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APPROXIMATION THEOREMS IN WEIGHTED ORLICZ SPACES

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Abstract

In the present work, some direct theorems of the approximation theory in the weighted Orlicz spaces with weights satisfying so called Muckenhoupt's condition are improved with respect to the fractional moduli of smoothness.

1. Introduction and Main Results

A convex and continuous function $\Phi : [0, \infty) \to [0, \infty)$ for which $\Phi(0) = 0$, $\Phi(x) > 0$ for $x > 0$, and

$$\lim_{x \to 0} \frac{\Phi(x)}{x} = 0, \quad \lim_{x \to 0} \frac{\Phi(x)}{x} = \infty,$$

is called a *Young function*.

The *complementary Young function* ψ of Φ is defined by

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$$\psi(y) := \max\{xy - \Phi(x) : x \geq 0\},$$

for $y \geq 0$.

A Young function Φ said to satisfy Δ_2 *condition* $(\Phi \in \Delta_2)$, if there is a constant $c > 0$ such that

$$\Phi(2x) \leq c\Phi(x),$$

for all $x \in \mathbb{R}$.

A nonnegative function $\varphi : [0, \infty) \to [0, \infty)$ is said to be *quasiconvex*, if there exists a convex Young function Φ and a constant $c \geq 1$ such that, for all $x \geq 0$,

$$\Phi(x) \leq \varphi(x) \leq \Phi(cx)$$

holds.

Let $\mathbf{T} := [-\pi, \pi]$. A measurable function $\omega : \mathbf{T} \to [0, \infty]$ is called a *weight function*, if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero.

A 2π-periodic weight function ω belongs to the *Muckenhoupt class* A_p, $p > 1$, if

$$\left(\frac{1}{|I|}\int_I \omega(x)dx\right)\left(\frac{1}{|I|}\int_I \omega^{-1/(p-1)}(x)dx\right)^{p-1} \leq C,$$

with a finite constant C independent of I, where I is any subinterval of $\mathbf{T}$ and $|I|$ denotes the length of I.

Let φ be a quasiconvex Young function. We denote by $\widetilde{L}_{\varphi,\omega}(\mathbf{T})$ the class of Lebesgue measurable functions $f : \mathbf{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathbf{T}} \varphi(|f(x)|)\omega(x)dx < \infty.$$

The linear span of the *weighted Orlicz class* $\widetilde{L}_{\varphi,\omega}(\mathbf{T})$, denoted by $L_{\varphi,\omega}(\mathbf{T})$, becomes a normed space with the *Orlicz norm*

$$\|f\|_{\varphi,\omega} := \sup\left\{\int_{\mathbf{T}} |f(x)g(x)|\omega(x)dx : \int_{\mathbf{T}} \psi(|g(x)|)dx \le 1\right\}, \tag{1}$$

where ψ is the complementary Young function of φ. We define the Luxemburg norm as

$$\|f\|_{(\varphi,\omega)} := \inf\left\{k > 0 : \int_{\mathbf{T}} \varphi(k^{-1}|f(x)|)\omega(x)dx \le 1\right\}. \tag{2}$$

There exist [7, p. 23] constants $c, C > 0$ such that

$$c\|f\|_{(\varphi,\omega)} \le \|f\|_{\varphi,\omega} \le C\|f\|_{(\varphi,\omega)}.$$

For a quasiconvex function φ, we define the indice $p(\varphi)$ of φ as

$$\frac{1}{p(\varphi)} := \inf\{\beta : \beta > 0,\ \varphi^{\beta} \text{ is quasiconvex}\}.$$

If $\omega \in A_{p(\varphi)}$, then it can be easily seen that $L_{\varphi,\omega}(\mathbf{T}) \subset L_1(\mathbf{T})$ and $L_{\varphi,\omega}(\mathbf{T})$ becomes a Banach space with the Orlicz norm. The Banach space $L_{\varphi,\omega}(\mathbf{T})$ is called *weighted Orlicz space*.

Throughout this paper, the constant c denotes a generic constant, i.e., a constant, whose values can change even between different occurrences in a chain of inequalities.

Detailed information about Orlicz spaces, defined with respect to the convex Young function φ, can be found in [13]. Orlicz spaces, considered in this work, are investigated in the books [7] and [18].

For a given $f \in L^1(\mathbf{T})$, assuming

$$\int_{\mathbf{T}} f(x)dx = 0, \tag{3}$$

we define α *-th fractional* $(\alpha > 0)$ *integral* of f as [24, v. 2, p. 134]

$$I_\alpha(x, f) := \sum_{k \in \mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i \alpha \operatorname{sign} k},$$

and $\mathbb{Z}^* := \{\pm 1, \pm 2, \pm 3, \ldots\}$.

Let $\alpha > 0$ be given. We define *fractional derivative* of a function $f \in L^1(\mathbf{T})$, satisfying (3), as

$$f^{(\alpha)}(x) := \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{\alpha-[\alpha]}(x, f),$$

if the right hand side exists.

It is well-known that the Steklov's mean operator

$$\sigma_t f(x) := \frac{1}{2t} \int_{-t}^{t} f(x+u) du, \quad x \in \mathbf{T}, \quad 0 < t < \pi,$$

is bounded in $L_{\varphi, \omega}(\mathbf{T})$ when $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, and φ^α is quasiconvex for some $\alpha \in (0, 1)$ [7, Theorem 6.4.4, p. 250]. Under these conditions, for $x, t \in \mathbf{T}$, $r > 0$, and $f \in L_{\varphi, \omega}(\mathbf{T})$, we define

$$\sigma_t^r f(x) := (I - \sigma_t)^r f(x)$$

$$= \sum_{k=0}^{\infty} (-1)^k [C_k^r] \frac{1}{(2t)^k} \int_{-t}^{t} \ldots \int_{-t}^{t} f(x + u_1 + \ldots + u_k) du_1 \ldots du_k,$$

where $[C_k^r] := \dfrac{r(r-1)\ldots(r-k+1)}{k!}$ for $k > 1$, $[C_1^r] := r$ and $[C_0^r] := 1$ are binomial coefficients.

Since [21, (1.51), p. 14,]

$$|[C_k^r]| \le \frac{c}{k^{r+1}}, \quad k \in \mathbb{Z}^+,$$

we have

$$\sum_{k=0}^{\infty} |[C_k^r]| < \infty,$$

and therefore,

$$\|\sigma_t^r f(x)\|_{\varphi,\omega} \le c\|f\|_{\varphi,\omega} < \infty, \tag{4}$$

if $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, and φ^α is quasiconvex for some $\alpha \in (0, 1)$.

For $r > 0$, we define the *fractional modulus of smoothness of index r* for $f \in L_{\varphi,\omega}(\mathbf{T})$ as

$$\Omega_{\varphi,\omega}^r(f, \delta) := \sup_{0<h_i, t\le\delta} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f \right\|_{\varphi,\omega}, \tag{5}$$

when $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, and φ^α is quasiconvex for some $\alpha \in (0, 1)$. Since the operator σ_t is bounded in $L_{\varphi,\omega}(\mathbf{T})$, we have by (4) that

$$\Omega_{\varphi,\omega}^r(f, \delta) \le c\|f\|_{\varphi,\omega},$$

if $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, and φ^α is quasiconvex for some $\alpha \in (0, 1)$.

Remark. Let $r > 0$, $f \in L_{\varphi,\omega}(\mathbf{T})$, $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, and φ^α be quasiconvex for some $\alpha \in (0, 1)$. Then, the modulus of smoothness has the following properties:

(i) $\Omega_r(f, \delta)_{\varphi,\omega}$ is nonnegative and non-decreasing function of $\delta \ge 0$.

(ii) $\Omega_r(f_1 + f_2, \delta)_{\varphi,\omega} \le \Omega_r(f_1, \delta)_{\varphi,\omega} + \Omega_r(f_2, \delta)_{\varphi,\omega}$.

(iii) $\lim_{\delta\to 0} \Omega_r(f, \delta)_{\varphi,\omega} = 0$.

We denote by $E_n(f)_{\varphi,\omega}$ the best approximation of $f \in L_{\varphi,\omega}(\mathbf{T})$ by trigonometric polynomials of degree not exceeding n, i.e.,

$$E_n(f)_{\varphi,\omega} = \inf\left\{\|f - T_n\|_{\varphi,\omega} : T_n \in \prod_n,\ n = 1, 2, \ldots\right\},$$

where $\prod_n$ denotes the class of trigonometric polynomials of degree at most n. Note that the existence of $T_n^* \in \prod_n$ such that

$$E_n(f)_{\varphi,\omega} = \|f - T_n^*\|_{\varphi,\omega},$$

follows, for example, from [4, Theorem 1.1, p. 59].

Let also

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx),$$

be the Fourier series of $f \in L^1(\mathbf{T})$. In addition, we put

$$S_n(x, f) := \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx), \quad n = 0, 1, 2, \ldots.$$

In this paper, we investigate some direct problems of approximation theory in the weighted Orlicz spaces. In the literature, many results on such approximation problems have been obtained in weighted and non-weighted Lebesgue spaces. The corresponding results in the non-weighted Lebesgue spaces can be found in the books [4] and [22]. The best approximation problems by trigonometric polynomials in weighted Lebesgue spaces with weights belonging to the Muckenhoupt class were investigated in [8] and [14]. Detailed information on weighted polynomial approximation can be found in the books [5] and [15].

Approximation by trigonometric polynomials and other related problems in the Orlicz and weighted Orlicz spaces were studied in [1, 6, 9, 10, 11, 16, 17, 20, 23].

Since every convex function is quasiconvex, the Orlicz spaces considered by us in this paper are more general than the Orlicz spaces studied in the above mentioned works. Therefore, the results obtained in this paper are new also in the non-weighted cases.

Our main results are the following:

Theorem 1. *If* $r > 0$, $f \in L_{\varphi, \omega}(\mathbf{T})$, $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, *and* φ^{α} *be quasiconvex for some* $\alpha \in (0, 1)$, *then the estimate*

$$E_n(f)_{\varphi, \omega} \preceq \Omega^r_{\varphi, \omega}\left(f, \frac{1}{n}\right), \tag{6}$$

holds for $n = 1, 2, \ldots$.

This theorem was proved in [2] for $f \in L_{p, \omega}(\mathbf{T})$, $1 < p < \infty$, in [16] for $r \in \mathbb{N}$.

Theorem 2. *Let* $\varphi\left(t^{\frac{1}{p_0}}\right)$ *be a Young function for some* $p_0 > 1$ *satisfying the* Δ_2 *condition. Let* $f \in L_{\varphi, \omega}(\mathbf{T})$, $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, *and* φ^{α} *be quasiconvex for some* $\alpha \in (0, 1)$. *For* $r > 0$ *and the system of numbers* $\lambda_{\nu}^{(n)} = 1 - \left(\frac{\nu}{n}\right)^{2r} (\nu \le n)$;

$$R_n(f, \lambda)_{\varphi, \omega} := \left\| f(x) - \left[\frac{a_0}{2} + \sum_{\nu=1}^{n} \lambda_{\nu}^{(n)} (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x) \right] \right\|_{\varphi, \omega} \preceq \Omega^r_{\varphi, \omega}\left(f, \frac{1}{n}\right).$$

Theorem 3. *Let* $\varphi\left(t^{\frac{1}{p_0}}\right)$ *be a Young function for some* $p_0 > 1$ *satisfying the* Δ_2 *condition. Let* $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, *and* φ^α *be quasiconvex for some* $\alpha \in (0, 1)$. *For* $k > 0$ *and a sequence of functions* $\lambda_\nu(r) = 1 - (\nu|r - r_0|)^{2k}$, $\left(\nu \le \left[\frac{1}{|r - r_0|}\right]\right)$ *specified on some set E of the real axis and in addition for any fixed* $r \in E$ *and any function* $f \in L_{\varphi,\omega}(\mathbf{T})$, *the series* $\frac{a_0}{2} + \sum_{\nu=1}^{\infty} \lambda_\nu(r)(a_\nu \cos \nu x + b_\nu \sin \nu x)$ *convergences to a metric of the space* $L_{\varphi,\omega}(\mathbf{T})$, *then*

$$R_r(f, \lambda)_{\varphi,\omega} := \left\| f(x) - \left[\frac{a_0}{2} + \sum_{\nu=1}^{\infty} \lambda_\nu(r)(a_\nu \cos \nu x + b_\nu \sin \nu x)\right]\right\|_{\varphi,\omega}$$

$$\le C\Omega^k_{\varphi,\omega}(f, |r - r_0|).$$

The non-weighted analogues of these theorems were proved in [16] for $r \in \mathbb{N}$.

2. Auxiliary Result

Lemma 1. *If* $0 < \alpha \le \beta$, $\varphi \in \Delta_2$, $\omega \in A_{p(\varphi)}$, $f \in L_{\varphi,\omega}(\mathbf{T})$, *and* φ^α *is quasiconvex for some* $\alpha \in (0, 1)$, *then*

$$\Omega^\beta_{\varphi,\omega}\left(f, \frac{1}{n}\right) \preceq \Omega^\alpha_{\varphi,\omega}\left(f, \frac{1}{n}\right). \tag{7}$$

Proof. If $0 < \alpha \le \beta$, $\alpha, \beta \in \mathbb{N}$, then it is easy to see from (5) that

$$\Omega^\beta_{\varphi,\omega}\left(f, \frac{1}{n}\right) \preceq \Omega^\alpha_{\varphi,\omega}\left(f, \frac{1}{n}\right). \tag{8}$$

Now, we assume $0 < \alpha \le \beta < 1$. In this case, putting $\Psi(\cdot) := \sigma^\alpha_t f(\cdot)$, we have

$$\sigma_t^{\beta-\alpha}\Psi(\cdot) = \sum_{j=0}^{\infty}(-1)^j C_j^{\beta-\alpha}\frac{1}{(2t)^j}\int_{-t}^{t}\dots\int_{-t}^{t}\Psi(\cdot + u_1 + \dots + u_j)du_1\dots du_j$$

$$= \sum_{j=0}^{\infty}(-1)^j C_j^{\beta-\alpha}\frac{1}{(2t)^j}\int_{-t}^{t}\dots\int_{-t}^{t}[\sum_{k=0}^{\infty}(-1)^{\alpha-k}C_k^{\alpha}\frac{1}{(2t)^k}$$

$$\times\int_{-t}^{t}\dots\int_{-t}^{t}f(\cdot + u_1 + \dots + u_j + u_{j+1} + \dots + u_{j+k})$$

$$\times du_1\dots du_j du_{j+1}\dots du_{j+k}]$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k C_j^{\beta-\alpha}C_k^{\alpha}$$

$$\times\left[\frac{1}{(2t)^{j+k}}\int_{-t}^{t}\dots\int_{-t}^{t}f(\cdot + u_1 + \dots + u_{j+k})du_1\dots du_{j+k}\right]$$

$$= \sum_{\nu=0}^{\infty}(-1)^{\nu}C_{\nu}^{\beta}\frac{1}{(2t)^{\nu}}\int_{-t}^{t}\dots\int_{-t}^{t}f(\cdot + u_1 + \dots + u_{\nu})du_1\dots du_{\nu} = \sigma_t^{\beta}f(\cdot)\text{a.e..}$$

Then by (4),

$$\left\|\sigma_t^{\beta}f(\cdot)\right\|_{\varphi,\omega} = \left\|\sigma_t^{\beta-\alpha}\Psi(\cdot)\right\|_{\varphi,\omega} \preceq \|\sigma_t^{\alpha}f(\cdot)\|_{\varphi,\omega},$$

and

$$\Omega_{\varphi,\omega}^{\beta}\left(f, \frac{1}{n}\right) \preceq \Omega_{\varphi,\omega}^{\alpha}\left(f, \frac{1}{n}\right). \tag{9}$$

Remaining cases will follow from (8) and (9).

Lemma 2. *Let* $\varphi\left(t^{\frac{1}{p_0}}\right)$ *be a Young function for some* $p_0 > 1$ *satisfying the* Δ_2 *condition. Let* $f_n(x)(n = 1, 2, \dots)$ *be a sequence of* 2π

periodic functions in $L_{\varphi,\omega}(\mathbf{T})$, $\omega \in A_{p(\varphi)}$; *and let* $S_{n,k_n}(x)$ *be the k-th partial sum of Fourier series of the function* $f_n(x)$, $k = k_n$ *is a function of n. Then, there exists a positive constant C such that*

$$\int_{\mathbf{T}} \varphi\left(\left(\sum_{n=0}^{\infty}|S_{n,k_n}(x)|^2\right)^{1/2}\right)\omega(x)dx \le C\int_{\mathbf{T}} \varphi\left(\left(\sum_{n=0}^{\infty}|f_n(x)|^2\right)^{1/2}\right)\omega(x)dx,$$

with a constant C is independent of $f_n(x)$.

This lemma is proved by taken in [12, Theorem 1]

$$f := \sum_{n=0}^{\infty}|S_{n,k_n}(x)|^2 \quad \text{and} \quad g := \sum_{n=0}^{\infty}|f_n(x)|^2.$$

Lemma 3. *Let* $\varphi\left(t^{\frac{1}{p_0}}\right)$ *be a Young function for some* $p_0 > 1$ *satisfying the* Δ_2 *condition. Let* $\lambda_0, \lambda_1, \ldots$ *be a sequence of numbers such that*

$$|\lambda_l| \le M, \quad \sum_{\nu=2^l}^{2^{l+1}-1}|\lambda_\nu - \lambda_{\nu+1}| \le M \quad (l = 0, 1, 2, \ldots).$$

Then the series $a_0\lambda_0/2 + \sum_{\nu=0}^{\infty}\lambda_\nu(a_\nu \cos \nu x + b_\nu \sin \nu x)$, *where* a_ν, b_ν *are the Fourier coefficients of a function* $f(x) \in L_{\varphi,\omega}[0, 2\pi]$, *is a Fourier series of some function* $h(x) \in L_{\varphi,\omega}[0, 2\pi]$ *and the following inequality is valid*:

$$\int_{\mathbf{T}} \varphi(|h(x)|)\omega(x)dx \le C\int_{\mathbf{T}} \varphi(|f(x)|)\omega(x)dx.$$

This lemma is proved by the similar way of [23, Lemma 2.4].

3. Proofs of Main Results

Proof of Theorem 1. Since [3, Theorem 2]

$$E_n(f)_{\varphi,\omega} \preceq \Omega^r_{\varphi,\omega}\left(f, \frac{1}{n}\right), \quad n, r \in \mathbb{N},$$

we obtain by (7)

$$E_n(f)_{\varphi,\omega} \preceq \Omega^{[r]+1}_{\varphi,\omega}\left(f, \frac{1}{n}\right) \preceq \Omega^r_{\varphi,\omega}\left(f, \frac{1}{n}\right), \quad n \in \mathbb{N},$$

and the assertion (6) follows.

Proof of Theorem 2. Let $2^m \le n < 2^{m+1}$. By virtue of the property of the norm,

$$\left\| f(x) - \left[\sum_{\nu=0}^{n} \lambda_\nu^{(n)} A_\nu(x)\right]\right\|_{\varphi,\omega} \le \left\|\sum_{\nu=0}^{n}\left(1 - \lambda_\nu^{(n)}\right)A_\nu(x)\right\|_{\varphi,\omega} + \left\|\sum_{\nu=n+1}^{\infty} A_\nu(x)\right\|_{\varphi,\omega} = I_1 + I_2,$$

where $A_\nu(x) := a_\nu \cos \nu x + b_\nu \sin \nu x$. From [3, Lemma 3],

$$\|S_n(f, x)\|_{\varphi,\omega} \preceq \|f(x)\|_{\varphi,\omega},$$

$$\|f(x) - S_n(f, x)\|_{\varphi,\omega} \preceq E_n(f)_{\varphi,\omega}. \tag{10}$$

It follows from the latter inequality and (6) that

$$I_2 = \left\|\sum_{\nu=n+1}^{\infty} A_\nu(x)\right\|_{\varphi,\omega} \preceq E_n(f)_{\varphi,\omega} \preceq \Omega^r_{\varphi,\omega}\left(f, \frac{1}{n}\right).$$

Now, we estimate

$$I_1 = \left\|\sum_{\nu=0}^{n} \frac{1 - \lambda_\nu^{(n)}}{\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r} A_\nu(x)\left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r\right\|_{\varphi,\omega}.$$

We let $\mu_{\nu,r}^{(n)} = \left\{ \begin{array}{ll} \dfrac{1-\lambda_\nu^{(n)}}{\left(1-\dfrac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r}, & \text{for } \nu \le n, \\ 0, & \text{for } \nu > n \end{array} \right\}$. For the sequence $\left(\mu_{\nu,r}^{(n)}\right)$, the conditions of Lemma 3 are fulfilled [2]. Then, according to Lemma 3,

$$I_1 = \left\| \sum_{\nu=0}^{n} \mu_{\nu,r}^{(n)} A_\nu(x) \left(1 - \frac{\sin\frac{\nu}{n}}{\frac{\nu}{n}}\right)^r \right\|_{\varphi,\omega} \preceq \left\| (I - \sigma_{1/n})^r f \right\|_{\varphi,\omega}$$

$$= \left\| (I - \sigma_{1/n})^{[r]} (I - \sigma_{1/n})^{r-[r]} f \right\|_{\varphi,\omega}$$

$$\preceq \sup_{0<h_i,\, t \le \frac{1}{n}} \left\| \prod_{i=1}^{[r]} (I - \sigma_{h_i})(I - \sigma_t)^{r-[r]} f \right\|_{\varphi,\omega} \preceq \Omega_{\varphi,\omega}^r\left(f, \frac{1}{n}\right),$$

and the Theorem 2 is proved.

Proof of Theorem 3. By virtue of the property of the norm,

$$R_r(f,\lambda)_{\varphi,\omega} \le \left\| \sum_{\nu=1}^{\left[\frac{1}{|r-r_0|}\right]} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{\varphi,\omega} + \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{\varphi,\omega}$$

$$= I_1' + I_2'.$$

We estimate

$$I_1' = \left\| \sum_{\nu=1}^{\left[\frac{1}{|r-r_0|}\right]} \frac{1-\lambda_\nu(r)}{\left(1-\frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k} A_\nu(x) \left(1 - \frac{\sin\nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{\varphi,\omega}, \quad k > 0.$$

Let us assume

$$\mu_{\nu,r} = \left\{ \begin{array}{ll} \dfrac{1-\lambda_\nu(r)}{\left(1-\dfrac{\sin \nu|r-r_0|}{\nu|r-r_0|}\right)^k}, & \text{for } \nu \le \left[\dfrac{1}{|r-r_0|}\right] \\ 0, & \text{for } \nu > n \end{array} \right\}.$$

For the sequence $\left\{\mu_{\nu,r}^{(n)}\right\}$, the conditions of Lemma 3 are fulfilled. Consequently, according to Lemma 3, for $2^m \le \left[\dfrac{1}{|r-r_0|}\right] < 2^{m+1}$,

$$I_1' \le \left\| \sum_{\nu=1}^{2^{m+1}} \mu_{\nu,r} A_\nu(x) \left(1-\frac{\sin \nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{\varphi,\omega}$$

$$\preceq \left\| \sum_{\nu=1}^{\infty} A_\nu(x) \left(1-\frac{\sin \nu|r-r_0|}{\nu|r-r_0|}\right)^k \right\|_{\varphi,\omega} \preceq \Omega_{\varphi,\omega}^k(f, |r-r_0|).$$

Let us now estimate I_2'. It is easily seen that the conditions of Lemma 3 are fulfilled for the system of numbers $\{1-\lambda_\nu(r)\}$. Then, according to Lemma 3,

$$I_2' = \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} (1-\lambda_\nu(r)) A_\nu(x) \right\|_{\varphi,\omega} \preceq \left\| \sum_{\nu=\left[\frac{1}{|r-r_0|}\right]+1}^{\infty} A_\nu(x) \right\|_{\varphi,\omega}.$$

Hence, as in the Theorem 2, it follows

$$I_2' \preceq \Omega_{\varphi,\omega}^k(f, |r-r_0|).$$

The latter completes the proof of the theorem.

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