. \tag{2.15}$$

The inequality (2.13) follows from this inequality together with the fact that

$$\forall y \in \mathbb{K}, \quad T_0^\alpha(f)(y) = f(y). \tag{2.16}$$

• If $p \in [1,\ +\infty[,$ then we have for all $x \in \mathbb{K}^*,$

$$\forall y \in \mathbb{K}^*, \quad |T_x^\alpha(f)(y)| \leq \int_{|x-y|}^{x+y} |f(t)|W_\alpha(x,\ y,\ t)dm(t). \tag{2.17}$$

Using Hölder inequality and Fubini-Tonnelli's theorem, we deduce that

$$\int_{\mathbb{K}} |T_x^{\alpha}(f)(y)|^p \, dm(y) \leq \int_{\mathbb{K}} \left(\int_{|x-y|}^{x+y} W_{\alpha}(x, y, t) dm(y) \right) |f(t)|^p \, dm(t). \quad (2.18)$$

The relation (2.13) is then obtained by using relation (2.7).

We conclude that $f \to T_x^{\alpha}(f)$ is continuous from $L_{\alpha}^p(\mathbb{K}, dm)$, $p \in [1, +\infty]$, into itself.

In the other hand, it is clear that the function $f \to T_x^{\alpha}(f)$ is linear. This completes the proof.

□

Remark 2.5. For all $x \in \mathbb{K}$ and $f \in \mathcal{C}_*(\mathbb{K})$ nonnegative, the function $T_x^{\alpha}(f)$ is not necessary nonnegative.

Proposition 2.6. *Let f be in $\mathcal{D}_*(\mathbb{R})$ with $supp(f) \subset [-a, a]$, $a \geq 0$, and g in $\mathcal{D}_*(\mathbb{R})$ with $supp(g) \subset [-b, b]$, $b \geq 0$. The the function $f \star g$ belongs to $\mathcal{D}_*(\mathbb{R})$ such that $supp(f \star g) \subset [-a - b, a + b]$.*

Proposition 2.7. *Let f be in $L_{\alpha}^p(\mathbb{K}, dm)$, $p \in [1, +\infty]$, and g in $L_{\alpha}^q(\mathbb{K}, dm)$, $q \in [1, +\infty]$. Then the function $f \star g$ belongs to $L_{\alpha}^r(\mathbb{K}, dm)$, $r \in [1, +\infty]$, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, and we have*

$$\|f \star g\|_{\alpha, r} \leq \|f\|_{\alpha, p} \|g\|_{\alpha, q}. \quad (2.19)$$

Proof.

• In the case of 1 or $+\infty$ satisfying the relation $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, the result is immediately deduced from Proposition 2.4.

• If $p, q, r \in]1, +\infty[$, we put

$$\frac{1}{a} = \frac{1}{p} - \frac{1}{r} \text{ and } \frac{1}{b} = \frac{1}{q} - \frac{1}{r}, \quad (2.20)$$

which gives

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{r}=1. \tag{2.21}$$

On the other hand, we have

$$|T_x^\alpha(f)(y)|\,|g(y)| = |T_x^\alpha(f)(y)|^{p/a}|g(y)|^{q/b}((T_x^\alpha(f)(y))^p(g(y))^q)^{1/r}. \tag{2.22}$$

By using this relation, Hölder inequality and Proposition 2.4, we obtain the desired result.

□

Remark 2.8. From [8], $(\mathbb{K}, \star)$ is a generalized hypergroup in the sense of Jewett [5]. We call it Laguerre generalized hypergroup. The involution of $(\mathbb{K}, \star)$ is the identity and m is its Haar measure.

3. Generalized Fourier Transform on the Laguerre Generalized Hypergroup

Notations. We denote by

• $\mathcal{S}_*(\mathbb{K})$ the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}$, even and rapidly decreasing together with their derivatives, i.e., satisfying

$$\forall m,\, n \in \mathbb{N}, \quad p_{m,n}(\psi) = \sup_{x\in\mathbb{K}} (1+x^2)^m \left|\frac{d^n}{dx^n}\psi(x)\right| < +\infty. \tag{3.1}$$

Equipped with the topology defined by the semi-norms $p_{m,n}$, $m, n \in \mathbb{N}$, the space $\mathcal{S}_*(\mathbb{K})$ is a Fréchet space.

• $\mathcal{S}(\mathbb{N})$ the space of complex sequences $\{\psi(n)\}_{n\in\mathbb{N}}$, which are rapidly decreasing, i.e., satisfying

$$\forall p \in \mathbb{N}, \quad q_p(\psi) = \sup_{n\in\mathbb{N}} n^p|\psi(n)| < +\infty. \tag{3.2}$$

Provided with the topology defined by the semi-norms q_p, $p \in \mathbb{N}$, the space $\mathcal{S}(\mathbb{N})$ is a Fréchet space.

• $l_\alpha^p(\mathbb{N})$, $p \in [1, +\infty[$, the space of complex sequences $\{\psi(n)\}_{n\in\mathbb{N}}$, such that

$$\|\psi\|_{l_\alpha^p} = \left(\sum_{n=0}^{+\infty} N_\alpha(n)\,|\psi(n)|^p\right)^{1/p} < +\infty, \tag{3.3}$$

with

$$N_\alpha(n) = \|\varphi_n^\alpha\|_{\alpha,2}^2 = \frac{\Gamma(n+\alpha+1)}{n!}. \tag{3.4}$$

Definition 3.1. The generalized Fourier transform $\mathcal{F}$ of a function $f \in L_\alpha^1(\mathbb{K}, dm)$ is given by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(f)(n) = \int_{\mathbb{K}} f(x)\varphi_n^\alpha(x)dm(x). \tag{3.5}$$

Proposition 3.2. *For all f, g in $L_\alpha^1(\mathbb{K}, dm)$ and $x \in \mathbb{K}$, we have the following results*:

(1) $f \star g = g \star f$,

(2) $\forall n \in \mathbb{N}$, $|\mathcal{F}(f)(n)| \le \|f\|_{\alpha,1}$,

(3) $\forall n \in \mathbb{N}$, $\mathcal{F}(T_x^\alpha(f))(n) = \varphi_n^\alpha(x)\mathcal{F}(f)(n)$,

(4) $\forall n \in \mathbb{N}$, $|\mathcal{F}(T_x^\alpha(f))(n)| \le \|f\|_{\alpha,1}$,

(5) $\forall n \in \mathbb{N}$,

$$\mathcal{F}(f \star g)(n) = \mathcal{F}(f)(n)\mathcal{F}(g)(n). \tag{3.6}$$

Notations. We denote by

• Δ_+ and Δ_- the operators defined on the space of functions defined on $\mathbb{N}$ by

$$\Delta_+ g(n) = g(n+1) - g(n),$$

$$\Delta_- g(n) = \begin{cases} g(n) - g(n-1 \ right), & n \geq 1, \\ g(0), & n = 0. \end{cases}$$

- Λ the operator defined on the space of functions defined on $\mathbb{N}$ by

$$\Lambda g(n) = n\Delta_+\Delta_- g(n) + (\alpha+1)\Delta_+ g(n).$$

Lemma 3.3. *For all* $n \in \mathbb{N}$ *and* $x \in \mathbb{K}$, *we have*

$$-x^2 \varphi_n^\alpha(x) = \Lambda \varphi_n^\alpha(x). \tag{3.7}$$

Proof. For all $n \in \mathbb{N}$, we have from ([2], p. 190), the following recurrent relation:

$$x^2 L_n^\alpha(x^2) = (2n+\alpha+1)L_n^\alpha(x^2) - (n+1)L_{n+1}^\alpha(x^2) - (n+\alpha)L_{n-1}^\alpha(x^2). \tag{3.8}$$

We also have

$$(n+1)L_{n+1}^\alpha(0) = (n+\alpha+1)L_n^\alpha(0). \tag{3.9}$$

The result follows from these relations and the definition of φ_n^α (1.1). □

We put $\Delta_1 = \Delta + x^2$. From ([1], p. 457), we have the following result:

Proposition 3.4. *Let* f *be in* $\mathcal{C}_*(\mathbb{K})$ *such that for all* $m \in \mathbb{N}$, $\Delta_1^m f$ *belongs to* $\mathcal{C}_*^k(\mathbb{K})$, *then* $f \in \mathcal{C}_*^{m+k}(\mathbb{K})$.

Proposition 3.5. *Let* f *be in* $\mathcal{C}_*(\mathbb{K})$. *Then the function* f *belongs to* $\mathcal{S}_*(\mathbb{K})$, *if and only if*

- $\forall m \in \mathbb{N}$, $x \to \Delta^m f(x)$ *belongs to* $\mathcal{C}_*^k(\mathbb{K})$,
- $\forall p, q \in \mathbb{N}$, *we have*

$$\widetilde{p_{p,q}}(f) = \sup_{x \in \mathbb{K}} (1+x^2)^p \, |\Delta^q f(x)| < +\infty.$$

Proof. The result follows from Proposition 3.4 and the fact that f belongs to $\mathcal{S}_*(\mathbb{K})$, if and only if the function $\frac{1}{x}\frac{df}{dx}$ belongs to $\mathcal{S}_*(\mathbb{K})$. □

Remark 3.6. Equipped with the topology defined by the semi-norms $\widetilde{p_{p,q}}$, the space $\mathcal{S}_*(\mathbb{K})$ is a Fréchet space. This topology is equivalent to the topology defined by the semi-norms $p_{m,n}$.

Proposition 3.7. *The generalized Fourier transform $\mathcal{F}$ is linear and continuous from $\mathcal{S}_*(\mathbb{K})$ into $\mathcal{S}(\mathbb{N})$.*

Proof. It is clear that $\mathcal{F}$ is linear.

On the other hand, for $f \in \mathcal{S}_*(\mathbb{K})$, we have the relation

$$\forall n \in \mathbb{N}, \quad \langle f, \Delta\varphi_n^\alpha \rangle_{L^2(\mathbb{K},dm)} = \int_{\mathbb{K}} f(x)\overline{\Delta\varphi_n^\alpha(x)}dm(x),$$

where $\langle .,. \rangle_{L^2(\mathbb{K},dm)}$ is the scalar product of $L^2(\mathbb{K},dm)$.

As

$$\langle f, \Delta\varphi_n^\alpha \rangle_{L^2(\mathbb{K},dm)} = \langle \Delta f, \varphi_n^\alpha \rangle_{L^2(\mathbb{K},dm)},$$

then using relations (3.5) and (2.1), we obtain

$$-4\left(n + \frac{\alpha+1}{2}\right)\mathcal{F}(f)(n) = \langle \Delta f, \varphi_n^\alpha \rangle_{L^2(\mathbb{K},dm)}.$$

For all $r \in \mathbb{N}^*$, we deduce from this relation and Hölder's inequality that

$$\forall n \in \mathbb{N}, \quad |\mathcal{F}(f)(n)| \leq 4^{-r}\left(n + \frac{\alpha+1}{2}\right)^{-r} \|\Delta^r f\|_{2,\alpha}\|\varphi_n^\alpha\|_{2,\alpha}.$$

Using the result

$$\|\varphi_n^\alpha\|_{2,\alpha} \underset{\alpha\to+\infty}{\simeq} \frac{1}{(n+1)^\alpha},$$

we deduce that there exists a constant $k_1 > 0$, such that

$$\forall r \geq 0, \quad \sup_{n\in\mathbb{N}} n^r|\mathcal{F}(f)(n)| \leq k_1\|\Delta^r f\|_{2,\alpha}, \tag{3.10}$$

or there exists a constant $k_2 > 0$, such that

$$\|\Delta^r f\|_{2,\alpha} \leq k_2\left(\sup_{x\in\mathbb{K}}(1+x^2)^{\left[\frac{\alpha}{2}\right]+2}|\Delta^r f(x)|\right).$$

We obtain the result by using this inequality together with the relation (3.10) and Remark 3.6. □

Definition 3.8. We define the operator $\mathcal{F}^{-1}$ on $l^1_\alpha(\mathbb{N})$ by

$$\mathcal{F}^{-1}(g)(x) = \sum_{n=0}^{+\infty} N_\alpha(n) g(n) \varphi_n^\alpha(x), \tag{3.11}$$

where $N_\alpha(n)$ is defined as in (3.4).

Proposition 3.9. *Let g be in $l^1_\alpha(\mathbb{N})$. Then the function $\mathcal{F}^{-1}(g)$ is continuous on $\mathbb{K}$ and we have*

$$\|\mathcal{F}^{-1}(g)\|_{\alpha,\infty} \leq \|g\|_{l^1_\alpha}. \tag{3.12}$$

Proof. The result follows from dominated convergence theorem and relation (2.4).

□

Proposition 3.10. *Let g be in $\mathcal{S}(\mathbb{N})$, then we have*

(1)

$$-x^2\mathcal{F}^{-1}(g)(x) = \mathcal{F}^{-1}(\Lambda g)(x), \quad \forall x \in \mathbb{K}. \tag{3.13}$$

(2) *$\mathcal{F}^{-1}(g)$ belongs to $\mathcal{C}^2_*(\mathbb{K})$ and we have*

$$\Delta\mathcal{F}^{-1}(g)(x) = \mathcal{F}^{-1}\left(-4\left(n+\frac{\alpha+1}{2}\right)g\right)(x), \quad \forall x \in \mathbb{K}. \tag{3.14}$$

Proof.

(1) From relation (3.11), we have for all $x \in \mathbb{K}$

$$-x^2\mathcal{F}^{-1}(g)(x) = \sum_{n=0}^{+\infty} N_\alpha(n) g(n)(-x^2\varphi_n^\alpha(x)).$$

Using relation (3.7), we obtain

$$-x^2\mathcal{F}^{-1}(g)(x) = \sum_{n=0}^{+\infty} N_\alpha(n) g(n) \Lambda \varphi_n^\alpha(x). \tag{3.15}$$

On the other hand, we have the relation

$$\sum_{n=0}^{+\infty} L_n^\alpha(0) g(n) \Lambda \varphi_n^\alpha(x) = \sum_{n=0}^{+\infty} L_n^\alpha(0) \Lambda g(n) \varphi_n^\alpha(x), \tag{3.16}$$

then we deduce that

$$-x^2\mathcal{F}^{-1}(g)(x) = \sum_{n=0}^{+\infty} N_\alpha(n) \Lambda g(n) \varphi_n^\alpha(x),$$

which completes the proof.

(2) From ([2], p. 189), we have

$$\frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x).$$

From this relation and Proposition 3.2, we deduce that $\mathcal{F}^{-1}(g)$ belongs to $\mathcal{C}_*^2(\mathbb{K})$.

The relation (3.14) is obtained from relation (2.1).

□

Proposition 3.11. *The operator* $\mathcal{F}^{-1}$ *is linear and continuous from* $\mathcal{S}(\mathbb{N})$ *into* $\mathcal{S}_*(\mathbb{K})$ *and we have*

$$\forall g \in \mathcal{S}(\mathbb{N}), \quad \mathcal{F}(\mathcal{F}^{-1}(g)) = g. \tag{3.17}$$

Proof. It is clear that $\mathcal{F}^{-1}$ is linear.

Let g be in $\mathcal{S}(\mathbb{N})$. From Proposition 3.10, we have $\forall p, q \geq 0$,

$$(1+x^2)^p \Delta^q \mathcal{F}^{-1}(g)(x) = (1+x^2)^p \mathcal{F}^{-1}\left(-4^q\left(n+\frac{\alpha+1}{2}\right)^q g\right)(x), \quad x \in \mathbb{K}.$$

By using Proposition 3.10 and relation (3.11), we obtain $\forall p, q \geq 0$,

$$(1+x^2)^p \Delta^q \mathcal{F}^{-1}(g)(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\left(-4^q\left(n+\frac{\alpha+1}{2}\right)^q\right)(I-\Lambda)^p g(n)\varphi_n^\alpha(x),\ x \in \mathbb{K},$$

then we deduce that there exists a constant $c > 0$ and $r \in \mathbb{N}$, such that

$$\forall p,\ q \geq 0, \quad \sup_{x\in\mathbb{K}} \left|(1+x^2)^p \Delta^q \mathcal{F}^{-1}(g)(x)\right| \leq c q_r(g),$$

which implies that $\mathcal{F}^{-1}(g)$ belongs to $\mathcal{S}_*(\mathbb{K})$ and the function $\mathcal{F}^{-1}$ is continuous from $\mathcal{S}(\mathbb{N})$ into $\mathcal{S}_*(\mathbb{K})$.

On the other hand, from relations (3.5) and (3.11), we have

$$\mathcal{F}(\mathcal{F}^{-1}(g))(n') = \int_0^{+\infty}\left(\sum_{n=0}^{+\infty} N_\alpha(n) g(n)\varphi_n^\alpha(x)\right)\varphi_{n'}^\alpha(x)dm(x).$$

Using Fubini theorem, we deduce that

$$\mathcal{F}(\mathcal{F}^{-1}(g))(n') = \sum_{n=0}^{+\infty} N_\alpha(n) g(n)\int_0^{+\infty} \varphi_n^\alpha(x)\varphi_{n'}^\alpha(x)dm(x).$$

From relation (3.4) and the fact that φ_n^α, $n \in \mathbb{N}$, form an orthogonal basis in the space of square integrable functions on $\mathbb{K}$, with respect to $dm(x)$, we obtain

$$\mathcal{F}(\mathcal{F}^{-1}(g))(n') = g(n'),$$

which completes the proof. □

Theorem 3.12. *The generalized Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{K})$ into $\mathcal{S}(\mathbb{N})$. It's inverse is given by relation* (3.11).

Proof. The result is deduced from Propositions 3.7 and 3.11. □

Theorem 3.13. (1) *For all $f \in \mathcal{S}_*(\mathbb{K})$, we have the following Plancherel formula*:

$$\|f\|_{\alpha,2}^2 = \|\mathcal{F}(f)\|_{l_\alpha^2}^2, \tag{3.18}$$

(2) *The generalized Fourier transform* $\mathcal{F}$ *can be extended to an isometric isomorphism from* $L^2_\alpha(\mathbb{K}, dm)$ *onto* $l^2_\alpha(\mathbb{N})$.

Proof.

(1) From Theorem 3.12, we have for all $f \in \mathcal{S}_*(\mathbb{K})$,

$$\mathcal{F}^{-1}(\mathcal{F}(f))(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(x),$$

then

$$\int_0^{+\infty} \mathcal{F}^{-1}(\mathcal{F}(f))(x)\overline{f(x)}dm(x) = \int_0^{+\infty}\left(\sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(x)\right)\overline{f(x)}dm(x).$$

Using Fubini theorem and relation (3.5), we obtain

$$\int_0^{+\infty} \mathcal{F}^{-1}(\mathcal{F}(f))(x)\overline{f(x)}dm(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\overline{\mathcal{F}(f)(n)},$$

then we deduce that

$$\|f\|^2_{\alpha,2} = \|\mathcal{F}(f)\|^2_{l^2_\alpha}.$$

(2) The result is deduced from the density of the space $\mathcal{S}(\mathbb{N})$ in $l^2_\alpha(\mathbb{N})$.

□

Lemma 3.14. *Let f be in* $L^1_\alpha(\mathbb{K}, dm)$ *and g be in* $L^2_\alpha(\mathbb{K}, dm)$. *Then we have*

$$\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g). \tag{3.19}$$

Proof. By using Proposition 3.2, we deduce that the formula (3.19) holds, if f belongs to $L^1_\alpha(\mathbb{K}, dm)$ and g in $\mathcal{S}_*(\mathbb{K})$. We deduce the result by using the density of $\mathcal{S}_*(\mathbb{K})$ in $L^2_\alpha(\mathbb{K}, dm)$ together with Proposition 2.7 and Theorem 3.13.

Proposition 3.15. *Let* ψ *be a nonnegative function in* $\mathcal{D}_*(\mathbb{K})$ *with support in* $[-1, 1]$ *such that* $\|\psi\|_{\alpha,1} = 1$. *For all* $\varepsilon > 0$, *we put*

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^{2\alpha+2}}\psi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{K},$$

then we have

(1) $\forall n \in \mathbb{N}$, $\lim_{\varepsilon\to 0} \mathcal{F}(\psi_\varepsilon)(n) = 1$,

(2) $\|\psi_\varepsilon\|_{\alpha,1} = 1$, $\forall \varepsilon > 0$,

(3) $\lim_{\varepsilon\to 0} \int_{x\geq\eta} \psi_\varepsilon(x)dm(x) = 0$, $\forall \eta > 0$.

Proposition 3.16. *Let* ψ *be a function in* $\mathcal{D}_*(\mathbb{K})$ *satisfying the hypotheses of Proposition* 3.15. *Then, we have for all f in* $L^1_\alpha(\mathbb{K}, dm)$

$$\lim_{\varepsilon\to 0} \|f \star \psi_\varepsilon - f\|_{\alpha,1} = 0. \tag{3.20}$$

Proof. From Proposition 3.15, we have

$$\forall x \in \mathbb{K}, \quad f(x) = \int_{\mathbb{K}} f(x)\psi_\varepsilon(y)dm(y).$$

Using this relation and relation (2.10), we deduce that

$$\|f \star \psi_\varepsilon - f\|_{\alpha,1} \leq \int_{\mathbb{K}}\int_{\mathbb{K}} |T^\alpha_x(f)(x) - f(x)|\psi_\varepsilon(y)dm(y)dm(x).$$

Since

$$T^\alpha_0(f)(x) = f(x), \quad \forall x \in \mathbb{K}, \tag{3.21}$$

then we deduce from the fact that the mapping $y \to T^\alpha_y(f)$ is continuous from $\mathbb{K}$ into $L^1_\alpha(\mathbb{K}, dm)$, that for all $\nu > 0$, there exists $\eta > 0$ such that

$$\forall 0 < y < \eta, \quad \int_{\mathbb{K}} |T^\alpha_x(f)(x) - f(x)|dm(x) < \nu,$$

then we have from Proposition 3.15 and Fubini-Tonnelli's theorem

$$\|f \star \psi_\varepsilon - f\|_{\alpha,1} \leq \nu + \int_{y\geq\eta} \psi_\varepsilon(y)\left(\int_{\mathbb{K}} |T_x^\alpha(f)(x) - f(x)| dm(x)\right) dm(y). \quad (3.22)$$

On the other hand, from Proposition 2.4, we have

$$\int_{\mathbb{K}} |T_x^\alpha(f)(x) - f(x)| dm(x) \leq 2\|f\|_{\alpha,1}. \quad (3.23)$$

The relation (3.20) is obtained by using relations (3.22) and (3.23) together with Proposition 3.15. □

Theorem 3.17. *Let f be in $L_\alpha^1(\mathbb{K}, dm)$ such that $\mathcal{F}(f)$ belongs to $l_\alpha^1(\mathbb{N})$. Then, we have the following inversion formula*:

$$f(x) = \mathcal{F}^{-1}(\mathcal{F}(f))(x), \text{ a.e.,} \quad x \in \mathbb{K}.$$

Proof. Let ψ_ε, $\varepsilon > 0$, be the function in $\mathcal{D}_*(\mathbb{K})$ satisfying the hypothesis of Proposition 3.15.

Since the function ψ_ε belongs to $\mathcal{D}_*(\mathbb{K})$, then Theorem 3.12 gives

$$\forall x \in \mathbb{K}, \quad \psi_\varepsilon(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(\psi_\varepsilon)(n)\varphi_n^\alpha(x).$$

Using the definition of convolution (2.10), we deduce that

$$\forall x \in \mathbb{K}, \quad f \star \psi_\varepsilon(x) = \int_{\mathbb{K}} T_x^\alpha(f)(y)\left(\sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(\psi_\varepsilon)(n)\varphi_n^\alpha(y)\right) dm(y).$$

From Fubini's theorem and relation (3.5), we obtain

$$\forall x \in \mathbb{K}, \quad f \star \psi_\varepsilon(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\mathcal{F}(\psi_\varepsilon)(n)\varphi_n^\alpha(x).$$

Using Proposition 3.15 together with the dominated convergence theorem, we deduce that

$$\forall x \in \mathbb{K}, \quad \lim_{\varepsilon\to 0} f \star \psi_\varepsilon(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(x).$$

On the other hand, from Proposition 3.16, we deduce that we can extract from $\{f \star \psi_\varepsilon\}_{\varepsilon>0}$ a subsequence, which converges to f almost everywhere. This completes the proof. □

4. Linear Wavelet Packets on the Laguerre Generalized Hypergroup

4.1. Linear wavelets on the Laguerre generalized hypergroup

Definition 4.1. We consider a family of functions ψ_ρ^L, $\rho \in]0, +\infty[$, of $L^2_\alpha(\mathbb{K}, dm)$. We say that ψ_ρ^L is a linear wavelet if it satisfies the following conditions:

(1) For all $x \in \mathbb{K}$, the function $\rho \to \psi_\rho^L(x)$ is measurable.

(2) The integral $\int_0^{+\infty} |\mathcal{F}(\psi_\rho^L)(n)| \frac{d\rho}{\rho}$ is finite and independent of n.

(3) $\forall n \in \mathbb{N}$, $\int_0^{+\infty} \mathcal{F}(\psi_\rho^L)(n) \frac{d\rho}{\rho} = 1$.

Example 4.2. The function ψ_ρ^L, $\rho \in]0, +\infty[$ defined by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_\rho^L)(n) = \left(n + \frac{\alpha+1}{2}\right)^2 \rho e^{-\rho\left(n+\frac{\alpha+1}{2}\right)^2},$$

is a linear wavelet in $L^2_\alpha(\mathbb{K}, dm)$. Let ψ_ρ^L, $\rho \in]0, +\infty[$, be a linear wavelet in $L^2_\alpha(\mathbb{K}, dm)$. For all $y \in \mathbb{K}$, we consider the function $\psi^L_{\rho, y}$ defined by

$$\forall x \in \mathbb{K}, \quad \psi^L_{\rho, y}(x) = T_y^\alpha(\psi_\rho^L)(x), \tag{4.1}$$

where T_y^α, $y \in \mathbb{K}$, are the generalized translation operators given by relation (2.9). This function satisfies the following properties:

Proposition 4.3. (1) *For all* $y \in \mathbb{K}$, *the function* $\psi_{\rho,y}^L$ *belongs to* $L_\alpha^2(\mathbb{K}, dm)$ *and we have*

$$\|\psi_{\rho,y}^L\|_{\alpha,2} \leq \|\psi_\rho^L\|_{\alpha,2}. \tag{4.2}$$

(2) *For all* $y \in \mathbb{K}$, *we have the relation*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_{\rho,y}^L)(n) = \varphi_n^{(\alpha)}(y)\mathcal{F}(\psi_\rho^L)(n). \tag{4.3}$$

Definition 4.4. Let ψ_ρ^L, $\rho \in]0, +\infty[$, be a linear wavelet in $L_\alpha^2(\mathbb{K}, dm)$. We define the linear wavelet transform ϕ_ψ^L for regular functions f on $\mathbb{K}$, by

$$\phi_\psi^L(f)(\rho, y) = \int_{\mathbb{K}} f(x)\psi_{\rho,y}^L(x)dm(x), \quad \rho \in]0, +\infty[, \quad y \in \mathbb{K}. \tag{4.4}$$

This transform can also be written in the form

$$\phi_\psi^L(f)(\rho, y) = f \star \psi_\rho^L(y), \tag{4.5}$$

where $\star$ is the convolution product defined by relation (2.10).

Proposition 4.5. *Let* ψ_ρ^L, $\rho \in]0, +\infty[$, *be a linear wavelet in* $L_\alpha^2(\mathbb{K}, dm)$. *Then for all* f *in* $L_\alpha^q(\mathbb{K}, dm)$, $q \in [1, 2]$, *the function* $y \to \phi_\psi^L(f)(\rho, y)$ *belongs to* $L_\alpha^r(\mathbb{K}, dm)$, $r \in [1, +\infty]$, *with* $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$ *and we have*

$$\|\phi_\psi^L(f)(\rho, .)\|_{\alpha,r} \leq \|f\|_{\alpha,q}\|\psi_\rho^L\|_{\alpha,2}. \tag{4.6}$$

Proof. The result is immediate by relation (4.5) and Proposition 4.3.

□

Theorem 4.6. *Let* ψ_ρ^L, $\rho > 0$, *be a linear wavelet in* $L_\alpha^2(\mathbb{K}, dm)$ *and* ϕ_Ψ^L *be the linear wavelet transform associated to* ψ_ρ^L. *For all* $f \in (L_\alpha^1 \cap L_\alpha^2)(\mathbb{K}, dm)$ *such that* $\mathcal{F}(f) \in l_\alpha^1(\mathbb{N})$, *we have the following reconstruction formula*:

$$f(y) = \int_0^{+\infty} \phi_\psi^L(f)(\rho,\ y)\frac{d\rho}{\rho}, \quad y \in \mathbb{K}, \tag{4.7}$$

where for each $y \in \mathbb{K}$, *the integral is absolutely convergent.*

Proof. From Proposition 4.3 and Parseval's formula, we deduce that for all $y \in \mathbb{K}$,

$$\phi_\psi^L(f)(\rho,\ y) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\mathcal{F}(\psi_\rho^L)(n)\varphi_n^\alpha(y). \tag{4.8}$$

Using relation (2.4) and Fubini-Tonnelli's theorem, we obtain for all $y \in \mathbb{K}$,

$$\int_0^{+\infty} |\phi_\psi^L(f)(\rho,\ y)|\frac{d\rho}{\rho} \le \sum_{n=0}^{+\infty} N_\alpha(n)|\mathcal{F}(f)(n)|\left(\int_0^{\infty} |\mathcal{F}(\psi_\rho^L)(n)|\frac{d\rho}{\rho}\right).$$

Then using Definition 4.1, we deduce that the integral given by relation (4.7) is absolutely convergent.

On the other hand, using relation (4.8) and Fubini's theorem, we obtain for all $y \in \mathbb{K}$,

$$\int_0^{+\infty} \phi_\psi^L(f)(\rho,\ y)\frac{d\rho}{\rho} = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(y)\left(\int_0^{\infty} |\mathcal{F}(\psi_\rho^L)(n)|\frac{d\rho}{\rho}\right).$$

The result is then deduced from Definition 4.1 and Theorem 3.17. □

4.2. Linear wavelet packets on the Laguerre generalized hypergroup

In this section, we define and study linear wavelet packets and the linear wavelet packet transforms, on the Laguerre generalized hypergroup $(\mathbb{K},\ \star)$, and we prove for this transform a reconstruction formula.

Let ψ_ρ^L, $\rho > 0$, be a given linear wavelet in $L_\alpha^2(\mathbb{K},\ dm)$. We consider a scale sequence $\{r_j\}_{j\in\mathbb{Z}}$ in $]0,\ +\infty[$, which is strictly decreasing such that

$$\lim_{j\to-\infty} r_j = +\infty \text{ and } \lim_{j\to+\infty} r_j = 0.$$

By assuming that, for each $j \in \mathbb{Z}$, the function $n \to \int_{r_{j+1}}^{r_j} \mathcal{F}(\psi_\rho^L)(n)\frac{d\rho}{\rho}$ is in $l_\alpha^2(\mathbb{N})$, then using Theorem 3.13 and Riesz-Fischer theorem, we deduce that there exists a function $\psi_j^{L,P} \in L_\alpha^2(\mathbb{K}, dm)$ such that

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_j^{L,P})(n) = \int_{r_{j+1}}^{r_j} \mathcal{F}(\psi_\rho^L)(n)\frac{d\rho}{\rho}. \tag{4.9}$$

Definition 4.7. The sequence $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ defined by relation (4.9) is called a *linear wavelet packet*.

Remark 4.8. For each $j \in \mathbb{Z}$, the function $\Psi_j^{L,P}$ is called a linear wavelet packet member of step j.

Example 4.9. Let ψ_ρ^L, $\rho \in]0, +\infty[$, be the linear wavelet given by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_\rho^L)(n) = \left(n + \frac{\alpha+1}{2}\right)^2 \rho e^{-\rho\left(n+\frac{\alpha+1}{2}\right)^2}.$$

From relation (4.9), we deduce that the sequence $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ defined by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}\left(\psi_j^{L,P}\right)(n) = e^{-r_{j+1}\left(n+\frac{\alpha+1}{2}\right)^2} - e^{-r_j\left(n+\frac{\alpha+1}{2}\right)^2},$$

is a linear wavelet packet.

Proposition 4.10. *We have the following properties*:

(1) $\forall n \in \mathbb{N}$,

$$\sum_{j=-\infty}^{+\infty} \left|\mathcal{F}\left(\psi_j^{L,P}\right)(n)\right| \le \int_0^{+\infty} \left|\mathcal{F}(\psi_\rho^L)(n)\right| \frac{d\rho}{\rho}. \tag{4.10}$$

(2) $\forall n \in \mathbb{N}$,

$$\sum_{j=-\infty}^{+\infty} \mathcal{F}\left(\psi_j^{L,P}\right)(n) = 1. \tag{4.11}$$

Proof. We obtain these results from relation (4.9) and Definition 4.1.

□

Let $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ be a linear wavelet packet. We consider for all $j \in \mathbb{Z}$ and $y \in \mathbb{K}$, the function $\psi_{j,y}^{L,P}$ given by

$$\forall x \in \mathbb{K}, \quad \psi_{j,y}^{L,P}(x) = T_y^{\alpha}\left(\psi_j^{L,P}\right)(x), \tag{4.12}$$

where T_y^{α}, $y \in \mathbb{K}$, are the generalized translation operators given by relation (2.11). Then we have the following results.

Proposition 4.11. *The following properties hold*:

(1) *For all* $j \in \mathbb{Z}$ *and* $y \in \mathbb{K}$, *the function* $\psi_{j,y}^{L,P}$ *belongs to* $L_{\alpha}^{2}(\mathbb{K}, dm)$ *and we have*

$$\left\|\psi_{j,y}^{L,P}\right\|_{\alpha,2} \leq \left\|\psi_j^{L,P}\right\|_{\alpha,2}. \tag{4.13}$$

(2) *For all* $j \in \mathbb{Z}$ *and* $y \in \mathbb{K}$, *we have the relations*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}\left(\psi_{j,y}^{L,P}\right)(n) = \varphi_n^{\alpha}(y)\mathcal{F}\left(\psi_j^{L,P}\right)(n), \tag{4.14}$$

and

$$\forall n \in \mathbb{N}, \quad \sum_{j=-\infty}^{+\infty}\left|\mathcal{F}\left(\psi_{j,y}^{L,P}\right)(n)\right| \leq \int_0^{+\infty}\left|\mathcal{F}\left(\psi_{\rho}^{L}\right)(n)\right|\frac{d\rho}{\rho}. \tag{4.15}$$

Proof. We deduce these results from Proposition 2.4 and relations (4.12), (4.10), and (2.4). □

Definition 4.12. Let $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ be a linear wavelet packet. The linear wavelet packet transform $\phi_{\psi}^{L,P}$ is defined for regular functions f on $\mathbb{K}$, by

$$\phi_{\psi}^{L,P}(f)(j, y) = \int_{\mathbb{K}} f(x)\psi_{j,y}^{L,P}(x)dm(x), \quad j \in \mathbb{Z}, \quad y \in \mathbb{K}. \tag{4.16}$$

This transform can also be written in the form

$$\phi_{\psi}^{L,P}(f)(j,\ y) = f \star \psi_j^{L,P}, \tag{4.17}$$

where $\star$ is the generalized convolution product defined by relation (2.10).

Proposition 4.13. *Let* $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ *be a linear wavelet packet. For all* $j \in \mathbb{Z}$ *and* $f \in L_{\alpha}^{q}(\mathbb{K},\ dm)$, $q \in [1,\ 2]$, *the function* $y \to \phi_{\psi}^{L,P}(f)(j,\ y)$ *belongs to* $L_{\alpha}^{r}(\mathbb{K},\ dm)$, *with* $r \in [1,\ +\infty]$ *such that* $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$, *and we have*

$$\left\|\phi_{\psi}^{L,P}(f)(j,.)\right\|_{\alpha,\ r} \leq \|f\|_{\alpha,\ q}\left\|\psi_j^{L,P}\right\|_{\alpha,\ 2}. \tag{4.18}$$

Proof. The result is obtained from relation (4.17) and Proposition 2.7. □

Theorem 4.14. *Let* $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ *be a linear wavelet packet and* $\phi_{\Psi}^{L,P}$ *be its associated linear wavelet packet transform. For all* $f \in (L_{\alpha}^{1} \cap L_{\alpha}^{2})$ $(\mathbb{K},\ dm)$ *such that* $\mathcal{F}(f)$ *belongs to* $l_{\alpha}^{1}(\mathbb{N})$, *we have the following reconstruction formula*:

$$f(y) = \sum_{j=-\infty}^{j=+\infty} \phi_{\Psi}^{L,P}(f)(j,\ y), \quad y \in \mathbb{K}, \tag{4.19}$$

where, for all $y \in \mathbb{K}$, *this series is absolutely convergent.*

Proof. Using Definition 4.12, Proposition 4.11, and Parseval's identity, we deduce that for all $j \in \mathbb{Z}$ and $y \in \mathbb{K}$,

$$\phi_{\Psi}^{L,P}(f)(j,\ y) = \sum_{n=0}^{+\infty} N_{\alpha}(n)\mathcal{F}(f)(n)\mathcal{F}(\psi_j^{L,P})(n)\varphi_n^{\alpha}(y), \tag{4.20}$$

since

$$\forall n \in \mathbb{N}, \quad \forall y \in \mathbb{K}, \quad |\varphi_n^{\alpha}(y)| \leq 1, \tag{4.21}$$

then Fubini-Tonnelli's theorem and Proposition 4.10 give

$$\sum_{j=-\infty}^{+\infty} \left|\phi_{\psi}^{L,P}(f)(j,\ y)\right| \leq \sum_{n=0}^{+\infty} N_{\alpha}(n)\left|\mathcal{F}(f)(n)\right|\left(\sum_{j=-\infty}^{+\infty}\left|\mathcal{F}\left(\psi_j^{L,P}\right)(n)\right|\right), \tag{4.22}$$

yields to

$$\sum_{j=-\infty}^{+\infty} \left|\phi_{\psi}^{L,P}(f)(j,\ y)\right| \leq \sum_{n=0}^{+\infty} N_{\alpha}(n)\left|\mathcal{F}(f)(n)\right|\left(\int_0^{+\infty}\left|\mathcal{F}(\psi_{\rho}^{L})(n)\right|\frac{d\rho}{\rho}\right). \tag{4.23}$$

Then from Definition 4.1, we conclude that for all $y \in \mathbb{K}$, the series (4.19) converges absolutely.

On the other hand, for all $y \in \mathbb{K}$, relation (4.20) implies that for every M, $N \in \mathbb{K}$, we have

$$\sum_{j=-N}^{M} \phi_{\psi}^{L,P}(f)(j,\ y) = \sum_{n=0}^{+\infty} N_{\alpha}(n)\mathcal{F}(f)(n)\left(\sum_{j=-N}^{M}\mathcal{F}(\psi_j^{L,P})(n)\right)\varphi_n^{\alpha}(y).$$

Using Proposition 4.10 and dominated convergence theorem, we obtain

$$\lim_{M,N\to+\infty}\sum_{j=-N}^{M}\phi_{\psi}^{L,P}(f)(j,\ y) = \sum_{n=0}^{+\infty} N_{\alpha}(n)\mathcal{F}(f)(n)\varphi_n^{\alpha}(y).$$

Relation (4.19) is deduced from Theorem 3.17. □

4.3. Discrete linear scaling function on the Laguerre generalized hypergroup

We consider the linear wavelet packet $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$ studied in the Subsection 4.2. We suppose that for all $J \in \mathbb{Z}$, the function $n \to \int_{r_J}^{+\infty}\mathcal{F}(\psi_{\rho}^{L})(n)\frac{d\rho}{\rho}$ belongs to $l_{\alpha}^{2}(\mathbb{N})$. Then from Theorem 3.13, we deduce that there exists a function $\phi_J^{L,P}$ in $L_{\alpha}^{2}(\mathbb{K},\ dm)$, such that

$$\forall n \in \mathbb{N},\quad \mathcal{F}\left(\phi_J^{L,P}\right)(n) = \int_{r_J}^{+\infty}\mathcal{F}(\psi_{\rho}^{L})(n)\frac{d\rho}{\rho}. \tag{4.24}$$

Definition 4.15. The sequence $\left\{\psi_J^{L,P}\right\}_{J\in\mathbb{Z}}$ defined by relation (4.24) is called a *discrete linear scaling function.*

Example 4.16. Let ψ_ρ^L, $\rho \in]0, +\infty[$, be the linear wavelet given by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_\rho^L)(n) = \left(n + \frac{\alpha+1}{2}\right)^2 \rho e^{-\rho\left(n+\frac{\alpha+1}{2}\right)^2}.$$

From relation (4.24), we deduce that the sequence $\left\{\psi_J^{L,P}\right\}_{J\in\mathbb{Z}}$ defined by

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_J^{L,P})(n) = e^{-r_J\left(n+\frac{\alpha+1}{2}\right)^2},$$

is a discrete linear scaling function. From relations (4.24), (4.9), and Proposition 4.10, we deduce that the functions $\left\{\psi_J^{L,P}\right\}_{J\in\mathbb{Z}}$ satisfy the following properties:

Proposition 4.17. (1) *For all* $J \in \mathbb{Z}$, *we have*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}(\psi_J^{L,P})(n) = \sum_{j=-\infty}^{J-1} \mathcal{F}(\psi_j^{L,P})(n). \tag{4.25}$$

(2) *For all* $J \in \mathbb{Z}$, *we have*

$$\forall n \in \mathbb{N}, \quad \left|\mathcal{F}(\psi_J^{L,P})(n)\right| \le \int_0^{+\infty} |\mathcal{F}(\psi_\rho^L)(n)| \frac{d\rho}{\rho}. \tag{4.26}$$

(3) *We have*

$$\forall n \in \mathbb{N}, \quad \lim_{J\to+\infty} \mathcal{F}(\psi_J^{L,P})(n) = 1. \tag{4.27}$$

Let $\left\{\psi_J^{L,P}\right\}_{J\in\mathbb{Z}}$ be a discrete linear scaling function. We consider for all $J \in \mathbb{Z}$ and $y \in \mathbb{K}$, the function $\psi_{J,y}^{L,P}$ defined by

$$\forall x \in \mathbb{K}, \quad \psi_{J,y}^{L,P}(x) = T_y^\alpha\left(\psi_J^{L,P}\right)(x), \tag{4.28}$$

where T_x^α, $x \in \mathbb{K}$, are the generalized translation operators given by relation (2.9).

The function $\psi_{J,y}^{L,P}$ satisfies the following properties:

Proposition 4.18. (1) *The function* $\psi_{J,y}^{L,P}$ *belongs to* $L_\alpha^2(\mathbb{K}, dm)$, *and we have*

$$\left\|\psi_{J,y}^{L,P}\right\|_{\alpha,2} \leq \left\|\psi_J^{L,P}\right\|_{\alpha,2}. \tag{4.29}$$

(2) *We have the relations*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}\left(\psi_{J,y}^{L,P}\right)(n) = \varphi_n^\alpha(y)\mathcal{F}(\psi_J^{L,P})(n), \tag{4.30}$$

and

$$\forall n \in \mathbb{N}, \quad \left|\mathcal{F}\left(\psi_{J,y}^{L,P}\right)(n)\right| \leq \int_0^{+\infty} |\mathcal{F}(\psi_\rho^L)(n)| \frac{d\rho}{\rho}. \tag{4.31}$$

Proof. (1) We have the result from Proposition 2.4.

(2) We obtain these results from Proposition 3.2 and relations (4.28), (4.26), and (2.4).

□

Proposition 4.19. (1) *For all* $J \in \mathbb{Z}$, *we have*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}\left(\psi_J^{L,P}\right)(n) + \sum_{j=J}^{+\infty} \mathcal{F}(\psi_j^{L,P})(n) = 1. \tag{4.32}$$

(2) *For all* $J \in \mathbb{Z}$, *we have*

$$\forall n \in \mathbb{N}, \quad \mathcal{F}\left(\psi_J^{L,P}\right)(n) = \mathcal{F}\left(\psi_{J+1}^{L,P}\right)(n) - \mathcal{F}\left(\psi_J^{L,P}\right)(n). \tag{4.33}$$

(3) *We have*

$$\forall n \in \mathbb{N}, \quad \sum_{j=-\infty}^{+\infty} \left(\mathcal{F}\left(\psi_{j+1}^{L,P}\right)(n) - \mathcal{F}(\psi_j^{L,P})(n)\right) = 1. \tag{4.34}$$

Proof. (1) From relation (4.9), we have

$$\int_0^{+\infty} \mathcal{F}(\psi_\rho^L)(n)\frac{d\rho}{\rho} = \sum_{j=-\infty}^{+\infty} \mathcal{F}(\psi_j^{L,P})(n).$$

On the other hand, we have

$$\sum_{j=-\infty}^{+\infty} \mathcal{F}(\psi_j^{L,P})(n) = \sum_{j=-\infty}^{J-1} \mathcal{F}(\psi_j^{L,P})(n) + \sum_{j=J}^{+\infty} \mathcal{F}(\psi_j^{L,P})(n).$$

We obtain the result from this relation and relations (4.11) and (4.25).

(2) The result is deduced from relation (4.25).

(3) Relation (4.34) is deduced from relations (4.33) and (4.11).

□

Theorem 4.20. *Let* $\left\{\psi_J^{L,P}\right\}_{J\in\mathbb{Z}}$ *be a discrete linear scaling function, which corresponds to the linear wavelet packet* $\left\{\psi_j^{L,P}\right\}_{j\in\mathbb{Z}}$. *For all* f *in* $(L_\alpha^1 \cap L_\alpha^2)(\mathbb{K}, dm)$ *such that* $\mathcal{F}(f)$ *belongs to* $l_\alpha^1(\mathbb{N})$, *we have the following reconstruction formulas*:

$$f(y) = \lim_{J\to+\infty} \int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P} dm(x), \quad y \in \mathbb{K}, \tag{4.35}$$

and

$$f(y) = \int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P} dm(x) + \sum_{j=J}^{+\infty} \phi_\psi^L(f)(j, y), \quad y \in \mathbb{K}, \tag{4.36}$$

where for all $y \in \mathbb{K}$, *the series converge absolutely*.

Proof.

• From Parseval's formula and relation (4.30), we obtain for all $y \in \mathbb{K}$,

$$\int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P}dm(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\mathcal{F}\left(\psi_J^{L,P}\right)(n)\varphi_n^\alpha(y). \qquad (4.37)$$

Using relations (4.26), (4.27), (2.4), and the dominated convergence theorem, we obtain

$$\lim_{J\to+\infty}\int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P}(x)dm(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(y),$$

then Theorem 3.17 gives relation (4.35).

● As in the proof of Theorem 4.14, by applying Definition 4.12, we deduce that the series $\sum_{j=J}^{+\infty}\phi_\psi^L(f)(j,\ y)$ converges absolutely and we have for all $y \in \mathbb{K}$,

$$\sum_{j=J}^{+\infty}\phi_\psi^L(f)(j,\ y) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\left(\sum_{j=J}^{+\infty}\mathcal{F}(\psi_j^{L,P})(n)\right)\varphi_n^\alpha(y). \qquad (4.38)$$

On the other hand, by using relation (4.25), we deduce that the equality (4.37) can be written in the form

$$\int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P}dm(x) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\left(\sum_{j=-\infty}^{J-1}\mathcal{F}(\psi_j^{L,P})(n)\right)\varphi_n^\alpha(y), \qquad (4.39)$$

then for all $y \in \mathbb{K}$, we obtain from relations (4.38) and (4.39)

$$\int_{\mathbb{K}} f(x)\psi_{J,y}^{L,P}dm(x) + \sum_{j=J}^{+\infty}\phi_\psi^L(f)(j,\ y) = \sum_{n=0}^{+\infty} N_\alpha(n)\mathcal{F}(f)(n)\varphi_n^\alpha(y).$$

Relation (4.36) is then deduced from Theorem 3.17.

□

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