**Journal of Mathematical Sciences: Advances and Applications**
Volume 14, Number 1, 2012, Pages 51-62

# ON PRENORMAL OPERATORS

**ADNAN A. A. JIBRIL**

Preparatory Year Deanship
King Faisal University
Al-Ahsa
Saudi Arabia
e-mail: ajibril@kfu.edu.sa

**Abstract**

In this paper, we introduce the class of prenormal operators acting on a complex Hilbert space $H$. We study some basic properties of such operators. We study the relation between the class of all prenormal operators and some other classes of operators acting on $H$.

## 1. Introduction

Let $H$ be a complex Hilbert space and $L(H)$ be the algebra of all bounded linear operators acting on $H$. If $T \in L(H)$, then $T = A + iB$ is its Cartesian decomposition, where $A$ and $B$ are selfadjoint operator in $L(H)$ and $T^*$ is its adjoint. Many classes of operator in $L(H)$ are defined according to the relation between $T$ and $T^*$. For example, $T \in L(H)$ is normal if and only if $TT^* = T^*T$; 2-normal if and only if $T^2T^* = T^*T^2$, [3]; and skew-normal if and only if $T^2 = {T^*}^2$. In this paper, we introduce

2010 Mathematics Subject Classification: 47B15.

Keywords and phrases: operator, normal, adjoint, isometry, partial isometry, idempotent, skew-normal.

Received March 18, 2012

the class of operators for which $(TT^*)^2 = (T^*T)^2$. We call such operators prenormal operators and we denote the set of all prenormal operators in $L(H)$ by $(PN)$. In Section 2 of this paper, we study some of the basic properties of operators in $(PN)$. In Section 3, we study the relation between the class $(PN)$ and some other classes of operators in $L(H)$.

## 2. Some Basic Properties of Prenormal Operators

In this section, we investigate some basic properties of prenormal operators. We start by an easily proved proposition:

**Proposition 2.1.** *If* $T \in (PN)$, *then so are*

(i) *kT for any real number k* (*i.e.*, $(PN)$ *is arcwise connected*);

(ii) $T^*$;

(iii) $T^{-1}$ *if it exists*;

(iv) *any* $S \in L(H)$ *that is unitarily equivalent to T*;

(v) *the restriction*, $T|M$, *of T to any closed subspace of H that reduces T*.

**Proof.** Immediate from the definition.

**Remark 2.1.** Unitary equivalence in Proposition 2.1 cannot be replaced by similarity as the following example shows:

**Example 2.1.** Consider the operators $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$ acting on the two-dimensional Hilbert space $R^2$. It is clear that $T \in (PN)$. Now by direct calculation, one can shows that $(SS^*)^2 = \begin{pmatrix} 20 & -40 \\ -40 & 80 \end{pmatrix} \neq \begin{pmatrix} 50 & 50 \\ 50 & 50 \end{pmatrix} = (S^*S)^2$. Thus $S \notin (PN)$. Now, let $X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and direct calculation shows that $S = X^{-1}TX$.

**Proposition 2.2.** *If* $T \in L(H)$ *such that* $T$ *is unitarily equivalent to* $T^*$, *then* $T \in (PN)$.

**Proof.** By assumption, there is a unitary operator $U \in L(H)$ such that $T^* = U^*TU$, which implies that $T = U^*T^*U$. Now

$$= U^*(T^*T)^2U \ldots \tag{A}$$

$$= U^*(TT^*)^2U \ldots. \tag{B}$$

From (A) and (B) above, we have $(TT^*)^2 = (T^*T)^2$. Thus $T \in (PN)$.

**Proposition 2.3.** $T \in L(H)$ *and* $T^*TT^* = T^3$, *then* $T \in (PN)$.

**Proof.** Suppose that $T^3 = T^*TT^*$. Multiplying the last equation from the right by $T$ and then from the left by $T$, we get $T^4 = (TT^*)^2 = (T^*T)^2$. Thus $T \in (PN)$.

The converse of Proposition 2.3 is not in general true. We show this by an example.

**Example 2.2.** Consider the operator $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ acting on $R^2$, then direct calculation shows that $T^*T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = TT^*$, which implies that $(TT^*)^2 = (T^*T)^2$. Thus $T \in (PN)$. Now by direct calculation, one can shows that

$$T^3 = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \neq \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} = T^*TT^*.$$

**Corollary 2.1.** *If* $T = A + iB$ *such that* $AB^2 = -B^2A$ *and* $BA^2 = -A^2B$, *then* $T \in (PN)$.

**Proof.** Let $T = A + iB$, then $T^* = A - iB$. Now by direct calculation, one can shows that

$$T^3 = (A^3 - AB^2 - BAB - B^2A) + i(A^2B + ABA + BA^2 - B^3)\ldots, \quad \text{(i)}$$

$$T^*TT^* = (A^3 + B^2A + AB^2 - BAB) + i(ABA - BA^2 - A^2B - B^3)\ldots. \text{(ii)}$$

Now using $AB^2 = -B^2A$ and $BA^2 = -A^2B$ in the last two equations, we get $T^3 = T^*TT^*$. Thus $T \in (PN)$.

The following example shows that $(PN)$ is not convex:

**Example 2.3.** Consider the two operators $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ acting on $R^2$, then it is clear that $S \in (PN)$ and direct calculation shows that $TT^* = 2I = T^*T$, which implies that $T \in (PN)$. Now consider the operator $\frac{1}{2}S + \frac{1}{2}T = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & 1 \end{pmatrix} = A(\text{say})$. By direct calculation, one can shows that

$$(A^*A)^2 = \begin{pmatrix} \frac{5}{16} & \frac{9}{8} \\ \frac{9}{8} & \frac{17}{8} \end{pmatrix} \neq \begin{pmatrix} \frac{41}{16} & \frac{9}{4} \\ \frac{9}{4} & 2 \end{pmatrix} = (AA^*)^2.$$

Thus $\frac{1}{2}S + \frac{1}{2}T \notin (PN)$, which implies that $(PN)$ is not convex.

In [4], the author introduced the mutually normal relation between operators in $L(H)$ : $T$ and $F$ in $L(H)$ are mutually normal, if and only if $TT^* = F^*F$ and $T^*T = FF^*$. In [4], it is proved that many properties are shared between mutually normal operators.

**Proposition 2.4.** *If $T$ and $F$ in $L(H)$ are mutually normal such that $T \in (PN)$, then $F \in (PN)$.*

**Proof.** Straightforward.

**Proposition 2.5.** *The class $(PN)$ is closed in the strong operator topology.*

**Proof.** Let $\{T_n\}$ be a sequence of operators in $(PN)$ that converges strongly to $T \in L(H)$. Then, we have $\|T_n x - Tx\| \to 0$ as $n \to \infty$ for each $x \in H$. Thus,

$$\begin{aligned}\|T_n^* x - T^* x\| &= \|(T_n^* - T^*)x\| \\ &= \|(T_n - T)^* x\| \\ &\leq \|(T_n - T)^*\| \| x \| \\ &= \|T_n - T\| \| x \| \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus $\boldsymbol{T_n^*} \xrightarrow{s} \boldsymbol{T^*}$ (strongly). Since the product of operators is sequentially continuous in the strong operator topology ([2], p. 62), $\boldsymbol{T_n^*} T_n \xrightarrow{s} T^* T$ and $T_n \boldsymbol{T_n^*} \xrightarrow{s} TT^*$, which implies that $(\boldsymbol{T_n^*} T_n)^2 \xrightarrow{s} (T^* T)^2$ and $(T_n \boldsymbol{T_n^*})^2 \xrightarrow{s} (TT^*)^2$. Since $\{T_n\}$ is in $(PN)$, $(\boldsymbol{T_n^*} \boldsymbol{T_n})^2 = (T_n \boldsymbol{T_n^*})^2$. Thus $(T_n \boldsymbol{T_n^*})^2 \xrightarrow{s} (TT^*)^2$ and $(T_n \boldsymbol{T_n^*})^2 \xrightarrow{s} (T^* T)^2$, which implies that $(TT^*)^2 = (T^* T)^2$. Thus $T \in (PN)$, which implies that $(PN)$ is closed in the strong operator topology.

$(PN)$ is not closed under addition or multiplication as shown by the following example:

**Example 2.4.** Consider the two operators $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ acting on $R^2$, then direct calculation shows that $TT^* = I = T^*T$. Thus $T \in (PN)$. Also, it is clear that $F \in (PN)$. Now consider the operator $T + F = \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} = A$ (say), then and by direct calculation, one can shows that $(AA^*)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} \neq \begin{pmatrix} 10 & 9 \\ -6 & -5 \end{pmatrix} = (A^*A)^2$. Thus $T + F \notin (PN)$. Also $TF = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = A$ (say). Thus $(A^*A)^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = (AA^*)^2$. Thus $TF \notin (PN)$.

Notice that $TF = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} = FT$.

## 3. Relation between the Class $(PN)$ and Some Other Classes of Operators in $L(H)$

In this section, we investigate the relation between the class $(PN)$ and some other classes of operators in $L(H)$. First, we notice that if $T \in L(H)$ is normal, then $T \in (PN)$. The converse of this is not in general true as we will show in Example 3.9.

**Example 3.1.** Consider the operator $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ acting on $R^2$, then direct calculation shows that $T^*T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = TT^* \neq I$, which implies that $(TT^*)^2 = (T^*T)^2$. Thus $T \in (PN)$ and $T$ is not an isometry.

**Proposition 3.1.** *If $T \in (PN)$ is an isometry, then it is a unitary.*

**Proof.** Since $T \in (PN)$, $(T^*T)^2 = (TT^*)^2$. Since $T$ is an isometry, the last equation gives $(T^*T)^2 = (TT^*)^2 = I$. Thus $(TT^*)^2 = TT^*TT^* = TT^* = I$. Thus $T$ is unitary.

**Remark 3.1.** It follows from Proposition 3.1 that there are isometric operators, which are not in $(PN)$, since otherwise all isometric operators are unitary operators, which is not true. Also Example 2.3 serves as an example of a prenormal operator, which is not an isometry.

It follows from Example 3.1 and from Remark 3.1 that:

**Proposition 3.2.** *The class* $(PN)$ *and the class of all isometries are independent.*

In [6], Kamei introduced the class of skew-normal operators acting on $H$. If $T = A + iB \in L(H)$, then $T$ is called *skew-normal* if $AB = -BA$. It follows immediately from the definition that $T$ is skew-normal, if and only if $T^2$ is Hermitian.

In the following, we give an example of a skew-normal operator, which is not prenormal:

**Example 3.2.** Consider the operator $T = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$ acting on $R^2$, then direct calculation shows that $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $T^2$ is Hermitian, which implies that $T$ is skew-normal. Now by direct calculation, one can shows that $(TT^*)^2 = \begin{pmatrix} 16 & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \neq \begin{pmatrix} \frac{1}{16} & 0 \\ 0 & 16 \end{pmatrix} = (T^*T)^2$. Thus $T \notin (PN)$.

**Example 3.3.** Consider the operator $T = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$ acting on $R^2$, then direct calculation shows that $T^*T = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = TT^*$. Thus $T \in (PN)$. However and by direct calculation, one can shows that $T^2 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, which is not Hermitian. Thus $T$ is not skew-normal.

It follows from Examples 3.2 and 3.3 that:

**Proposition 3.3.** *The class* $(PN)$ *and the class of all skew-normal operators are independent.*

In the following, we give a condition under which a skew-normal operator becomes prenormal:

**Proposition 3.4.** *If* $T \in L(H)$ *is skew-normal and* $T^2$ *is unitarily equivalent to* $T^*$, *then* $T \in (PN)$.

**Proof.** Since $T^2$ is unitarily equivalent to $T^*$, there is a unitary operator $U \in L(H)$ such that $T^* = U^*T^2U$, which implies (since $T^2$ is Hermitian) that $T = U^*T^2U$. Thus $T$ is Hermitian, which implies that $T \in (PN)$.

The following two examples show that $(PN)$ and the class of all nilpotent operators $(T^2 = 0)$ are independent:

**Example 3.4.** Consider the operator $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acting on $R^2$, then it was shown before that $T \in (PN)$. Now by direct calculation, we have $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq 0$. Thus $T$ is not a nilpotent.

**Example 3.5.** Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $R^2$, then it can be easily shown that $(TT^*)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = (T^*T)^2$. Thus $T \notin (PN)$. However and again, by direct calculation, we have $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which means that $T$ is nilpotent.

It follows from Examples 3.4 and 3.5 that:

**Proposition 3.5.** *The class* $(PN)$ *and the class of all nilpotent operators are independent.*

The following two examples show that $(PN)$ and the class of all idempotent $(T^2 = T)$ operators are independent:

**Example 3.6.** Consider the operator $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $R^2$, then it can be easily shown that $T^2 = T$. Thus $T$ is idempotent. Now by direct calculation, one can shows that $(TT^*)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = (T^*T)^2$. Thus $T \notin (PN)$.

**Example 3.7.** Consider the operator $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acting on $R^2$, then it was shown before that $T \in (PN)$. Now by direct calculation, we have $T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \neq T$. Thus $T$ is not an idempotent.

It follows from Examples 3.6 and 3.7 that:

**Proposition 3.6.** *The class $(PN)$ and the class of all idempotent operators are independent.*

In [5], the author introduced the class of $n$-normal operators in $L(H)$: $T \in L(H)$ is $n$-normal, if $T^nT^* = T^*T^n$ for some positive integer $n$. The class of all $n$-normal operators is denoted by $[nN]$. In this section, we will concentrate on the relation between the class $(PN)$ and the two classes $[2N]$ and $[3N]$.

**Proposition 3.7.** *If $T \in L(H)$ is skew-normal, then $T \in [2N]$.*

**Proof.** Since $T$ is skew-normal, then $T^{*2} = T^2$. Multiplying the last equation on the left and then on the right by $T$, we get $T^2T^* = T^{*3} = T^*T^2$. Thus $T \in [2N]$.

**Remark 3.2.** It follows from Proposition 3.7 that the operator in Example 3.2 is a 2-normal operator, which is not in $(PN)$.

Now, we give an example of a 3-normal operator, which is not in $(PN)$.

**Example 3.8.** Consider the operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $R^2$, then we have $T^3T^* = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = T^*T^3$. Thus $T \in [3N]$. However and by direct calculation, one can shows that $(TT^*)^2 = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = (T^*T)^2$. Thus $T \notin (PN)$.

**Proposition 3.8.** *If $T \in [2N] \cap [3N]$, then $T \in (PN)$.*

**Proof.** Let $T = A + iB$ be the Cartesian decomposition of $T$, then by direct calculation, we have

$$(TT^*)^2 = ((A^2$$

$$i[(A^2 + B^2)(BA - AB) + (BA - AB)(A^2 + B^2)]\ldots, \quad (*)$$

$$(T^*T)^2 = ((A^2$$

$$i[(A^2 + B^2)(AB - BA) + (AB - BA)(A^2 + B^2)]\ldots. \quad (**)$$

It is clear that $Re((TT^*)^2) = Re((T^*T)^2)$.

Now suppose that $T \in [2N] \cap [3N]$, then we have

$$A^2B = BA^2,\ B^2A = AB^2,\ A^3B = BA^3, \text{ and } B^3A = AB^3.$$

Using the last four equations, one can shows that $Im((TT^*)^2) = Im((T^*T)^2) = 0$. Thus $(TT^*)^2 = (T^*T)^2$, which means that $T \in (PN)$.

**Corollary 3.1.** *If $T \in L(H)$ and if $T^2$ and $T^3$ are normal operators, then $T \in (PN)$.*

**Proof.** Since $T^2$ and $T^3$ are normal operators, then, by Fuglede's theorem [1], we have $T^{*2}T = TT^{*2}$ and $T^{*3}T = TT^{*3}$, which implies that $T^2T^* = T^*T^2$ and $T^3T^* = T^{*3}$. Thus $T \in [2N] \cap [3N]$, which implies that $T \in (PN)$.

In the following, we give an example of an operator in $(PN)$, which is not normal:

**Example 3.9.** Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $R^2$, then it can be easily shown that $TT^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = T^*T$.

Thus $T$ is not normal. Also, it can be shown that $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T^3$, which implies that $T \in [2N] \cap [3N]$. Thus $T \in (PN)$.

The following is a condition on a prenormal operator to become normal.

**Proposition 3.9.** *If* $T \in (PN)$ *is a partial isometry*, *then* $T$ *is normal*.

**Proof.** Since $T$ is a partial isometry, then, by ([2], p. 153), $TT^*T = T$. Now multiplying the last equation on the left by $T^*$ and then on the right by $T^*$, we have $(T^*T)^2 = T^*T$ and $(TT^*)^2 = TT^*$. Since $T \in (PN)$, the left hand sides of the last two equations are equal. Thus the right sides are equal, which means that $T$ is normal.

## References


[1] B. Fuglede, A commutativity theorem for normal operators, Proc. N. A. S. 36 (1950), 35-40

[2] P. R. Halmos, A Hilbert Space Problem Book, Springer-Verlag, New York, 1982.

[3] A. A. S. Jibril, On 2-normal operators, Dirasat, Natural and Engineering Sciences 23(2) (1996), 190-195.

[4] A. A. S. Jibril, On mutually normal operators, Abhath Al-Yarmouk Basic Sciences and Engineering 8(1) (1999), 53-60.

[5] A. A. S. Jibril, On $n$-power normal operators, The Arabian Journal for Sciences and Engineering 33(2A) (2008), 247-251.

[6] E. Kamei, Operators with skew-commutative Cartesian parts, Math. Japonica 25(4) (1980), 431-432.


■